MA 104 Graded Homework 3 Solutions  
Dr. L. Zhao, Sections B7, C7, D7  
Issued: Monday 26 April 2004  
Due: Beginning of class, Thursday 13 May 2004

(1) (5 Points) A box with a square base and open top must have a volume of 32,000 cubic centimeters. Find the dimensions of the box that minimize that amount the material used in constructing the box.

Let $l$, $w$ and $h$ by the length, width and height of the box in centimeters, respectively. Since the box has a square base, we have $l = w$ and since the box is to have volume 32000 cubic centimeters, we have

$$lwh = l^2h = 32000 \implies h = 32000l^{-2}.$$ 

The surface area, $A$, of the box is

$$A = lw + 2lh + 2wh = l^2 + 4lh = l^2 + \frac{128000}{l}.$$ 

We thus set $A'(l) = 0$ and get

$$\frac{dA}{dl} = 2l - \frac{128000}{l^2} = 0.$$ 

The only real root to this equation is $l = 40$. We certainly can discard the complex roots. To verify that this is indeed of minimum, we have

$$\frac{d^2A}{dl^2} = 2 + \frac{256000}{l^3}.$$ 

It is clear that $\frac{d^2A}{dl^2} = 6 > 0$ if $l = 40$. Hence $l = 40$ centimeters is a minimum, and in that case $w = l = 40$ centimeters and $h = 32000l^{-2} = 20$ centimeters. Hence the dimension of the box must be $40 \times 40 \times 20$ centimeters.

