(1) A manufacturer has been selling 1000 television sets a week at $450 each and it is determined that for each $10 rebate offered to the buyer, the number of sets sold will increase by 100 per week.

(a) (5 points) Find the demand function.

Let $p(x)$ be the demand function with $x$ as the number of TV sets demanded. As given in the problem, $p(x)$ going down by $10$ would cause $x$ to increase by 100. Therefore the slope of the graph for the demand function is $\frac{-10}{100} = -\frac{1}{10}$. We also know that $p(1000) = 450$. Hence we have

$$\frac{-1}{10} = \frac{p(x) - 450}{x - 1000} \implies p(x) = -\frac{1}{10}x + 550.$$ 

(b) (15 Points) If the cost function is $C(x) = 68000 + 150x$, how should it set the size of the rebate in order to maximize its profit.

First of all, setting the demand function $p(x) = 0$ and solve, we get $x = 5500$ which is the largest value $x$ can be. Let $P(x)$ be the profit function. We must have

$$P(x) = x \times p(x) - C(x) = x \times (-\frac{1}{10}x + 550) - (68000 + 150x) = -\frac{1}{10}x^2 + 400x - 68000.$$ 

Differentiating, we get

$$P'(x) = -\frac{x}{5} + 400,$$

which yields a critical point of $x = 2000$. Again, testing the critical points and the end points, we have

$$P(0) = -68000, \quad P(2000) = 332000, \quad P(5500) = -893000.$$ 

Clearly, the profit and the biggest at $x = 2000$ at which point the price is $p(2000) = 350$ dollars. That mean the rebate should be $450 - 350 = 100$ dollars.

On Mathematica this is done first by issuing the commands

```mathematica
TVCost[x_] := 68000 + 150*x
p[x_] := -1/10*x+550
Solve[p[x] == 0, x]
Prof[x_] := x*p[x] - TVCost[x]
Solve[Prof'[x] == 0, x]
```

which define the necessary functions and find the critical point which is $x = 2000$. The first `Solve` command gives the upper limit for the $x$ value. Note that the capital letters "C" and "P" are reserved in Mathematica. Then issuing
Prof[0]  
Prof[2000]  
Prof[5500]  
gives the values of the profit at $x = 0, 2000$ and 5500. It will be clear that $x = 2000$ is a maximum. Lastly, issuing  
\[450 - p[2000]\]  
gives the value that the rebate must be.

(2) (15 Points) Supposed you are given the task of enclosing a rectangular area with 40 feet of fencing. What is the maximum area that can be enclosed?

Suppose that the length and width of the rectangle are $x$ and $y$, respectively. It is concluded from the given information that the parameter of the rectangle is 40 feet, meaning $2 \times (x + y) = 40$ or $y = 20 - x$. Hence the area of the rectangle, $A$, is  
\[A = xy = x(20 - x) = -x^2 + 20x.\]

Differentiating, we get $A'(x) = -2x + 20$. We see that $A'(x) = 0$ when $x = 10$. It should be understood that $x$ must be positive and $x$ cannot exceed 20 as $y$ would be negative if $x > 20$. Hence, testing the three points, we get  
\[A(0) = A(20) = 0, ~ A(10) = -10^2 - 20 \times 10.\]

Hence, the maximum area is attained when $x = 10$ in which case $y = 20 - 10 = 10$. Hence the large area that can be enclosed is in fact a ten feet by ten feet square.

In Mathematica, issuing the following commands will yield  
\[y[x_] = 20 - x\]  
\[A[x_] = x*y[x]\]  
\[Solve[y[x] == 0, x]\]  
\[Solve[A'[x] == 0, x]\]

will define the appropriate function and find the largest value $x$ can be and the critical point for the area function, which will be $x = 10$. Then issuing the commands  
\[A[0]\]  
\[A[10]\]  
\[A[20]\]
gives the areas of at the points of interest. It will be clear that $A(10)$ is the largest. Lastly,  
\[y[10]\]
gives the value for $y$ corresponding to $x = 10$.

For the cognoscenti, among the rectangles with a fixed parameter, the one with the largest area is **always** a square!