ELSEVIER

Discrete Mathematics 237 (2001) 185–186

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

An improved finiteness theorem for graphical t-designs

Yeow Meng Chee *

CoolConnect Pte Ltd, 45 Club Street, Singapore 069422, Singapore

Received 9 July 1999; accepted 20 December 1999

Abstract

We prove that there exist only finitely many nontrivial graphical t-(v,k,λ) designs when $k \leqslant 4t/3$. This improves a previous result of Betten et al. (Discrete Math. 197/198 (1999) 83–109).

We use the notation and terminology of [1] and assume that the reader is familiar with the concept of graphical t-designs [2]. All polynomials in this note are polynomials in n.

Betten et al. [1, Theorem 10] have shown that there exist only finitely many nontrivial graphical t-$(\binom{n}{2},k,\lambda)$ designs when $k=t+1$. In this note, we show that this finiteness result remains true when the condition $k=t+1$ is relaxed to $k \leqslant 4t/3$.

Let $t \geqslant 3$ and $k \leqslant 4t/3$. Let $I(t)$ denote the graph consisting of t independent edges and define $\mathscr{K}$ to be the set of all graphs, each having k edges and contains $I(t)$ as a subgraph. Then if $G \in \mathscr{K}$, G must contain at least $t/3$ isolated edges.

By Alltop's Lemma (see [1, Lemma 2]), the entry in row G and column H of the polynomial Kramer–Mesner matrix is a polynomial whose degree is the difference in the sizes of the supports of G and H. Hence, the entry in row $I(t)$ and column $I(k)$ of the polynomial Kramer–Mesner matrix is a polynomial of degree $2(k-t)$. The other entries in row $I(t)$ are polynomials of degree strictly less than $2(k-t)$.

Without loss of generality, assume that $I(k)$ is a block of a graphical t-$(\binom{n}{2},k,\lambda)$ design $\mathscr{D}$. The columns indexed by graphs in $\mathscr{K}\setminus\{I(k)\}$ each has an entry a polynomial of degree $2(k-t)$, precisely in the row indexed by the graph obtained by removing $k-t \leqslant t/3$ isolated edges from the graph indexing the corresponding column. Hence for large n, all graphs in $\mathscr{K}\setminus\{I(k)\}$ must also be blocks of $\mathscr{D}$. This forces $\mathscr{D}$ to be the complete design and establishes the following result.

* Corresponding author. 12 Jambol Place, Singapore 119339, Singapore.
E-mail address: ymchee@coolconnect.com (Y.M. Chee).

0012-365X/01/$ - see front matter
PII: S0012-365X(00)00059-5

Theorem 1. *There exist only finitely many nontrivial graphical* t-$(\binom{n}{2}, k, \lambda)$ *designs when* $k \leqslant 4t/3$.

References

[1] A. Betten, M. Klin, R. Laue, A. Wassermann, Graphical t-designs vai polynomial Kramer–Mesner matrices, Discrete Math. 197/198 (1999) 83–109.

[2] Y.M. Chee, Graphical designs, in: C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996, pp. 366–369 (Chapter. IV.23).