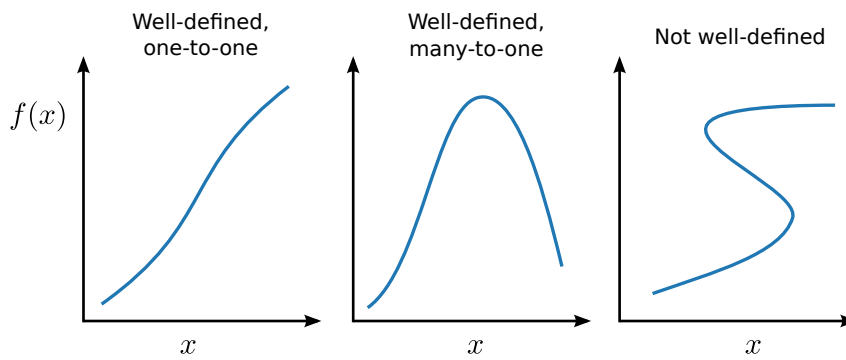# 1. Mathematical Functions

This is a course on complex methods in the physical sciences. But before launching into a discussion on complex numbers, let us undertake a brief review of real mathematical functions.

1.1 Real functions

A mathematical function, denoted f, takes an **input** x (also called an **argument**), and returns an **output** $f(x)$. For now, we consider the case where both x and $f(x)$ are real numbers. The set of possible inputs is the function's **domain**, and the set of possible outputs is the **range**.

Every function must have a well-defined output: for any x in the domain, $f(x)$ must be a specific, unambiguous number. In other words, a function must be either a one-to-one (injective) map or a many-to-one map; it cannot be one-to-many or many-to-many:

Simple examples of functions are those based on elementary algebra operations:

$$f(x) = x + 2 \qquad \text{(a one-to-one function)}$$
$$f(x) = x^2 \qquad \text{(a many-to-one function)}$$

1.2 The exponential function

The exponential function, denoted by "exp", is one of the most important functions in mathematics. We will deal with it frequently, in many different contexts.

It is very closely tied to the concept of power operations. Let us start by examining what it means to take a number x to the power of y:

$$f(x) = x^y. \tag{1.1}$$

For values of y in the natural numbers $\mathbb{N} \equiv \{1, 2, 3, \dots\}$, the power operation simply means multiplying x by itself y times. For example, $x^4 = x \cdot x \cdot x \cdot x$. But what about non natural number powers, like x^{-1} or $x^{1/2}$ or x^π? As we shall see, the exponential will help us answer this question.
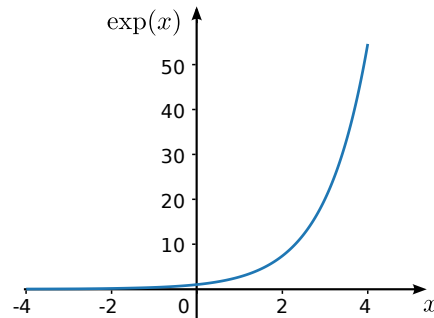
We define the exponential function as the following limiting infinite series:

$$\exp(x) \equiv 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}, \quad \text{for } x \in \mathbb{R}. \tag{1.2}$$

(Note: the "$\equiv$" symbol is used to indicate a definition.)

Note that the infinite series in Eq. (1.2) uses natural number powers only. The domain of this function is the set of real numbers, $\mathbb{R}$, and its range is the set of positive numbers, $\mathbb{R}^+$.

Here is a graph of the exponential function:

From this, we see that $\exp(x)$ increases *very* quickly with increasing x. Going in the other direction, as x decreases the value of $\exp(x)$ approaches zero rapidly.

Moreover,

1. $\exp(0) = 1$. (This follows from the definition of the exponential.)

2. For all $x, y \in \mathbb{R}$,
$$\exp(x + y) = \exp(x)\,\exp(y) \tag{1.3}$$

   Try proving this as an exercise (see Section 1.8). The key ingredients for the proof are (i) the above definition of the exponential and (ii) the binomial theorem.

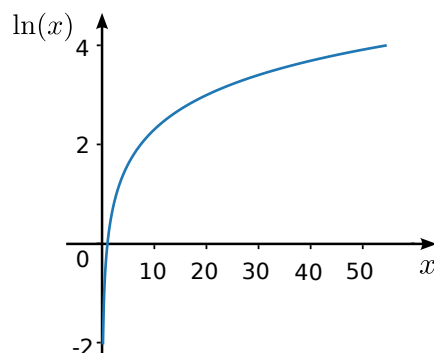3. As a corollary of properties 1 and 2,
$$\exp(-x) = 1/\exp(x). \tag{1.4}$$

1.3 The logarithm function

Since the exponential is a one-to-one function, its inverse is a well-defined function. We call this the **natural logarithm**:

$$\ln(x) \equiv y \;\; \text{such that} \;\; \exp(y) = x. \tag{1.5}$$

For brevity, we will use "logarithm" to refer to the natural logarithm, unless otherwise stated; the "non-natural" logarithms are not important in this course. From the definition of the logarithm, it can be seen that the domain is $\mathbb{R}^+$ and the range is $\mathbb{R}$ (the domain of the logarithm is the exponential's range, and vice versa). Its graph is shown below:

Observe that $\ln(x)$ increases extremely slowly with x, which is precisely the opposite of the exponential's behavior.

