Optimal Equitable Symbol Weight Codes for Power Line Communications

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***Abstract*—The use of multiple frequency shift keying modulation with permutation codes addresses the problem of permanent narrowband noise disturbance in a power line communications (PLC) system. Equitable symbol weight codes was recently demonstrated to optimize the performance against narrowband noise in a general coded modulation scheme. This paper establishes the first infinite family of optimal equitable symbol weight codes with code lengths greater than alphabet size and whose relative narrowband noise error-correcting capabilities do not diminish to zero as the length grows. These families of codes meet the Plotkin bound. The construction method introduced is combinatorial and reveals interesting interplay with an extension of the concept of generalized balanced tournament designs from combinatorial design theory.**

I. Introduction

Power line communications (PLC) is a technology that enables data transmission over high voltage electric power lines. Started in the 1950s in the form of ripple control for load and tariff management in power distribution, this low bandwidth one-way communication system evolved to a two-way communication system in the 1980s. With the emergence of the Internet in the 1990s, research into broadband PLC gathered pace as a promising technology for Internet access and local area networking, since the electrical grid infrastructure provides "last mile" connectivity to premises and capillarity within premises. Recently, there has been renewed interests in high-speed narrowband PLC due to applications in sustainable energy strategies, specifically in smart grids (see [1]–[4]).

However, overcoming permanent narrowband disturbance has remained a challenging problem [5]–[7]. Vinck [5] addressed this problem by showing that multiple frequency shift keying (MFSK) modulation, in conjunction with the use of a permutation code having minimum (Hamming) distance d, is able to correct up to $d-1$ errors due to narrowband noise. Since then, more general codes such as constant-composition codes, frequency permutation arrays, and injection codes (see [8]–[23]) have been considered as possible replacements for permutation codes in PLC. Versfeld *et al.* [24], [25] later introduced the notion of 'same-symbol weight' (henceforth, termed as *symbol weight*) of a code as a measure of the capability of a code in dealing with narrowband noise. They also showed empirically that low symbol weight cosets of Reed-Solomon codes outperform normal Reed-Solomon codes in the presence of narrowband noise and additive white Gaussian noise.

Unfortunately, symbol weight alone is not sufficient to capture a code's performance against narrowband noise. Chee *et al.* [26] introduced an additional parameter that more precisely captures the narrowband noise error-correcting capability. This parameter is related to *symbol equity*, the uniformity of frequencies of symbols in each codeword. Codes designed taking into account this new parameter have been shown to perform better than general ones [26].

Relatively little is known about optimal equitable symbol weight codes, other than those that correspond to injection codes (which include permutation codes) and frequency permutation arrays (FPAs). In particular, only six infinite families of optimal equitable symbol weight codes with code length greater than alphabet size are known. These have all been constructed by Ding and Yin [15], and Huczynska and Mullen [21] as frequency permutation arrays and they meet the Plotkin bound. One drawback with the code parameters of these families is that the relative narrowband noise error-correcting capability diminishes to zero as its length grows.

In this paper, we construct the first infinite families of optimal equitable codes, whose code lengths are larger than alphabet size and whose relative narrowband noise error-correcting capabilities tend to a positive constant as code length grows. These families of codes all attain the Plotkin bound. Our results are based on the construction of equivalent combinatorial objects called generalized balanced tournament designs.

Owing to space constraints, some proofs are omitted here. These will appear in the full version of the paper.

II. Preliminaries

A. Equitable Symbol Weight Codes

For positive integer n, denote the set $\{1,2\ldots,n\}$ by $[n]$.

Let Σ be a set of q *symbols*. A *q-ary code* of *length* n over the *alphabet* Σ is a subset $\mathcal{C}\subseteq\Sigma^n$. Elements of $\mathcal{C}$ are called *codewords*. The *size* of $\mathcal{C}$ is the number of codewords in $\mathcal{C}$. For $i\in[n]$, the ith coordinate of a codeword $\mathsf{u}\in\mathcal{C}$ is denoted u_i, so that $\mathsf{u}=(\mathsf{u}_1,\mathsf{u}_2,\ldots,\mathsf{u}_n)$.

