*J. Symbolic Computation* (1992) **13**, 677–681

# On Graphical Quintuple Systems

YEOW MENG CHEE

*Planning and Infrastructure Department, National Computer Board, 71 Science Park Drive, S0511, Republic of Singapore*

(*Received* 25 *June* 1991)

In this paper, we prove with the aid of symbolic computational tools, that there does not exist a non-trivial graphical 4-$(v, 5, \lambda)$ design for any $v$ and $\lambda$.

## 1. Background

A $t$-$(v, k, \lambda)$ design is a set $\Omega$ of $v$ *points* together with some $k$-element subsets of $\Omega$ called *blocks* such that any $t$-element subset of $\Omega$ occurs in exactly $\lambda$ blocks. Formally, let $\Omega$ be a finite set. We denote by $\Sigma_k(\Omega)$ the set of all $k$-element subsets of $\Omega$. An ordered pair $(\Omega, \mathscr{D})$ is called a $t$-$(v, k, \lambda)$ *design* if $|\Omega| = v$ and $\mathscr{D} \subseteq \Sigma_k(\Omega)$ such that for every $T \in \Sigma_t(\Omega)$,

$$|\{B \in \mathscr{D}: B \supseteq T\}| = \lambda.$$

It is well-known that the following *divisibility conditions* are necessary for the existence of a $t$-$(v, k, \lambda)$ design:

$$\lambda\binom{v-i}{t-i} \equiv 0 \bmod \binom{k-i}{t-i}, \qquad 0 \le i \le t.$$

A $t$-$(v, k, \lambda)$ design $(\Omega, \mathscr{D})$ with $\mathscr{D} = \varnothing$ or $\mathscr{D} = \Sigma_k(\Omega)$ is said to be *trivial*. One can show by elementary counting arguments that in a trivial $t$-$(v, k, \lambda)$ design, we must have either $\lambda = 0$ or $\lambda = \binom{v-t}{k-t}$. The *complement* of a $t$-$(v, k, \lambda)$ design $(\Omega, \mathscr{D})$ is the ordered pair $(\Omega, \Sigma_k(\Omega)\backslash\mathscr{D})$. It is easy to show that the complement of a $t$-$(v, k, \lambda)$ design is a $t$-$(v, k, \binom{v-t}{k-t} - \lambda)$ design. A $(k-1)$-$(v, k, \lambda)$ design is also commonly called a *k-tuple system*.

Let $\Omega$ be the set of $v = \binom{p}{2}$ labelled edges of the undirected complete graph $K_p$. An ordered pair $(\Omega, \mathscr{D})$ is a *graphical* $t$-$(v, k, \lambda)$ *design* if

(i) $(\Omega, \mathscr{D})$ is a $t$-$(v, k, \lambda)$ design, and
(ii) if $B \in \mathscr{D}$, then all subgraphs of $K_p$ isomorphic to $B$ are also in $\mathscr{D}$.

One may think of $\mathscr{D}$ as a collection of $k$-edge subgraphs of $K_p$ such that every $t$-edge subgraph of $K_p$ is a subgraph of exactly $\lambda$ elements of $\mathscr{D}$, and such that $\mathscr{D}$ is closed under isomorphism of graphs. We note that for every $t$, $k$, and $v = \binom{p}{2}$, there always exists a trivial graphical $t$-$(v, k, \lambda)$ design, by taking $\mathscr{D} = \varnothing$ or $\mathscr{D}$ to be the set of all $k$-edge subgraphs of $K_p$.