Using Eq. (1.3), we can prove that the logarithm satisfies the product and quotient rules

$$\ln(xy) = \ln(x) + \ln(y) \tag{1.6}$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y). \tag{1.7}$$

1.4   Non-natural powers

Having defined the exponential and logarithm, we have the tools needed to address the issue raised earlier, of how to define non-natural powers. First, observe that

$$\text{For } y \in \mathbb{N}, \quad \ln(x^y) = \underbrace{\ln(x)\ln(x)\cdots\ln(x)}_{y \text{ times}} = y\ln(x). \tag{1.8}$$

Hence, by applying the exponential to each side of Eq. (1.8),

$$x^y = \exp[y\ln(x)] \quad \text{for } y \in \mathbb{N}. \tag{1.9}$$

We can generalize the above equation so that it holds for any positive x and real y, not just $y \in \mathbb{N}$. In other words, we treat this as our *definition* of the power operation for non-natural powers:

$$x^y \equiv \exp[y\ln(x)] \quad \text{when } x \in \mathbb{R}^+ \text{ and } y \notin \mathbb{N}. \tag{1.10}$$

By this definition, the power operation always gives a positive result. Moreover, for $y \in \mathbb{N}$, the formula is consistent with the definition based on multiplying x by itself y times.

This generalization of the power operation leads to several important consequences:

1. $x^0 = 1$ for $x \in \mathbb{R}^+$.

2. Negative powers are reciprocals:

$$x^{-y} = \exp[-y\ln(x)] = \exp[-\ln(x^y)] = \frac{1}{x^y}. \tag{1.11}$$

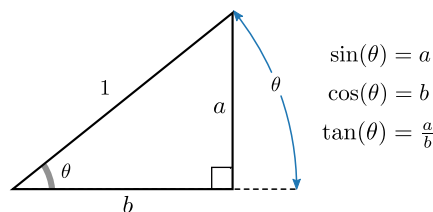3. The output of the exponential function is equivalent to a power operation:

$$\exp(y) = e^y \quad \text{where } e \equiv \exp(1) = 2.718281828459\ldots \tag{1.12}$$

(This follows by plugging in $x = e$ and using the fact that $\ln(e) = 1$.)

4. For $x \le 0$, the meaning of x^y for non-natural y is ill-defined, since the logarithm does not accept negative inputs. (Later, we will see how this limitation can be bypassed.)
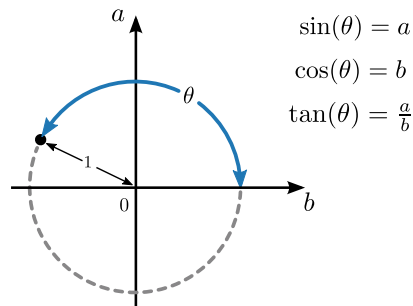
1.5   Trigonometric functions

Another extremely important group of functions are the fundamental trignonometric functions sin, cos, and tan. These can be defined in terms of the geometric ratios of the sides of right-angled triangles, as shown below:

$$\sin(\theta) = a$$
$$\cos(\theta) = b$$
$$\tan(\theta) = \frac{a}{b}$$

If we use this basic definition, the domain is $\theta \in [0, \pi/2)$, where the input angle θ is given in radians.

We can generalize the definition using the following scheme, which allows for negative values of a and/or b:

With this, the domain is extended to $\theta \in [0, 2\pi)$. We can further extend the domain to all real numbers, $\theta \in \mathbb{R}$, by treating input values modulo 2π as equivalent, i.e., $f(\theta+2\pi) = f(\theta)$. Then the trigonometric functions become many-to-one functions.

According to the Pythagorean theorem,

$$\big[\sin(\theta)\big]^2 + \big[\cos(\theta)\big]^2 = 1. \tag{1.13}$$

Using this, we can go on to prove a variety of identities, like the addition identities

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \tag{1.14}$$
$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2). \tag{1.15}$$

As you may recall, the trigonometric proofs for these identities involve drawing complicated triangle diagrams, cleverly applying the Pythagorean formula, and other tricks. There are two problems with such geometrical proofs: (i) they require some ingenuity in the construction of the triangle diagrams, and (ii) it may not be obvious whether the proofs work if the angles lie outside $[0, \pi/2)$. Happily, there is a solution to both problems. As we shall see, trigonometric identities can be proven algebraically with complex numbers.