Denote the *frequency* of symbol $\sigma\in\Sigma$ in codeword $\mathsf{u}\in\Sigma^n$ by $w_\sigma(\mathsf{u})$, that is, $w_\sigma(\mathsf{u})=|\{\mathsf{u}_i=\sigma : i\in[n]\}|$.

An element $\mathsf{u}\in\Sigma^n$ is said to have *equitable symbol weight* if $w_\sigma(\mathsf{u})\in\{\lfloor n/q\rfloor,\lceil n/q\rceil\}$ for any $\sigma\in\Sigma$. If all the codewords of $\mathcal{C}$ have equitable symbol weight, then the code $\mathcal{C}$ is called an *equitable symbol weight code*.

Consider the usual Hamming distance defined on codewords and codes and let d denote the minimum distance of a code $\mathcal{C}$. Consider the following parameter.

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Definition 2.1: Let $\mathcal{C}$ be a q-ary code with minimum distance d. The *narrowband noise error-correcting capability* of $\mathcal{C}$ is

$$c(\mathcal{C}) = \min\{e : E_{\mathcal{C}}(e) \geq d\},$$

where $E_{\mathcal{C}}$ is a function $E_{\mathcal{C}} : [q] \to [n]$, given by

$$E_{\mathcal{C}}(e) = \max_{\substack{\Gamma \subseteq \Sigma \\ |\Gamma|=e}} \max_{\mathsf{c} \in \mathcal{C}} \left\{ \sum_{\sigma \in \Gamma} w_\sigma(\mathsf{c}) \right\}.$$

The *relative narrowband noise error-correcting capability* is then given by $\gamma(\mathcal{C}) = c(\mathcal{C})/n$.

Chee *et al.* [26] established that a code $\mathcal{C}$ can correct up to $c(\mathcal{C}) - 1$ narrowband noise errors and demonstrated that an equitable symbol weight code maximizes the quantity $c(\mathcal{C})$, for fixed n, d and q. Henceforth, only equitable symbol weight codes are considered. A q-ary equitable symbol weight code of length n having minimum distance d is denoted $(n, d)_q$*-equitable symbol weight code*. The maximum size of an $(n, d)_q$-equitable symbol weight code is denoted $A_q^E(n, d)$. Any $(n, d)_q$-equitable symbol weight code of size $A_q^E(n, d)$ is said to be *optimal*.

Taken as a q-ary code of length n, an optimal $(n, d)_q$-equitable symbol weight code satisfies the Plotkin bound [27].

Theorem 2.1 (Plotkin Bound): For n, q, d such that $d \leq n$ and $qd > (q-1)n$, $A_q^E(n, d) \leq qd/(qd - (q-1)n)$.

In this paper, equitable symbol weight codes whose sizes attain the Plotkin bound are constructed.

Main Theorem: Let $q \geq 2$. Then the following holds.

(i) $A_q^E(2q-1, 2q-2) = 2q$ for all $q \geq 3$;
(ii) $A_q^E(3q-1, 3q-3) = 3q$ for all $q \geq 3$;
(iii) $A_q^E(4q-1, 4q-4) = 4q$ for all $q \geq 5$, except possibly when $q \in \{28, 32, 33, 34, 37, 38, 39, 44\}$;
(iv) $A_q^E\left(\frac{3q-1}{2}, \frac{3q-3}{2}\right) = 3q$ for all odd $q \geq 7$, $A_3^E(4, 3) = 6$ and $A_5^E(7, 6) = 14$.

Observe that any equitable symbol weight code $\mathcal{C}$ with the above parameters must have $c(\mathcal{C}) = q - 1$. It can be verified that $\gamma(\mathcal{C})$ tends to a positive constant as q grows. Known values of $A_q^E(n, d)$ are:

