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QUADRILATERAL-FREE STEINER TRIPLE SYSTEMS OF ORDERS 31 AND 37

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A hill-climbing algorithm is proposed for the construction of cyclic Steiner triple systems. This hill-climbing algorithm is coupled with a fast quadrilateral-detecting algorithm to generate cyclic quadrilateral-free Steiner triple systems of small orders. As a result, we are able to construct quadrilateral-free Steiner triple systems for orders 31 and 37, whose existence was previously not known.

1. INTRODUCTION

A *Steiner triple system* of *order* v, denoted STS(v), is a pair $(X, \mathscr{B})$, where X is a finite set of v elements called *points*, and $\mathscr{B}$ is a family of 3-subsets of X called *triples*, such that every 2-subset of X is contained in precisely one triple. It is well-known that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6) (see, e.g., [1]). An *automorphism* of an STS(v), say $(X, \mathscr{B})$, is a bijection $\pi : X \to X$ such that $\{x, y, z\} \in \mathscr{B}$ implies $\{\pi(x), \pi(y), \pi(z)\} \in \mathscr{B}$. A *cyclic* STS($v$) is one that admits a permutation consisting of a cycle of length v as an automorphism. The existence of cyclic STS(v)'s has been completely settled by Peltesohn [6] who proved that there exists a cyclic STS(v) for every $v \equiv 1$ or 3 (mod 6) except $v = 9$.

A *quadrilateral* (also known as *fragment* or *Pasch configuration*) in an STS(v) is a subset of four triples whose union contains exactly six points. A quadrilateral must have the following configuration: $\{a, b, c\}$, $\{a, d, e\}$ $\{f, b, d\}$, $\{f, c, e\}$. An STS(v) containing no quadrilaterals is said to be *quadrilateral-free*, and is denoted by QFSTS(v). A cyclic STS(v) containing no quadrilaterals is referred to as a *cyclic quadrilateral-free* STS(v), and is denoted by CQFSTS(v).

The existence of QFSTS(v)'s has been previously studied. Robinson [7] proved that if $v \equiv 19 \pmod{24}$, and v is prime, then there exists a QFSTS(v). Brouwer [2] established that there exists a QFSTS(v) for all $v \equiv 3 \pmod 6$, and showed that if $v \equiv 1 \pmod 6$, and $v = p^\alpha$, p prime, then there exists a QFSTS(v) whenever $\alpha \equiv 0 \pmod 2$ or $p \equiv 1$ or $3 \pmod 8$. It was also shown by Grannell, Griggs, and Phelan [4] that there exists a QFSTS(v) whenever the order of $-2 \pmod p$ is congruent to $2 \pmod 4$ for every prime divisor p of $v - 2$. Recently, two recursive constructions of QFSTS(v)'s were given by Stinson and Wei [9]:

(i) If there exist QFSTS(v) and QFSTS(w), then there exists a QFSTS(vw).

(ii) If there exists a QFSTS(v), $v \equiv 1 \pmod 4$, and v has an odd divisor exceeding 3, then there exists a QFSTS($3v$-2).

Recursive constructions are generally useful in settling asymptotic existence of designs. However, the ingredients required for Stinson and Wei's constructions are QFSTS(v)'s of small orders. The status of the existence of QFSTS(v)'s; for $v \equiv 1 \pmod 6$, $v < 50$ prior to this paper is given in the table below:

TABLE 1

QFSTS(v)'s WITH $v \equiv 1 \pmod 6$, $v < 50$

v	*Existence*	*Reference*
7	No	[2]
13	No	[2]
19	Yes	[2]
25	Yes	[2]
31	?	
37	?	
43	Yes	[7]
49	Yes	[2]

In this paper, we consider cyclic STS(v)'s as a possible source of these small QFSTS(v)'s. A hill-climbing algorithm is used to generate random cyclic STS(v)'s which are then checked for the existence of quadrilaterals. As a result, we are able to prove the existence of CQFSTS(v)'s for $v = 31$ and 37.

2. Construction of Cyclic STS(v)'s

The construction of cyclic STS(v)'s was investigated by Heffter [5] in the context of cyclic decompositions of complete graphs. He made the observation that the construction of a cyclic STS(v), $v = 6t + 1$, is equivalent to partitioning the set $\{1, 2, \ldots, 3t\}$ into triples with the property that in each triple, (2.1) the sum of two of the numbers is equal to the third, or (2.2) the sum of all three numbers is equal to v. This is called *Heffter's first difference problem* for t, and we denote the problem by $HDP_1(t)$. In the case when $v = 6t + 3$, the construction is equivalent to a partition of the set $\{1, 2, \ldots, 3t + 1\} \setminus \{2t + 1\}$ with the same properties (2.1) or (2.2). This is called *Heffter's second difference problem* for t, and is denoted by $HDP_2(t)$.