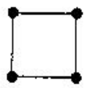
Kramer & Mesner (1976) seem to be the first to construct graphical $t$-$(v, k, \lambda)$ designs. The investigation of graphical $t$-$(v, k, \lambda)$ designs was subsequently carried out by many other researchers (Driessen (1978), Chouinard II *et al.* (1983), Kreher *et al.* (1990), Kramer

0747-7171/92/060677+05 $03.00/0 

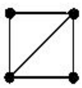
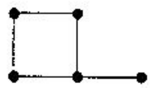
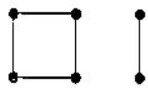
(1990), Chee (1990*a*, *b*; 1991)). In Chee (1991), the author proposed a symbolic computational approach to the problem of enumerating graphical $t$-$(v, k, \lambda)$ designs. As a result, all graphical triple systems and graphical quadruple systems are determined. In this paper, we prove that there do not exist any non-trivial graphical quintuple systems.

## 2. A Diophantine Equation

Suppose $(\Omega, \mathscr{D})$ is a non-trivial graphical $4$-$(\binom{p}{2}, 5, \lambda)$ design. Let $T_1 \in \Sigma_4(\Omega)$ be a subgraph of $K_p$ isomorphic to the graph consisting of a cycle of length four and $p-4$ isolated vertices. For convenience of presentation, isolated vertices are not shown in figures.

$T_1 \simeq$ 

The blocks in $\mathscr{D}$ containing $T_1$ must be isomorphic to one of the following graphs.

$B_1 \simeq$  $B_2 \simeq$  $B_3 \simeq$ 

If we denote by $\#(T \to B)$ the number of ways that a graph $T$ can be extended to a graph $B$, then

$$\#(T_1 \to B_1) = 2,$$

$$\#(T_1 \to B_2) = 4(p-4),$$

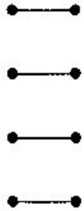
$$\#(T_1 \to B_3) = (p-4)(p-5)/2.$$

It follows from the isomorphism property that in any non-trivial graphical $4$-$(\binom{p}{2}, 5, \lambda)$ design, we must have

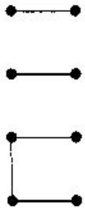
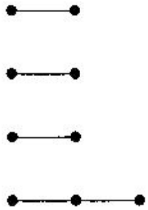
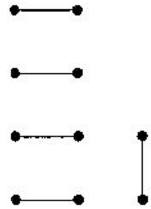
$$\lambda = 2x_1 + 4(p-4)x_2 + (p-4)(p-5)x_3/2$$

for some $(x_1, x_2, x_3) \in \{0, 1\}^3 \setminus \{(0,0,0), (1,1,1)\}$. The cases $(x_1, x_2, x_3) = (0, 0, 0)$ and $(x_1, x_2, x_3) = (1, 1, 1)$ are excluded since they lead to $\lambda = 0$ and $\lambda = \binom{p}{2} - 4$, thus giving trivial graphical quintuple systems.

Now let $p \geq 8$ and consider $T_2 \in \Sigma_4(\Omega)$ a subgraph of $K_p$ isomorphic to the graph consisting of a matching of size four together with $p-8$ isolated vertices.

$T_2 \simeq$ 

The blocks in $D$ containing $T_2$ must be isomorphic to one of the following graphs.

$B_4 \simeq$  $B_5 \simeq$  $B_6 \simeq$ 

In this case, we have

$$\#(T_2 \to B_4) = 24,$$

$$\#(T_2 \to B_5) = 8(p-8),$$

$$\#(T_2 \to B_6) = (p-8)(p-9)/2.$$

Since $B_1, B_2, \ldots, B_6$ are pairwise non-isomorphic, we have the following result.

LEMMA 1. *For any non-trivial graphical* $4\text{-}(\binom{p}{2}, 5, \lambda)$ *design with* $p \geq 8$, *we have*

$$2x_1 + 4(p-4)x_2 + (p-4)(p-5)x_3/2 = 24x_4 + 8(p-8)x_5 + (p-8)(p-9)x_6/2$$

*for some* $(x_1, x_2, x_3)$, $(x_4, x_5, x_6) \in \{0, 1\}^3 \backslash \{(0, 0, 0), (1, 1, 1)\}$.