(i) $A_q^E(3, 2) = q(q-1)$ for $q \geq 3$ [22].
(ii) $A_q^E(4, 2) = q(q-1)(q-2)$ for $q \geq 4$, $q \neq 7$ [22].
(iii) $A_q^E(n, 1) = q(q-1)(q-2)\cdots(q-n+1)$ (easy).
(iv) $A_n^E(n, 2) = n!$ for $n \geq 1$ (easy).
(v) $A_n^E(n, 3) = n!/2$ for $n \geq 1$ (easy).
(vi) $A_n^E(n, n-1) = n(n-1)$ for prime powers n [9].
(vii) $A_q^E(n, n-1) = q(q-1)$ for q sufficiently large [22].
(viii) $A_n^E(n, n-2) = n(n-1)(n-2)$ for prime powers $n-1$ [28].
(ix) $A_q^E(n, n) = q$ for $q \geq 2$ (easy).
(x) $A_q^E(q(q+1), q^2) = q^2$ for prime powers q [12].
(xi) $A_q^E(\frac{q(kq^2-1)}{k-1}, \frac{kq^2(q-1)}{k-1}) = kq^2$ for prime powers q, $2 \leq k \leq 5$, $(k, q) \neq (5, 9)$ [15].
(xii) $A_{q^{s-t}}^E(\frac{\mu q^{s-t}(q^{2s-t}-1)}{q^t-1}, \frac{\mu q^{2s-t}(q^{s-t}-1)}{q^t-1}) = q^{2s-t}$ for prime powers q, and $1 \leq t < s$, where $\mu = \prod_{i=1}^{t-1} \frac{q^{s-i}-1}{q^i-1}$ [15].
(xiii) $A_{q^s}^E(q^s(q^{2s+c}-1), q^{2s+c}(q^s-1)) = q^{2s+c}$ for prime powers q, and $s, c \geq 1$ [15].
(xiv) $A_q^E(\binom{kq}{k}, \frac{kq-k}{kq-1}\binom{kq}{k}) = kq$ for $q, k \geq 1$ [21].
(xv) $A_q^E(2q^2-q, 2q^2-2q) = 2q$ for even q, $q \notin \{2, 6\}$ [21].

In particular, only six infinite families of optimal codes with $n > q$ are known. However, the code parameters for these six families are such that their relative narrowband noise error-correcting capabilities diminish as q grows. This is undesirable for narrowband noise correction in PLC. Hence, Main Theorem provides the first infinite families of equitable symbol weight codes with code lengths greater than alphabet size and whose relative narrowband noise error-correcting capabilities tend to a positive constant as length grows.

Our construction of optimal equitable symbol weight codes employs tools from combinatorial design theory. The rest of this section introduces the necessary concepts and establishes their connections to equitable symbol weight codes.

B. Generalized Balanced Tournament Designs

A *set system* is a pair $\mathfrak{S} = (X, \mathcal{A})$, where X is a finite set of *points* and $\mathcal{A} \subseteq 2^X$. Elements of $\mathcal{A}$ are called *blocks*. The *order* of $\mathfrak{S}$ is the number of points in X, and the *size* of $\mathfrak{S}$ is the number of blocks in $\mathcal{A}$. Let K be a set of nonnegative integers. The set system $(X, \mathcal{A})$ is said to be K*-uniform* if $|A| \in K$ for all $A \in \mathcal{A}$.

A (v, k, λ)*-balanced incomplete block design*, or $\text{BIBD}(v, k, \lambda)$, is a $\{k\}$-uniform set system $(X, \mathcal{A})$ of order v, such that every pair of distinct points is contained in exactly λ blocks. The size of a $\text{BIBD}(v, k, \lambda)$ is $\frac{\lambda v(v-1)}{k(k-1)}$. A *resolvable* $\text{BIBD}(v, k, \lambda)$, or $\text{RBIBD}(v, k, \lambda)$, is a $\text{BIBD}(v, k, \lambda)$ $(X, \mathcal{A})$ such that $\mathcal{A}$ can be partitioned into *resolution classes*, each of which is a partition of X.

Definition 2.2: A *generalized balanced tournament design* (GBTD) is an $\text{RBIBD}(km, k, \lambda)$ $(X, \mathcal{A})$ whose $m \cdot \frac{\lambda(km-1)}{k-1}$ blocks are arranged into an $m \times \frac{\lambda(km-1)}{k-1}$ array such that:

(i) each point appears exactly once in each column,
(ii) each point appears either $\left\lfloor \frac{\lambda(km-1)}{m(k-1)} \right\rfloor$ or $\left\lceil \frac{\lambda(km-1)}{m(k-1)} \right\rceil$ times in each row.

We denote such a GBTD by $\text{GBTD}_\lambda(k, m)$.

