On Graphical Quintuple Systems

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In this paper, we prove with the aid of symbolic computational tools, that there does not exist a non-trivial graphical 4-(v, 5, \lambda) design for any v and \lambda.

1. Background

A t-(v, k, \lambda) design is a set \Omega of v points together with some k-element subsets of \Omega called blocks such that any t-element subset of \Omega occurs in exactly \lambda blocks. Formally, let \Omega be a finite set. We denote by \Sigma_k(\Omega) the set of all k-element subsets of \Omega. An ordered pair (\Omega, \mathcal{D}) is called a t-(v, k, \lambda) design if |

\begin{align*}
\frac{v-i}{t-i} &= \frac{k-i}{t-i}, \\
0 &\leq i \leq t.
\end{align*}

\]

A t-(v, k, \lambda) design (\Omega, \mathcal{D}) with \mathcal{D} = \mathcal{D} or \mathcal{D} = \Sigma_k(\Omega) is said to be trivial. One can show by elementary counting arguments that in a trivial t-(v, k, \lambda) design, we must have either \lambda = 0 or \lambda = \left(\frac{k}{t}\right) - 1. The complement of a t-(v, k, \lambda) design (\Omega, \mathcal{D}) is the ordered pair (\Omega, \Sigma_k(\Omega) \setminus \mathcal{D}). It is easy to show that the complement of a t-(v, k, \lambda) design is a t-(v, k, (\frac{k}{t}) - \lambda) design. A (k - 1)-t-(v, k, \lambda) design is also commonly called a k-tuple system.

Let \Omega be the set of u = (\frac{v}{t}) labelled edges of the undirected complete graph \mathcal{K}_v. An ordered pair (\Omega, \mathcal{D}) is a graphical t-(v, k, \lambda) design if

(i) (\Omega, \mathcal{D}) is a t-(v, k, \lambda) design, and
(ii) if B \in \mathcal{D}, then all subgraphs of \mathcal{K}_v isomorphic to B are also in \mathcal{D}.

One may think of \mathcal{D} as a collection of k-edge subgraphs of \mathcal{K}_v such that every t-edge subgraph of \mathcal{K}_v is a subgraph of exactly \lambda elements of \mathcal{D}, and such that \mathcal{D} is closed under isomorphism of graphs. We note that for every t, k, and \lambda, there always exists a trivial graphical t-(v, k, \lambda) design, by taking \mathcal{D} = \mathcal{D} or \mathcal{D} = \Sigma_k(\Omega) to be the set of all k-edge subgraphs of \mathcal{K}_v.

Kramer & Mesner (1976) seem to be the first to construct graphical t-(v, k, \lambda) designs. The investigation of graphical t-(v, k, \lambda) designs was subsequently carried out by many other researchers (Driessen (1978), Chouinard II et al. (1983), Kreher et al. (1990), Kramer et al. (1991)).
(1990), Chee (1990a, b; 1991). In Chee (1991), the author proposed a symbolic computational approach to the problem of enumerating graphical \( t-(n, k, \lambda) \) designs. As a result, all graphical triple systems and graphical quadruple systems are determined. In this paper, we prove that there do not exist any non-trivial graphical quintuple systems.

2. A Diophantine Equation

Suppose \((\Omega, \mathcal{D})\) is a non-trivial graphical \(4-(\ell, 5, \lambda)\) design. Let \(T_1 \in \Sigma_4(\Omega)\) be a subgraph of \(K_p\) isomorphic to the graph consisting of a cycle of length four and \(p-4\) isolated vertices. For convenience of presentation, isolated vertices are not shown in figures.

\[
T_1 \simeq \begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\]

The blocks in \(\mathcal{D}\) containing \(T_1\) must be isomorphic to one of the following graphs.

\[
B_1 \simeq \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad B_2 \simeq \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad B_3 \simeq \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

If we denote by \(#(T \rightarrow B)\) the number of ways that a graph \(T\) can be extended to a graph \(B\), then

\[
\#(T_1 \rightarrow B_1) = 2,
\]

\[
\#(T_1 \rightarrow B_2) = 4(p - 4),
\]

\[
\#(T_1 \rightarrow B_3) = (p - 4)(p - 5)/2.
\]