### 3. Non-existence Results

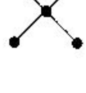
Given the six possibilities for $(x_1, x_2, x_3)$ and for $(x_4, x_5, x_6)$, we can easily derive a set $E$ of 36 quadratic equations involving only the variable $p$. We are interested in integers $\geq 8$ which obey at least one of these identities. Let $S$ be the set of such solutions. It is possible to determine $S$ by solving the 36 equations in $E$ manually. However, this laborious and error-prone task makes it more suitable for machines to handle. The symbolic computational system MAPLE (Char *et al.* (1988)) was used to solve the equations in $E$ over $\mathbf{Z}$. MAPLE yielded the result that $S = \{10, 12, 20\}$. Since the complement of a $4\text{-}(\binom{p}{2}, 5, \lambda)$ design is a $4\text{-}(\binom{p}{2}, 5, \binom{p}{2} - 4 - \lambda)$ design, we need only consider cases when $\lambda \leq \lfloor (\binom{p}{2} - 4)/2 \rfloor$. In addition to $S$ itself, we computed the possible values of $\lambda$ for each value of $p \in S$. Our computations with MAPLE are summarized in the following lemma.

LEMMA 2. *There exists a non-trivial graphical* $4\text{-}(\binom{p}{2}, 5, \lambda)$ *design with* $p \geq 8$ *only if* $(p, \lambda) \in \{(10, 17), (12, 30), (20, 66)\}$.

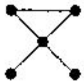
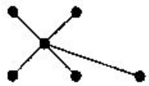
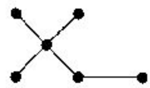
In the remainder of this section, we prove that there are no non-trivial graphical $4\text{-}(\binom{p}{2}, 5, \lambda)$ designs for any $p$ and $\lambda$.

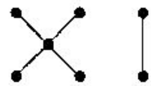
LEMMA 3. *There does not exist a graphical* 4-(45, 5, 17) *design.*

PROOF. Let $(\Omega, \mathscr{D})$ be a graphical 4-(45, 5, 17) design. Consider $T_3 \in \Sigma_4(\Omega)$ a subgraph of $K_{10}$ isomorphic to the graph consisting of a star on five vertices together with five isolated vertices.

$T_3 \simeq$ 

The blocks in $\mathscr{D}$ containing $T_3$ must be isomorphic to one of the following graphs.

$B_7 \simeq$  $B_8 \simeq$  $B_9 \simeq$ 

$B_{10} \simeq$ 

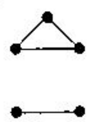
We have $\#(T_3 \to B_7) = 6$, $\#(T_3 \to B_8) = 5$, $\#(T_3 \to B_9) = 20$, $\#(T_3 \to B_{10}) = 10$, and there is no subset of $\{5, 6, 10, 20\}$ whose sum is 17.

Lemma 4. *There does not exist a graphical* 4-(66, 5, 30) *design.*

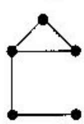
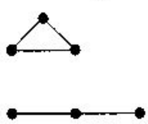
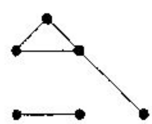
Proof. Consider the same graphs as in Lemma 3 (except in $K_{12}$ instead of $K_{10}$). We now have $\#(T_3 \to B_7) = 6$, $\#(T_3 \to T_8) = 7$, $\#(T_3 \to B_9) = 28$, $\#(T_3 \to B_{10}) = 21$, and there is no subset of $\{6, 7, 21, 28\}$ whose sum is 30.

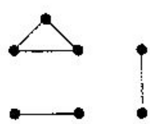
Lemma 5. *There does not exist a graphical* 4-(190, 5, 66) *design.*

Proof. Let $(\Omega, \mathscr{D})$ be a graphical 4-(190, 5, 66) design. Consider $T_4 \in \Sigma_4(\Omega)$ a subgraph of $K_{20}$ isomorphic to the graph consisting of a triangle, an edge that is vertex-disjoint from the triangle, and 15 isolated vertices.

$T_4 \simeq$ 

The blocks in $\mathscr{D}$ containing $T_4$ must be isomorphic to one of the following graphs.