We remark that our definition of a generalized balanced tournament design extends that of Lamken [29], which corresponds to the case $\lambda = k - 1$ in our definition.

C. Equivalence Between Equitable Symbol Weight Codes and GBTDs

Consider a $\text{GBTD}_\lambda(k, m)$ $(X, \mathcal{A})$ whose rows are indexed by $[m]$ and columns by $[n]$, where $n = \frac{\lambda(km-1)}{k-1}$. Given any point x and any column j, there is a unique row that contains x in column j. Hence, for each x, we correspond the codeword $\mathsf{c}(x) = (r_1, r_2, \ldots, r_n) \in [m]^n$, where r_j is the row where x appears in column j. So, $\mathcal{C} = \{\mathsf{c}(x) : x \in X\}$ is a code of length n over the alphabet $[m]$. Note that this correspondence is the one used by Semakov and Zinoviev [30] to show the equivalence between *equidistant codes* and RBIBDs.

For distinct points $x, y \in X$, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is the number of columns for which x and y are not both contained in the same row. Since there are exactly λ blocks containing both x and y, and no two such blocks can occur in the same column, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is $n-\lambda = \frac{\lambda k(m-1)}{k-1}$. This distance is independent of x and y, making $\mathcal{C}$ *equidistant*.

Next, we consider the composition of $\mathsf{c}(x)$ for $x \in X$. From the construction, the number of times a symbol i appears in $\mathsf{c}(x)$ is the number of cells in row i that contains x. By the definition of a $\mathrm{GBTD}_\lambda(k,m)$, this number is in $\{\lfloor n/m \rfloor, \lceil n/m \rceil\}$, so $\mathcal{C}$ is an equitable symbol weight code. We check that the size of $\mathcal{C}$ attains the Plotkin bound.

Finally, this construction of an equitable symbol weight code from a generalized balanced tournament design can easily be reversed. We record these observations as:

Theorem 2.2: There exists a $\mathrm{GBTD}_\lambda(k,m)$ if and only if there exists a $\left(\frac{\lambda(km-1)}{k-1}, \frac{\lambda k(m-1)}{k-1}\right)_m$-equitable symbol weight code of size km. Furthermore, this code is of equitable symbol weight and attains the Plotkin bound.

The following theorem summarizes the state-of-the-art results on the existence of $\mathrm{GBTD}_{k-1}(k,m)$.

Theorem 2.3 (Lamken [29], [31], [32] and Yin [33]):

(i) A $\mathrm{GBTD}_1(2,m)$ exists if and only if $m=1$ or $m \geq 3$.
(ii) A $\mathrm{GBTD}_2(3,m)$ exists if and only if $m=1$ or $m \geq 3$.
(iii) A $\mathrm{GBTD}_3(4,m)$ exists for all $m \geq 5$, except possibly when $m \in \{28,32,33,34,37,38,39,44\}$.

Main Theorem (i)–(iii) is now an immediate consequence of Theorem 2.2 and Theorem 2.3. The existence of $\mathrm{GBTD}_\lambda(k,m)$ when $\lambda \neq k-1$ has not been previously investigated. The smallest open case is when $k=3$ and $\lambda=1$, which we deal with here.

It follows readily from definition that a necessary condition for a $\mathrm{GBTD}_1(3,m)$ to exist is that m must be odd. A quick exhaustive computer search showed that a $\mathrm{GBTD}_1(3,3)$ and a $\mathrm{GBTD}_1(3,5)$ do not exist. In fact, we have $A_3^E(4,3)=6$ and $A_5^E(7,6)=14$, which do not meet the Plotkin bound. Hence, a $\mathrm{GBTD}_1(3,m)$ can exist only if m is odd and $m \notin \{3,5\}$. We prove that this necessary condition is also sufficient. A direct consequence of this is Main Theorem (iv). Since a $\mathrm{GBTD}_1(3,1)$ exists trivially, we focus on $m \geq 7$.

III. Proof Strategy

Our proof of the existence of a $\mathrm{GBTD}_1(3,m)$ for all odd $m \geq 7$ is technical and rather complex. This section outlines the general strategy used, and introduces some required combinatorial designs. As with most combinatorial designs, direct construction to settle their existence is often difficult. Instead, we develop a set of recursive constructions, building big designs from smaller ones. Direct methods are used to construct a large enough set of small designs on which the recursions can work to generate all larger designs.

