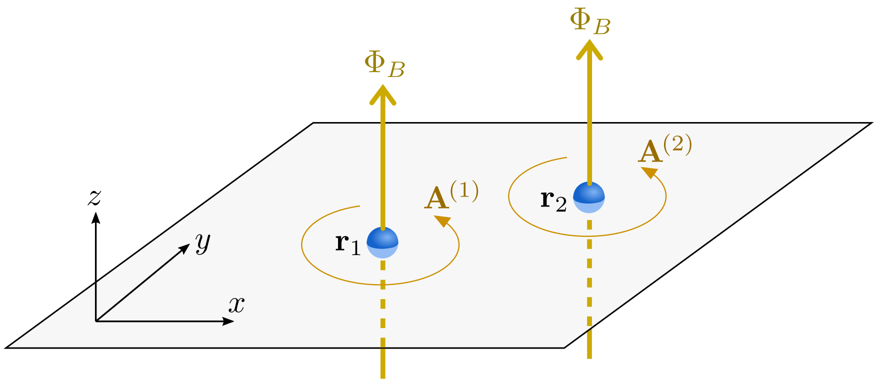
Appendix F: Anyons

One of the strangest consequences of magnetic vector potentials, introduced in Chapter 4, is that they can influence the statistics of identical particles. In two spatial dimensions—and *only* in 2D—vector potentials can give rise to a class of identical particles known as **anyons**, which act like neither the fermions nor bosons discussed in Chapter 3. Anyons have a form of particle exchange symmetry intermediate between the fermionic and bosonic cases.

E1. BOUND FLUX TUBES

The theory of anyons created by vector potentials was developed by Wilczek (1982). He considered a scenario with set of identical particles moving in 2D (the x-y plane), with each particle carrying a "flux tube" pointing along z, as shown in the figure below:



Each flux tube is an infinitely thin concentration of magnetic flux Φ_B, which can be described by a singular vector potential (as discussed in Chapter 4, Section I.C). If $\mathbf{r}_n$ is the center of the n-th particle, its vector potential is

$$\mathbf{A}^{(n)}(\mathbf{r}) = \frac{\Phi_B}{2\pi|\mathbf{r}-\mathbf{r}_n|}\,\mathbf{e}_\phi^{(n)}(\mathbf{r}), \tag{E.1}$$

where $\mathbf{e}_\phi^{(n)}(\mathbf{r})$ denotes the azimuthal unit vector at position $\mathbf{r}$ relative to the origin $\mathbf{r}_n$. The superscript (n) denotes that this vector potential is centered on the n-th flux tube.

Suppose the particles carrying these flux tubes also have electric charge $-e$. Each particle is acted upon by the vector potentials from all the other particles, which appear in the Hamiltonian according to the prescription

$$\hat{\mathbf{p}}_n \rightarrow \hat{\mathbf{p}}_n + e\sum_{m\neq n}\mathbf{A}^{(m)}(\hat{\mathbf{r}}_n), \tag{E.2}$$

where $\hat{\mathbf{p}}_n$ is the momentum operator for particle n. The fact that each particle's flux tube does not act on itself is similar to how electrostatic forces are handled (i.e., the electric field generated by a particle does not act on the particle itself). Assuming there are no other potentials and the particles are non-relativistic, the Hamiltonian is

$$\hat{H} = \frac{1}{2m}\sum_m \left|\,\hat{\mathbf{p}}_m + e\sum_{n\neq m}\mathbf{A}^{(n)}(\hat{\mathbf{r}}_m)\right|^2. \tag{E.3}$$

We will focus on the case of two particles. In the wavefunction representation,

$$\hat{H} = \frac{1}{2m}\left(\Big| -i\hbar\nabla_1 + e\mathbf{A}^{(2)}(\mathbf{r}_1)\Big|^2 + \Big| -i\hbar\nabla_2 + e\mathbf{A}^{(1)}(\mathbf{r}_2)\Big|^2\right), \tag{E.4}$$

where ∇_n (for $n = 1, 2$) is the gradient operator using partial derivatives on $\mathbf{r}_n$. The two-particle wavefunction $\psi(\mathbf{r}_1, \mathbf{r}_2)$ obeys either fermionic or bosonic exchange symmetry:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \sigma\psi(\mathbf{r}_2, \mathbf{r}_1), \tag{E.5}$$

where $\sigma = 1$ for bosons and $\sigma = -1$ for fermions.