This is done with the following in Mathematica.

```mathematica
A[l_] := l^2 + 128000/l
Solve[A'[l] == 0, l]
A''[40]
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(2) (5 Points) A water trough is 10 meters long and its cross-section has the shape of an isosceles trapezoid that is 30 cam wide at the bottom, 80 cam wide at the top, and has height 50 cam. If the trough is begin filled with water at the rate of 0.2 cubic meters per minute, how fast is the water level rising when the water is 30 centimeters deep?

If we extend the legs of the isosceles trapezoid until they meet, we will form a triangle if the distance from the bottom of the trapezoid to the point where the legs meet is $x$ m, then we have the following relations.

$$\frac{0.15}{x} = \frac{0.4}{0.5 + x} \implies x = 0.3.$$
Suppose the depth of water at time $t$ is $D(t)$. The cross section of the water at the time is also an isosceles trapezoid if the width of the top of the said trapezoid is $w$, then we have the following relation
\[
\frac{w/2}{0.15} = \frac{0.3 + D}{0.3} \implies w = 0.3 + D.
\]
Hence the volume of the water in the trough at time $t$ is
\[
V(t) = \frac{1}{2}(0.30 + w(t))D(t) = 5 \times (0.6 + D(t))D(t) \implies \frac{dV}{dt} = (3 + 10D(t))\frac{dD}{dt}.
\]
Insert $\frac{dV}{dt} = 0.2$ and $D(t) = 0.3$ and solve for $\frac{dD}{dt}$, we have that $\frac{dD}{dt} = \frac{1}{30}$.
This means that the water level is rising at $\frac{1}{30}$ meters per minute when the depth is 30 centimeters.

This is done with the following command in Mathematica.
\[
\text{Solve}[15/x == 40/(50 + x), x] \quad \text{V[t_] = 5*(0.6 + Dpth[t])*Dpth[t]} \\
\text{Solve}[V'[t] == 2, Dpth'[t]] /. \{Dpth[t] -> .3\}
\]

(3) (5 Points) Consider the function $f(x) = \int_0^x \frac{dt}{1 + t + t^2}$. Find the interval on which the graph of $f(x)$ is concave upward.

This amounts to determining where the $f''(x)$ is positive. By the fundamental theorem of calculus
\[
f'(x) = \frac{1}{1 + x + x^2}.
\]
Furthermore, we have
\[
f''(x) = \frac{-(1 + 2x)}{(1 + x + x^2)^2}.
\]
Note that the denominator of $f''(x)$ is never negative since it is a square. Moreover the denominator of $f''(x)$ is never zero as $1 + x + x^2 = 0$ only if $x = \frac{-1 \pm i\sqrt{2}}{2}$, complex roots. Hence the sign of $f''(x)$ is that of its numerator, namely $-(1 + 2x)$. The said numerator is positive when $x < -\frac{1}{2}$. Hence, the interval where $f(x)$ is concave upward is $(-\infty, -\frac{1}{2})$.

(4) Consider the second order differential equation
\[
\frac{d^2y}{dt^2} + py' + qy = 0.
\]
(a) (3 Points) Write the differential equation as a first-order linear system of differential equations.

Let $v = \frac{dy}{dt}$. The second order differential equations becomes
\[
\begin{cases}
\frac{dy}{dt} = v \\
\frac{dv}{dt} = -qv - pv
\end{cases}
\]
In matrix notation, we have
\[
\begin{pmatrix}
\frac{d y}{d t} \\
\frac{d v}{d t}
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}.
\]

(b) (7 Points) What conditions on \( p \) and \( q \) guarantee that the eigenvalues of the corresponding linear system are complex? Does this look familiar?

To compute the eigenvalues, we must solve the following equation.
\[
\det \begin{pmatrix} 0 - \lambda & 1 \\ -q & -p - \lambda \end{pmatrix} = 0 \implies \lambda^2 + p\lambda + q = 0 \implies \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.
\]

Hence we have complex eigenvalues only if \( p^2 - 4q < 0 \).

(5) A 10-kg object suspended from the end of a vertically hanging spring stretches the spring 9.8 cm. At \( t = 0 \), the resulting spring-mass system is disturbed from its resting state by an applied force, \( F(t) = 20e^{-t} \). The force \( F(t) \) is expressed in Newtons and is considered positive in the downward direction. It is also given that the magnitude of damping constant \( b \) is 10. Time is measured in seconds.

(a) (5 Points) Determine the spring constant \( k \) and what is the unit of the damping constant \( b \)?

9.8 cm is 0.098 m and hence \( 10 \times 9.8 = k \times 0.098 \implies k = 1000 \text{N/m} \). The damping constant has to have a unit of force divided by a unit of velocity. In our case it must be Newton Second per meter.

(b) (5 Points) Formulate and solve the initial value problem for \( y(t) \), where \( y(t) \) is the displacement of the object from its equilibrium rest state, measured positive in the downward direction.

The differential equation we must set up is
\[
10\frac{d^2 y}{d t^2} + 10\frac{d y}{d t} + 1000y = 20e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.
\]

The following command in Mathematica gives the solution
\[
\text{DSolve}[\{10y''[t] + 10y'[t] + 1000y[t] == 20*Exp[-t], y[0] == 0, y'[0] == 0\}, y[t], t] // \text{Simplify}
\]

We have
\[
y(t) = \frac{e^{-t}}{19950} \left[ 399 - 399e^{t/2} \cos(\sqrt{399}t/2) + \sqrt{399}e^{t/2} \sin(\sqrt{399}t/2) \right].
\]

(c) (5 Points) Plot the solution and determine the long term behavior of the oscillation of the object, if any. Does this long term behavior conform with common sense or expectation? Why or why not?

Plotting this function on Mathematica, we shall see that the long term behavior of the oscillation is that the amplitude of the oscillation will eventually decrease to zero. This conforms with common sense since the force function \( F(x) = 20e^{-t} \) is a function that decreases to zero rapidly as \( t \) gets large, the damping effect will take over rather quickly. Hence \( y(t) \) will tend to zero as \( t \) tends to infinity.
(6) (5 Points) Using a substitution and the formula

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}, \]

(i) to verify that the Gaussian distribution (bell curve, normal distribution)
\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \]
is indeed a probability density function. Recall that \( \mu \) and \( \sigma \geq 0 \) denote the mean and standard deviation of a probability density function, respectively.

First we must verify that \( f(x) \geq 0 \) which is easily disposed as \( f(x) \) is a positive constant multiplied by an exponential function, which is never negative.

Let \( u = \frac{x - \mu}{\sqrt{2\sigma}} \). We see that \( du = \frac{dx}{\sqrt{2\sigma}} \) and \( dx = \sqrt{2\sigma} \, du \). Moreover, \( \lim_{x \to \pm\infty} u = \pm\infty \).

We then have
\[
\int_{-\infty}^{\infty} f(x) \, dx = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \, dx = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[ -\left( \frac{(x - \mu)}{\sqrt{2\sigma}} \right)^2 \right] \, dx \\
= \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2\sigma} \, du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \, du = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.
\]

On a side note, Sir William Thomson, First Baron Kelvin of Largs, a well-known physicist, was one of the persons after whom the Kelvin and Rankine scale of temperature was named. Zero Kelvin, or complete absence of heat, is about -273.15 degrees Celsius or -459.67 degrees Fahrenheit.

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\(^1\)A mathematician is one to whom that(i) is as obvious as that twice two makes four is to you. Liouville was a mathematician.

— Sir William Thomson, First Baron Kelvin of Largs.