For our recursion to work, the GBTDs must possess more structure than stipulated in its definition. More specifically, we consider GBTDs that are *special* and $*$*colorable*.

Definition 3.1: A *special* $\mathrm{GBTD}_1(3,m)$ is a $\mathrm{GBTD}_1(3,m)$ such that there exists a *special* cell for which the block it contains is disjoint from all other blocks in the same row.

Definition 3.2: Let c be a positive integer. A c-$*$*colorable* $\mathrm{RBIBD}(v,3,1)$ is an $\mathrm{RBIBD}(v,3,1)$ with the property that its $v(v-1)/6$ blocks can be arranged in a $\frac{v}{3} \times \frac{v-1}{2}$ array, and each block can be colored with one of c colors so that

(i) each point appears exactly once in each column, and
(ii) in each row, blocks of the same color are pairwise disjoint.

Definition 3.3: A $\mathrm{GBTD}_1(3,m)$ is c-$*$*colorable* if each of its blocks can be colored with one of c colors so that in each row, blocks of the same color are pairwise disjoint.

Definition 3.4: A 3-$*$colorable $\mathrm{RBIBD}(v,3,1)$ is 3-$*$*colorable with property* Π if there exists a row r such that for each color i, there exists a point (called a *witness* for i) that does not appear in that row, or is not contained in any block in that row that is colored i. A $\mathrm{GBTD}_1(3,m)$ that is c-$*$*colorable with property* Π is similarly defined.

Proposition 3.1: If an $\mathrm{RBIBD}(v,3,1)$ is 2-$*$colorable, then it is 3-$*$colorable with property Π.

A few more classes of auxiliary designs are also required.

A. Group Divisible Designs, Transversal Designs

Definition 3.5: Let $(X,\mathcal{A})$ be a set system and let $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ be a partition of X into subsets, called *groups*. The triple $(X,\mathcal{G},\mathcal{A})$ is a *group divisible design* (GDD) when every 2-subset of X not contained in a group appears in exactly one block, and $|A \cap G| \leq 1$ for all $A \in \mathcal{A}$ and $G \in \mathcal{G}$.

We denote a GDD $(X,\mathcal{G},\mathcal{A})$ by K-GDD if $(X,\mathcal{A})$ is K-uniform. The *type* of a GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $\langle |G| : G \in \mathcal{G}\rangle$. The exponential notation is sometimes used to describe the type of a GDD: a GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_s^{t_s}$ is a GDD where there are exactly t_i groups of size g_i, $i \in [s]$.

Definition 3.6: A *transversal design* $\mathrm{TD}(k,n)$ is a $\{k\}$-GDD of type n^k. A *doubly resolvable* $\mathrm{TD}(k,n)$, denoted $\mathrm{DRTD}(k,n)$, is a $\mathrm{TD}(k,n)$ whose blocks can be arranged in an $n \times n$ array such that each point appears exactly once in each row and once in each column.

B. Frame Generalized Balanced Tournament Designs

Let $(X,\mathcal{G},\mathcal{A})$ be a $\{3\}$-GDD with $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ and $|G_i| \equiv 0 \bmod 6$ for all $i \in [s]$. Let $R = \frac{1}{3}\sum_{i=1}^{s} |G_i|$ and $C = \frac{1}{2}\sum_{i=1}^{s} |G_i|$. Suppose there exists a partition $[R] = \bigcup_{i=1}^{s} R_i$ and a partition $[C] = \bigcup_{i=1}^{s} C_i$ such that for each $i \in [s]$, we have $|R_i| = |G_i|/3$ and $|C_i| = |G_i|/2$. We say that $(X,\mathcal{G},\mathcal{A})$ is a *frame generalized balanced tournament design* (FGBTD) if its blocks can be arranged in an $R \times C$ array such that the following conditions hold:

(i) the cell (r,c) is empty when $(r,c) \in R_i \times C_i, i \in [s]$,
(ii) for any row $r \in R_i$, each point in $X \setminus G_i$ appears either once or twice and the points in G_i do not appear,
(iii) for any column $c \in C_i$, each point in $X \setminus G_i$ appears exactly once.