1.6   Hyperbolic functions

The hyperbolic functions are important functions defined in terms of exponentials:

$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right) \tag{1.16}$$

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \tag{1.17}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{1.18}$$

They have properties that are intriguingly similar to the trignometric functions. For example,

$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \tag{1.19}$$
$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \tag{1.20}$$

Because of these identities, it is sometimes more convenient to work with hyperbolic functions rather than exponentials. As we shall see, there is an intricate relationship between the hyperbolic and trigonometric functions.

1.7   Continuity (optional topic)

Continuity is an important concept in the theory of real functions. A continuous function is one whose output $f(x)$ does not undergo abrupt jumps when x changes by tiny amounts. A function can be continuous over its entire domain, or only a subset of its domain. For example, $\sin(x)$ is continuous for all x, whereas $f(x) = 1/x$ is discontinuous at $x = 0$. Another function that is discontinuous at $x = 0$ is the step function
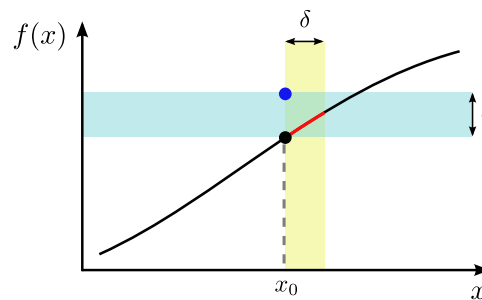
$$\Theta(x) = \left\{ \begin{array}{ll} 1, & \text{for } x \geq 0 \\ 0, & \text{otherwise.} \end{array} \right. \tag{1.21}$$

Mathematicians have even come up with functions that are discontinuous everywhere in their domain, but we won't deal with such things.
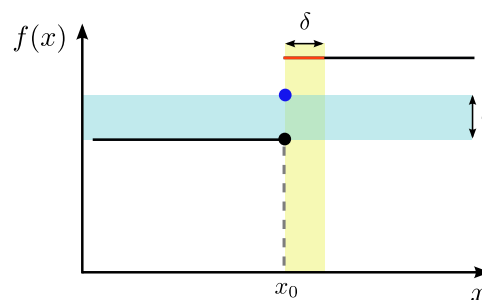
   The rigorous definition of continuity is as follows:

> A function f is continuous at a point x_0 if, for any $\epsilon > 0$, we can find a $\delta > 0$ such that setting x closer to x_0 than a distance of δ brings $f(x)$ closer to $f(x_0)$ than the specified distance ϵ.

That's a pretty complicated sentence! The plot below may help you to understand it:

At the selected point x_0, the value $f(x_0)$ is indicated by the black dot. Now, for a given choice of epsilon, $f(x_0) + \epsilon$ is indicated by the blue dot. One can choose δ so that for all $x_0 < x < x_0 + \delta$ (the region shaded in yellow), $f(x)$ is closer to $f(x_0)$ than the blue dot.

   A counter-example, with a function that has a discontinuity at some x_0, is shown below:

If we choose ϵ smaller than the gap, then no matter what value of $\delta > 0$ we try, any choice of $0 < x < \delta$ will give a value of $f(x)$ further than ϵ from $f(x_0)$. Hence, the continuity condition is violated for sufficiently small choices of ϵ, and f is **discontinuous** at x_0.

1.8   Exercises

1. An alternative definition of the exponential function is the limiting expression

$$\exp(x) \equiv \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n. \tag{1.22}$$

Prove that this is equivalent to the definition in terms of an infinite series,

$$\exp(x) \equiv 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}. \tag{1.23}$$

2. Prove that

$$\exp(x + y) = \exp(x)\,\exp(y) \tag{1.24}$$

using the definition of the exponential as an infinite series. Your proof must avoid using the concept of "raising to the power" of a non-natural number; this is to avoid circular logic, since this feature of the exponential function can be used in the generalized definition of the power operation (Section 1.4). [solution available]

3. One of the most important features of the exponential function $\exp(x)$ is that it becomes large *extremely* quickly with increasing x. To illustrate this behavior, consider the graph shown in Section 1.2, which plots the exponential up to $x = 4$. On your screen or page, the height of the graph should be around 4 cm. Suppose we keep to the same resolution, and plot up to $x = 10$; how high would the graph be? What if we plot up to $x = 20$?

4. Prove that $\exp(x) = e^x$. [solution available]

5. A "non-natural" logarithm of base c is defined as

$$\log_c(x) = y \quad \text{where} \;\; c^y = x. \tag{1.25}$$

Derive an expression for the non-natural logarithm in terms of the natural logarithm.

6. Prove, using trigonometry, that

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2). \tag{1.26}$$

You may assume that $\theta_1, \theta_2 \in [0, \pi/2]$.

7. Prove that

$$\cos(3x) = 4[\cos(x)]^3 - 3\cos(x) \tag{1.27}$$
$$\sin(3x) = 3\sin(x) - 4[\sin(x)]^3. \tag{1.28}$$