E2. GAUGE TRANSFORMATION

In Chapter 4, we discussed the gauge symmetry of a charged particle in an electromagnetic field. For simplicity, take a time-independent vector potential $\mathbf{A}$ and zero scalar potential. Given a single-particle wavefunction $\psi(\mathbf{r})$ describing a particle of charge $-e$, we know that the gauge transformed wavefunction

$$\psi'(\mathbf{r}) = \psi(\mathbf{r})\, \exp\left(-\frac{ie\Lambda(\mathbf{r})}{\hbar}\right) \tag{E.6}$$

solves the Schrödinger equation with the gauge transformed vector potential

$$\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla\Lambda(\mathbf{r}).$$

This symmetry can be generalized to the multi-particle case. For two-particle Hamiltonians of the form (E.4), one can show that the gauge transformed two-particle wavefunction

$$\psi'(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_1, \mathbf{r}_2)\, \exp\left(-\frac{ie\Lambda(\mathbf{r}_1, \mathbf{r}_2)}{\hbar}\right) \tag{E.7}$$

solves the Schrödinger equaton for the Hamiltonian

$$\hat{H}' = \frac{1}{2m}\left(\Big| -i\hbar\nabla_1 + e\mathbf{A}^{(2)}(\mathbf{r}_1) + e\nabla_1\Lambda\Big|^2 \;+\; \Big| -i\hbar\nabla_2 + e\mathbf{A}^{(1)}(\mathbf{r}_2) + e\nabla_2\Lambda\Big|^2\right). \tag{E.8}$$

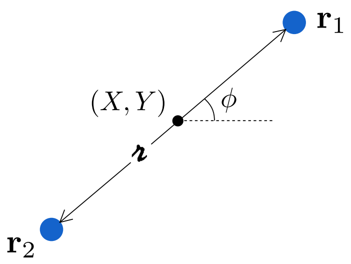
The derivation is left to the reader, and almost exactly follows the single-particle derivation from Chapter 4. The main thing to note is that Λ is an arbitrary function of $\mathbf{r}_1$ and $\mathbf{r}_2$; when calculating $\nabla_1\Lambda$, the partial derivatives with respect to $\mathbf{r}_1$ are taken with $\mathbf{r}_2$ fixed, and vice versa for $\nabla_2\Lambda$.

We are interested in the case where the $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ fields in Eq. (E.8) are the flux tube potentials of Eq. (E.1). Remarkably, it turns out that such potentials can be cancelled, or "gauged away", by a certain choice of $\Lambda(\mathbf{r}_1, \mathbf{r}_2)$. The resulting gauge transformed Hamiltonian is

$$\hat{H}' = -\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right), \tag{E.9}$$

describing a pair of free particles!

To find the $\Lambda(\mathbf{r}_1, \mathbf{r}_2)$ that achieves this, let us take a closer look at how to express the two-particle coordinates. These can, of course, be written in the Cartesian form (x_1, y_1, x_2, y_2). But we can also express them using a mix of center-of-mass coordinates and relative polar coordinates, $(X, Y, \mathscr{r}, \phi)$, as shown in this figure:



The two coordinate systems are related by

$$\begin{aligned} x_1 &= X + \frac{\mathscr{r}}{2}\cos\phi, & x_2 &= X - \frac{\mathscr{r}}{2}\cos\phi, \\ y_1 &= Y + \frac{\mathscr{r}}{2}\sin\phi, & y_2 &= Y - \frac{\mathscr{r}}{2}\sin\phi. \end{aligned} \tag{E.10}$$

From Eq. (E.10), we see that the transformation $\phi \rightarrow \phi \pm \pi$, with $(X, Y, \mathscr{r})$ constant, is equivalent to exchanging (x_1, y_1) and (x_2, y_2). In other words, the particles can be exchanged by a rotation of $\pm\pi$ around their fixed center of mass. The exchange symmetry condition (E.5) can therefore be written as

$$\psi(X, Y, \mathscr{r}, \phi \pm \pi) \;=\; \sigma\, \psi(X, Y, \mathscr{r}, \phi), \tag{E.11}$$

where $\sigma = 1$ for bosons and $\sigma = -1$ for fermions. Note, by the way, that this use of polar coordinates is specific to 2D space.

Now consider the gauge field

$$\Lambda(X, Y, \mathscr{r}, \phi) = -\frac{\Phi_B\, \phi}{2\pi}. \tag{E.12}$$

We claim that

$$\nabla_1 \Lambda = -\mathbf{A}^{(2)}(\mathbf{r}_1) \tag{E.13}$$

$$\nabla_2 \Lambda = -\mathbf{A}^{(1)}(\mathbf{r}_2), \tag{E.14}$$

which gauges away the vector potentials in Eq. (E.8).