Denote this FGBTD by $\mathrm{FGBTD}(3, T)$, where $T = \langle |G_i| : i \in [s] \rangle$.

IV. Recursive Constructions

Proposition 4.1 (Tripling Construction): Suppose there exist a 3-$*$colorable $\mathrm{RBIBD}(m, 3, 1)$ and a $\mathrm{DRTD}(3, m)$. Then there exists a 2-$*$colorable $\mathrm{GBTD}_1(3, m)$. Furthermore, if the $\mathrm{RBIBD}(m, 3, 1)$ is 3-$*$colorable with property Π, then the $\mathrm{GBTD}_1(3, m)$ is a special $\mathrm{GBTD}_1(3, m)$.

Corollary 4.1: Let $m > 3$ and suppose there exists an $\mathrm{RBIBD}(m, 3, 1)$ that is 3-$*$colorable with property Π. Then there exists a special $\mathrm{GBTD}_1(3, 3^k m)$, for all $k \geq 0$.

FGBTDs can also be used to produce GBTDs with the following recursive construction.

Proposition 4.2 (FGBTD Construction): Suppose an $\mathrm{FGBTD}(3, g_1 g_2 \cdots g_s)$ and a special $\mathrm{GBTD}_1(3, g_i + 1)$ for $i \in [s]$ exist. Then a special $\mathrm{GBTD}_1(3, \sum_{i=1}^{s} g_i + 1)$ exists.

Corollary 4.2: If an $\mathrm{FGBTD}(3, (3g)^t)$ and a special $\mathrm{GBTD}_1(3, g + 1)$ exist, a special $\mathrm{GBTD}_1(3, gt + 1)$ exists.

For Proposition 4.2 and Corollary 4.2 to be useful, we require large classes of FGBTDs. We give two recursive constructions for FGBTDs next.

Proposition 4.3 (Inflation): Suppose an $\mathrm{FGBTD}(3, T)$ and a $\mathrm{DRTD}(3, n)$ exist. Then an $\mathrm{FGBTD}(3, nT)$ exists.

Wilson's Fundamental Construction for GDDs [34] can be modified to admit the following specialization.

Proposition 4.4 (FGBTD from Truncated TD): Suppose there exists a $\mathrm{TD}(k + s, m)$, and $g_1, g_2, \ldots, g_s$ are positive integers at most m. If there exists an $\mathrm{FGBTD}(3, g^t)$ for each $t \in \{k, k+1, \ldots, k+s\}$, then there exists an $\mathrm{FGBTD}(3, T)$, where $T = (g \cdot m)^k (g \cdot g_1)(g \cdot g_2) \cdots (g \cdot g_s)$.

V. Direct Constructions

We construct some small GBTDs and FGBTDs that are required to seed the recursive constructions given in the previous section. This section summarizes the results.

Proposition 5.1: There exists a 2-$*$colorable special $\mathrm{GBTD}_1(3, q)$ for all prime powers $q \equiv 1 \bmod 6$.

Proposition 5.2: There exist a special $\mathrm{GBTD}_1(3, m)$ for $m \in \{1, 9, 11, 17, 23, 29, 35, 47, 53, 55\}$, a 3-$*$colorable $\mathrm{GBTD}_1(3, m)$ with property Π for $m \in \{9, 11, 23\}$ and a 3-$*$colorable $\mathrm{RBIBD}(15, 3, 1)$ with property Π.

Proposition 5.3: There exist an $\mathrm{FGBTD}(3, 6^t)$ for all $t \in \{5, 6, 7, 8\}$, an $\mathrm{FGBTD}(3, 24^t)$ for all $t \in \{5, 8\}$ and an $\mathrm{FGBTD}(3, 30^t)$ for all $t \in \{5, 7\}$.

VI. Spectrum of GBTDs

In this section, we apply recursive constructions in Section IV with small designs directly constructed in Section V to completely settle the existence of GBTDs.

Lemma 6.1: A special $\mathrm{GBTD}_1(3, 3^r q)$ exists for all $r \geq 0$ and $q \in Q$, where $Q = \{q : q \equiv 1 \bmod 6 \text{ is a prime power}\} \cup \{5, 9, 11, 23\}$, except when $(r, q) = (0, 5)$.