$B_{11} \simeq$  $B_{12} \simeq$  $B_{13} \simeq$ 

$B_{14} \simeq$ 

We have $\#(T_4 \to B_{11}) = 6$, $\#(T_4 \to B_{12}) = 30$, $\#(T_4 \to B_{13}) = 45$, $\#(T_4 \to B_{14}) = 105$, and there is no subset of $\{6, 30, 45, 105\}$ whose sum is 66.

Combining the results above gives the following.

Lemma 6. *There exist no non-trivial graphical* 4-$(\binom{p}{2}, 5, \lambda)$ *designs for any* $\lambda$ *and* $p \geq 8$.

Since the divisibility conditions force all 4-$(\binom{p}{2}, 5, \lambda)$ designs to be trivial for $p < 6$, we need only consider the two remaining values $p = 6$ and $p = 7$ to complete the solution of the existence problem for non-trivial graphical quintuple systems. Kramer & Mesner (1976) have established that there do not exist any non-trivial graphical 4-(15, 5, $\lambda$) designs. We now prove that there are no non-trivial graphical 4-(21, 5, $\lambda$) designs.

LEMMA 7. *There does not exist a non-trivial graphical* 4-$(21, 5, \lambda)$ *design for any* $\lambda$.

PROOF. Let $\Lambda$ be the set of integers $\lambda$ such that there exists a non-trivial graphical 4-$(21, 5, \lambda)$ design. By considering the number of ways that $T_1$ (in $K_7$) can be extended to each of $B_1$, $B_2$, and $B_3$, we have $\Lambda \subseteq \{2, 3, 5\}$. By considering the number of ways that $T_3$ (in $K_7$) can be extended to each of $B_7$, $B_8$, $B_9$, and $B_{10}$, we have $5 \notin \Lambda$. Finally, consideration of the number of ways that $T_4$ (in $K_7$) can be extended to each of $B_{11}$, $B_{12}$, $B_{13}$, and $B_{14}$ shows that $2, 3 \notin \Lambda$.

We can now state:

THEOREM 1. *There do not exist non-trivial graphical* 4-$(v, 5, \lambda)$ *designs for any* $v$ *and* $\lambda$.

## 4. Conclusion

In this paper, we proved that no non-trivial graphical quintuple systems exist. An immediate problem is suggested:

PROBLEM 1. *Determine if there are any, or find all, non-trivial graphical* $k$*-tuple systems for* $k \geq 6$.

We make the conjecture that there are no non-trivial graphical $k$-tuple systems for $k \geq 6$.

## References


Char, B. W., Geddes, K. O., Gonnet, G. H., Monagan, M. B., Watt, S. M. (1988). *MAPLE: Reference Manual.* 5th Edition, University of Waterloo.

Chee, Y. M. (1990*a*). The existence of a simple 3-(28, 5, 30) design. *Discrete Math.*, to appear.

Chee, Y. M. (1990*b*). New 3-(21, 7, $\lambda$) designs using graphs. Preprint.

Chee, Y. M. (1991). Graphical $t$-designs with block sizes three and four. *Discrete Math.* **91**, 201–206.

Chouinard II, L. G., Kramer, E. S., Kreher, D. L. (1983). Graphical $t$-wise balanced designs. *Discrete Math.* **46**, 227–240.

Driessen, L. H. M. E. (1978). $t$-Designs, $t \geq 3$. Ph. D. thesis, Eindhoven University of Technology, Netherlands.

Kramer, E. S. (1990). An $S_3(3, 5, 21)$ using graphs. *Discrete Math.* **81**, 223–224.

Kramer, E. S., Mesner, D. M. (1976). $t$-Designs on hypergraphs. *Discrete Math.* **15**, 263–296.

Kreher, D. L., Chee, Y. M., de Caen, D., Colbourn, C. J., Kramer, E. S. (1990). Some new simple $t$-designs. *J. Combin. Math. Combin. Comput.* **7**, 53–90.