It follows from the isomorphism property that in any non-trivial graphical \(4-(\ell, 5, \lambda)\) design, we must have

\[
\lambda = 2x_1 + 4(p - 4)x_2 + (p - 4)(p - 5)x_3/2
\]

for some \((x_1, x_2, x_3) \in \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}\). The cases \((x_1, x_2, x_3) = (0, 0, 0)\) and \((x_1, x_2, x_3) = (1, 1, 1)\) are excluded since they lead to \(\lambda = 0\) and \(\lambda = (\ell) - 4\), thus giving trivial graphical quintuple systems.

Now let \(p \geq 8\) and consider \(T_2 \in \Sigma_4(\Omega)\) a subgraph of \(K_p\) isomorphic to the graph consisting of a matching of size four together with \(p - 8\) isolated vertices.

\[
T_2 \simeq \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]
The blocks in $D$ containing $T_3$ must be isomorphic to one of the following graphs.

\[
\begin{array}{ccc}
  & \bullet & \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

\[
\begin{array}{ccc}
  & \bullet & \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

\[
\begin{array}{ccc}
  & \bullet & \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

In this case, we have

\[
\#(T_3 \to B_4) = 24,
\]

\[
\#(T_3 \to B_5) = 8(p - 8),
\]

\[
\#(T_3 \to B_6) = (p - 8)(p - 9)/2.
\]

Since $B_4, B_5, \ldots, B_6$ are pairwise non-isomorphic, we have the following result.

**Lemma 1.** For any non-trivial graphical $4-(\xi, 5, \lambda)$ design with $p \geq 8$, we have

\[
2x_1 + 4(p - 4)x_2 + (p - 4)(p - 5)x_3/2 = 24x_4 + 8(p - 8)x_5 + (p - 8)(p - 9)x_6/2
\]

for some $(x_1, x_2, x_3, x_4, x_5, x_6) \in \{0, 1\}^5 \setminus \{(0, 0, 0, 1, 1, 1)\}$.

3. Non-existence Results

Given the six possibilities for $(x_1, x_2, x_3)$ and for $(x_4, x_5, x_6)$, we can easily derive a set $E$ of 36 quadratic equations involving only the variable $p$. We are interested in integers $\geq 8$ which obey at least one of these identities. Let $S$ be the set of such solutions. It is possible to determine $S$ by solving the 36 equations in $E$ manually. However, this laborious and error-prone task makes it more suitable for machines to handle. The symbolic computational system MAPLE (Char et al. (1988)) was used to solve the equations in $E$ over $\mathbb{Z}$. MAPLE yielded the result that $S = \{10, 12, 20\}$. Since the complement of a $4-(\xi, 5, \lambda)$ design is a $4-(\xi, 5, (\xi) - 4 - \lambda)$ design, we need only consider cases when $\lambda \leq [(\xi) - 4]/2$. In addition to $S$ itself, we computed the possible values of $\lambda$ for each value of $p \in S$. Our computations with MAPLE are summarized in the following lemma.

**Lemma 2.** There exists a non-trivial graphical $4-(\xi, 5, \lambda)$ design with $p \geq 8$ only if $(p, \lambda) \in \{(10, 17), (12, 30), (20, 66)\}$.

In the remainder of this section, we prove that there are no non-trivial graphical $4-(\xi, 5, \lambda)$ designs for any $p$ and $\lambda$.

**Lemma 3.** There does not exist a graphical $4-(45, 5, 17)$ design.

**Proof.** Let $(\Omega, \mathcal{D})$ be a graphical $4-(45, 5, 17)$ design. Consider $T_3 \in \Sigma_{4}(\Omega)$ a subgraph of $K_{10}$ isomorphic to the graph consisting of a star on five vertices together with five isolated vertices.

\[
T_3 \simeq \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]
The blocks in \( \mathcal{B} \) containing \( T_3 \) must be isomorphic to one of the following graphs.

\[
\begin{align*}
B_7 \cong & \quad B_8 \cong \quad B_9 \cong \\
& \quad B_{12} \cong
\end{align*}
\]

We have \( \#(T_3 \rightarrow B_7) = 6 \), \( \#(T_3 \rightarrow B_8) = 5 \), \( \#(T_3 \rightarrow B_9) = 20 \), \( \#(T_3 \rightarrow B_{12}) = 10 \), and there is no subset of \( \{5, 6, 10, 20\} \) whose sum is 17.