Proof: Existence of a special $\mathrm{GBTD}_1(3, q)$ for all $q \in Q \setminus \{5\}$ is provided by Proposition 5.1 and Proposition 5.2. These GBTDs are all 3-$*$colorable with property Π. The lemma follows by considering these GBTDs as RBIBDs and applying Corollary 4.1. ■

Lemma 6.2: Let $s \in [2]$ and suppose there exists a $\mathrm{TD}(5+s, n)$. If $0 \leq g_i \leq n$, $i \in [s]$ and that there exists a special $\mathrm{GBTD}_1(3, m)$ for all $m \in \{2n+1\} \cup \{2g_i + 1 : i \in [s]\}$, then there exists a special $\mathrm{GBTD}_1(3, 10n + 1 + 2\sum_{i=1}^{s} g_i)$.

Proof: By Proposition 5.3, there exists an $\mathrm{FGBTD}(3, 6^t)$ for all $t \in \{5, 6, 7\}$. By Proposition 4.4, there exists an $\mathrm{FGBTD}(3, (6n)^5 (6g_1) \cdots (6g_s))$. Now apply Proposition 4.2 to obtain a special $\mathrm{GBTD}_1(3, 10n + 1 + 2\sum_{i=1}^{s} g_i)$. ■

Lemma 6.3: A special $\mathrm{GBTD}_1(3, m)$ exists for odd $m \geq 7$.

Proof: First, a special $\mathrm{GBTD}_1(3, m)$ can be constructed for odd m, $7 \leq m \leq 95$. Details are provided in Table I.

We then prove the lemma by induction on $m \geq 97$.

Let $E = \{t : t \geq 9\} \setminus \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$. Then a $\mathrm{TD}(7, n)$ exists for any $n \in E$ (see [35]). If there exists a special $\mathrm{GBTD}_1(3, m')$ for odd m', $7 \leq m' \leq 2n + 1$, then apply Lemma 6.2 with $3 \leq g_1, g_2 \leq n$ to obtain a special $\mathrm{GBTD}_1(3, m)$ for odd m, $10n + 7 \leq m \leq 14n + 1$.

Hence, take $n = 9$ to obtain a special $\mathrm{GBTD}_1(3, 97)$.

Suppose there exists a $\mathrm{GBTD}_1(3, m')$ for all odd $m' < m$. Then there exists $n \in E$ with $10n+7 \leq m \leq 14n+1$. Suppose otherwise. Then there exists $n_1 \in E$ such that $14n_1 + 1 < 10n_2 + 7$ for all $n_2 > n_1$ and $n_2 \in E$. This, together with the fact that $n_1 \geq 9$, implies that $n_2 - n_1 > 3$ for all $n_2 \in E$ and $n_2 > n_1$. However, a quick check on E gives a contradiction.

Since $n \in E$ and there exists a special $\mathrm{GBTD}_1(3, m')$ for all $m' \leq 2n + 1 < 10n + 7 \leq m$ (induction hypothesis), there exists a special $\mathrm{GBTD}_1(3, m)$ and induction is complete. ■

Lemma 6.3 shows that a $\mathrm{GBTD}_1(3, m)$ exists for all odd $m \neq 3, 5$. Main Theorem (iv) now follows.

VII. Conclusion

This paper establishes the first infinite families of equitable symbol weight codes, whose code lengths are greater than alphabet size and whose relative narrowband noise error-correcting capabilities tend to a positive constant as length grows. Such codes have applications in dealing with narrowband noise over a PLC channel. The construction method used is combinatorial and reveals interesting interplays with an extension of generalized balanced tournament designs, first introduced by Lamken [29]. These have enabled us to borrow ideas from combinatorial design theory to construct symbol weight codes. In return, questions on equitable symbol weight codes offer new problems to combinatorial design theory. We expect this symbiosis to deepen.