Without loss of generality, we may assume that a cyclic STS(v), say $(X, \mathscr{B})$, has point set $X = \mathbb{Z}_v$, and the cyclic automorphism is $i \to i + 1 \pmod v$. If $\{\{x_i, y_i, z_i\} : 1 \leqslant i \leqslant t\}$ is a solution to $HDP_1(t)$, then the set of base triples $\{\{0, x_i, x_i + y_i\} : 1 \leqslant i \leqslant t\}$ generates a cyclic STS($6t + 1$). It is useful to note that if $\{x, y, z\}$, $x < y < z$, is a triple in a solution to $HDP_1(t)$, then the corresponding base triple in the cyclic STS($6t + 1$) can be taken to be either $\{0, x, x + y\}$ or $\{0, -x, -(x + y)\}$. The set of base triples in the cyclic STS($6t + 3$) corresponding to a solution to $HDP_2(t)$ is the same as in $HDP_1(t)$ except with the addition of an extra base triple $\{0, 2t + 1, 4t + 2\}$.

3. A Hill-Climbing Algorithm for Generating Cyclic STS(v)'s

Hill-climbing techniques have been used successfully in the construction of several combinatorial designs, notably by Dinitz aad Stinson [3, 8]. In this section, we introduce similar techniques for the generation of random cyclic STS(v)'s.

Preprocessing is done to identify all the possible candidates, i.e. triples satisfying conditions (2.1) or (2.2). Hill-climbing is then carried out on this set of candidates C. Our algorithm works as follows:

begin

$b := 0$;

$B := \phi$;

while $(b < t)$ **do begin**

TMP := B;

choose a triple $T := \{x, y, z\}$ from C at random;

remove all triples in TMP that intersects T;

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    TMP := TMP ∪ {T};
    if (|TMP| ≥ b) then begin
      B := TMP;
      b := |B|;
    end
  end
end.
```

At termination, B will be a solution to Heffter's difference problem. For each triple $\{x, y, z\} \in B$, $x < y < z$, we randomly decide one of $\{0, x, x+y\}$ or $\{0, -x, -(x+y)\}$ to be included as a base triple of the associated cyclic STS(v).

This algorithm has been used successfully to generate random cyclic STS(v)'s with $v < 100$.

4. Quadrilateral-Checking in Cyclic STS(v)'s

The task of determining the existence of quadrilaterals in cyclic STS(v)'s is simplified owing to the transitive nature of the designs involved. Let $(X, \mathscr{B})$ be a cyclic STS(v). For every pair of points $x, y \in X$, there exists an automorphism π of $(X, \mathscr{B})$ such that $\pi(x) = y$ since $(X, \mathscr{B})$ is cyclic. Hence, we need only check if a fixed point $x \in X$ belongs to a quadrilateral. This is easy because we only have to check if for any two triples $\{x, a, b\}, \{x, c, d\} \in \mathscr{B}$, one of the following two situations arises:

$$\{a, c, y\}, \{b, d, y\} \in \mathscr{B} \quad \text{for some } y \in X; \tag{4.1}$$

$$\{a, d, y\}, \{b, c, y\} \in \mathscr{B} \quad \text{for some } y \in X. \tag{4.2}$$

If it does, the design is not quadrilateral-free; otherwise, the design is quadrilateral-free.

5. CQFSTS(v)'s with $v \in \{31, 37\}$

The hill-climbing algorithm described in section 3 is coupled with the quadrilateral-checking method discussed in section 4 in a generate-and-test fashion to yield a CQFSTS(v)-generating procedure. In this section, we list the base triples of a CQFSTS(v) for each $v \in \{31, 37\}$ that we have found using our algorithm.

TABLE 2

CQFSTS(v)

v	*Base triples*					
31	{0, 1, 3}	{0, 4, 11}	{0, 5, 15}	{0, 6, 18}	{0, 8, 17}	
37	{0, 4, 19}	{0, 1, 17}	{0, 32, 23}	{0, 31, 24}	{0, 2, 12}	{0, 34, 26}

The triples in each design can be obtained by developing the base triples with the cyclic automorphism $i \to i + 1 \pmod{v}$.

6. Conclusion

In this paper, CQFSTS(v)'s for $v \in \{31, 37\}$ are constructed by computer methods, thus settling the existence of QFSTS(v)'s of these previously unkown parameter situations. In view of this result, the existence of QFSTS(v)'s for all $v \leqslant 51$ has been settled. The smallest order for which existence of a QFSTS(v) is in doubt is now $v = 55$. However, our algorithm was unable to find a CQFSTS(55) after several hours of CPU time.

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