**Lemma 4.** There does not exist a graphical 4-(66, 5, 30) design.

**Proof.** Consider the same graphs as in Lemma 3 (except in \( K_{12} \) instead of \( K_{16} \)). We now have \( \#(T_3 \rightarrow B_7) = 6 \), \( \#(T_3 \rightarrow B_8) = 7 \), \( \#(T_3 \rightarrow B_9) = 28 \), \( \#(T_3 \rightarrow B_{10}) = 21 \), and there is no subset of \( \{6, 7, 21, 28\} \) whose sum is 30.

**Lemma 5.** There does not exist a graphical 4-(190, 5, 66) design.

**Proof.** Let \((\Omega, \mathcal{Z})\) be a graphical 4-(190, 5, 66) design. Consider \( T_4 \in \Sigma_4(\Omega) \) a subgraph of \( K_{20} \) isomorphic to the graph consisting of a triangle, an edge that is vertex-disjoint from the triangle, and 15 isolated vertices.

\[
T_4 \cong
\]

The blocks in \( \mathcal{B} \) containing \( T_4 \) must be isomorphic to one of the following graphs.

\[
\begin{align*}
B_{11} \cong & \quad B_{12} \cong \quad B_{13} \cong \\
& \quad B_{14} \cong
\end{align*}
\]

We have \( \#(T_4 \rightarrow B_{11}) = 6 \), \( \#(T_4 \rightarrow B_{12}) = 30 \), \( \#(T_4 \rightarrow B_{13}) = 45 \), \( \#(T_4 \rightarrow B_{14}) = 105 \), and there is no subset of \( \{6, 30, 45, 105\} \) whose sum is 66.

Combining the results above gives the following.

**Lemma 6.** There exist no non-trivial graphical 4-\((15, 5, \lambda)\) designs for any \( \lambda \) and \( p = 8 \).

Since the divisibility conditions force all 4-\((15, 5, \lambda)\) designs to be trivial for \( p < 6 \), we need only consider the two remaining values \( p = 6 \) and \( p = 7 \) to complete the solution of the existence problem for non-trivial graphical quintuple systems. Kramer & Mesner (1976) have established that there do not exist any non-trivial graphical 4-(15, 5, \( \lambda \)) designs. We now prove that there are no non-trivial graphical 4-(21, 5, \( \lambda \)) designs.
Lemma 7. There does not exist a non-trivial graphical 4-(21, 5, \(\lambda\)) design for any \(\lambda\).

Proof. Let \(\Lambda\) be the set of integers \(\lambda\) such that there exists a non-trivial graphical 4-(21, 5, \(\lambda\)) design. By considering the number of ways that \(T_1\) (in \(K_r\)) can be extended to each of \(B_2\), \(B_3\), and \(B_1\), we have \(\lambda \in \{2, 3, 5\}\). By considering the number of ways that \(T_3\) (in \(K_r\)) can be extended to each of \(B_7\), \(B_8\), \(B_9\), and \(B_{10}\), we have \(5 \not\in \Lambda\). Finally, consideration of the number of ways that \(T_4\) (in \(K_r\)) can be extended to each of \(B_{11}\), \(B_{12}\), \(B_{13}\), and \(B_{14}\) shows that \(2, 3 \not\in \Lambda\).

We can now state:

Theorem 1. There do not exist non-trivial graphical 4-(\(v\), 5, \(\lambda\)) designs for any \(v\) and \(\lambda\).

4. Conclusion

In this paper, we proved that no non-trivial graphical quintuple systems exist. An immediate problem is suggested:

Problem 1. Determine if there are any, or find all, non-trivial graphical \(k\)-tuple systems for \(k \geq 6\).

We make the conjecture that there are no non-trivial graphical \(k\)-tuple systems for \(k \geq 6\).

References