To see why, first consider $\nabla_1 \Lambda$. We need to be careful since ∇_1 is performed with respect to $\mathbf{r}_1$ for fixed $\mathbf{r}_2$, whereas Λ is expressed in Eq. (E.12) using the $(X, Y, \mathscr{r}, \phi)$ coordinates which are a mix of $\mathbf{r}_1$ and $\mathbf{r}_2$. Let us therefore define the coordinates $(\mathscr{r}', \phi', x_2', y_2')$, where $(\mathscr{r}', \phi')$ are the polar coordinates of $\mathbf{r}_1$ *relative* to $\mathbf{r}_2$, and (x_2', y_2') are the Cartesian coordinates of $\mathbf{r}_2$. We use primes to avoid mixing up the two sets of coordinates. The unprimed and primed coordinate systems are related by

$$\begin{aligned} X &= x_2' + \frac{\mathscr{r}'}{2}\cos\phi' \\ Y &= y_2' + \frac{\mathscr{r}'}{2}\sin\phi' \\ \mathscr{r} &= \mathscr{r}' \\ \phi &= \phi'. \end{aligned} \tag{E.15}$$

Using $(\mathscr{r}', \phi', x_2', y_2')$, we can express the gradient in polar form as

$$\nabla_1 \Lambda = \frac{\partial \Lambda}{\partial \mathscr{r}'}\, \mathbf{e}_{\mathscr{r}'} + \frac{1}{\mathscr{r}'} \frac{\partial \Lambda}{\partial \phi'}\, \mathbf{e}_{\phi'}, \tag{E.16}$$

where $\mathbf{e}_{\mathscr{r}'}$ and $\mathbf{e}_{\phi'}$ are the radial and azimuthal unit vectors relative to the origin $\mathbf{r}_2$. Using the chain rule, Eq. (E.12), and Eq. (E.15),

$$\frac{\partial \Lambda}{\partial \mathscr{r}'} = \frac{\partial \Lambda}{\partial X}\frac{\partial X}{\partial \mathscr{r}'} + \frac{\partial \Lambda}{\partial Y}\frac{\partial Y}{\partial \mathscr{r}'} + \frac{\partial \Lambda}{\partial \mathscr{r}}\frac{\partial \mathscr{r}}{\partial \mathscr{r}'} + \frac{\partial \Lambda}{\partial \phi}\frac{\partial \phi}{\partial \mathscr{r}'} = 0 \tag{E.17}$$

$$\frac{\partial \Lambda}{\partial \phi'} = \frac{\partial \Lambda}{\partial X}\frac{\partial X}{\partial \phi'} + \frac{\partial \Lambda}{\partial Y}\frac{\partial Y}{\partial \phi'} + \frac{\partial \Lambda}{\partial \mathscr{r}}\frac{\partial \mathscr{r}}{\partial \phi'} + \frac{\partial \Lambda}{\partial \phi}\frac{\partial \phi}{\partial \phi'} = -\frac{\Phi_B}{2\pi}. \tag{E.18}$$

Plugging this back into Eq. (E.16) and comparing it to Eq. (E.1), we obtain the claimed result Eq. (E.13). We can prove Eq. (E.14) in a similar way by setting up polar coordinates with $\mathbf{r}_1$ as the origin. Thus, we arrive at the gauge transformed Hamiltonian (E.9).

The gauge transformed two-particle wavefunction is

$$\psi'(X, Y, \mathscr{r}, \phi) = \exp\left(-\frac{ie\Lambda}{\hbar}\right) \psi(X, Y, \mathscr{r}, \phi) = e^{i\xi\phi}\, \psi(X, Y, \mathscr{r}, \phi), \tag{E.19}$$

where

$$\xi = -\frac{e}{\hbar}\left(-\frac{\Phi_B}{2\pi}\right) = \frac{\Phi_B}{h/e}. \tag{E.20}$$

The quantity h/e in the denominator is the magnetic flux quantum (introduced and discussed in Chapter 4), so ξ counts the number of magnetic flux quanta carried by each flux tube. Now, when the two particles are exchanged,

$$\psi'(X, Y, \mathscr{r}, \phi \pm \pi) = e^{i\xi(\phi\pm\pi)}\, \psi(X, Y, \mathscr{r}, \phi \pm \pi) \tag{E.21}$$

$$= \sigma\, e^{\pm i\xi\pi}\, \psi'(X, Y, \mathscr{r}, \phi). \tag{E.22}$$

Compared to Eq. (E.11), the gauge transformed wavefunction acquires an extra factor of $\exp(\pm i\xi\pi)$ under exchange. But notice that the value of Φ_B is arbitrary; if it is not an integer multiple of h/e, then ξ is not an integer, and the extra factor is not ± 1. In that case, the particles described by the wavefunction ψ' do not behave like fermions or bosons. Instead, they are an intermediate class of identical particles called *anyons*.

References

[1] F. Wilczek, *Quantum Mechanics of Fractional-Spin Particles*, Phys. Rev. Lett. **49**, 957 (1982).