TABLE I
EXISTENCE OF SPECIAL $\mathrm{GBTD}_1(3, m)$

Authority	m
Proposition 5.2	9, 11, 17, 23, 29, 35, 47, 53, 55
Lemma 6.1	7, 13, 15, 19, 21, 25, 27, 31, 33, 37, 39, 43, 45, 49, 57, 61, 63, 67, 69, 73, 75
Corollary 4.2 with $(g,t) \in \{(8,5),(5,10),(8,8),(7,10)\}$	41, 51, 65, 71
Lemma 6.2 with $n = 5$, $g_1 = 4$	59
Lemma 6.2 with $n = 7$, $g_1, g_2 \in \{0\} \cup \{t : 3 \leq t \leq 7\}$	77-95

ACKNOWLEDGEMENT

Research of Y. M. Chee, H. M. Kiah, and C. Wang is supported in part by the Singapore National Research Foundation under Research Grant NRF-CRP2-2007-03. Research of C. Wang is also supported in part by NSFC under Grant No. 10801064. The authors thank Charlie Colbourn and Punarbasu Purkayastha for helpful discussions, and the anonymous reviewers, whose comments improved the presentation of this paper.

REFERENCES

[1] A. Haidine, B. Adebisi, A. Treytl, H. Pille, B. Honary, and A. Portnoy, "High-speed narrowband PLC in smart grid landscape – state-of-the-art," in *ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications*, 2011, pp. 468–473.

[2] D. Dzung, I. Berganza, and A. Sendin, "Evolution of powerline communications for smart distribution: from ripple control to OFDM," in *ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications*, 2011, pp. 474–478.

[3] J. Liu, B. Zhao, L. Geng, Z. Yuan, and Y. Wang, "Communication performance of broadband PLC technologies for smart grid," in *ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications*, 2011, pp. 491–496.

[4] W. Zhang and L. Yang, "SC-FDMA for uplink smart meter transmission over low voltage power lines," in *ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications*, 2011, pp. 497–502.

[5] A. J. H. Vinck, "Coded modulation for power line communications," *AEÜ - Int J. Electron. Commun.*, vol. 54, no. 1, pp. 45–49, 2000.

[6] E. Biglieri, "Coding and modulation for a horrible channel," *IEEE Commun. Mag.*, vol. 41, no. 5, pp. 92–98, 2003.

[7] N. Pavlidou, A. J. H. Vinck, J. Yazdani, and B. Honary, "Power line communications: state of the art and future trends," *IEEE Commun. Mag.*, vol. 41, no. 4, pp. 34–40, 2003.

[8] Y. Luo, F.-W. Fu, A. J. H. Vinck, and W. Chen, "On constant-composition codes over Z_q," *IEEE Trans. Inform. Theory*, vol. 49, no. 11, pp. 3010–3016, 2003.

[9] C. J. Colbourn, T. Kløve, and A. C. H. Ling, "Permutation arrays for powerline communication and mutually orthogonal Latin squares," *IEEE Trans. Inform. Theory*, vol. 50, no. 6, pp. 1289–1291, 2004.

[10] W. Chu, C. J. Colbourn, and P. Dukes, "Constructions for permutation codes in powerline communications," *Des. Codes Cryptogr.*, vol. 32, no. 1-3, pp. 51–64, 2004.

[11] C. Ding and J. Yin, "Algebraic constructions of constant composition codes," *IEEE Trans. Inform. Theory*, vol. 51, no. 4, pp. 1585–1589, 2005.

[12] ——, "Combinatorial constructions of optimal constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 51, no. 10, pp. 3671–3674, 2005.

[13] C. Ding and J. Yuan, "A family of optimal constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 51, no. 10, pp. 3668–3671, 2005.

[14] W. Chu, C. J. Colbourn, and P. Dukes, "On constant composition codes," *Discrete Appl. Math.*, vol. 154, no. 6, pp. 912–929, 2006.

[15] C. Ding and J. Yin, "A construction of optimal constant composition codes," *Des. Codes Cryptogr.*, vol. 40, no. 2, pp. 157–165, 2006.

[16] S. Huczynska, "Powerline communication and the 36 officers problem," *Phil. Trans. R. Soc. A*, vol. 364, no. 1849, pp. 3199–3214, 2006.

[17] S. Huczynska and G. L. Mullen, "Frequency permutation arrays," *J. Combin. Des.*, vol. 14, no. 6, pp. 463–478, 2006.

[18] Y. M. Chee, A. C. H. Ling, S. Ling, and H. Shen, "The PBD-closure of constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 53, no. 8, pp. 2685–2692, 2007.

[19] J. Lin, J. Chang, R. Chen, and T. Kløve, "Distance-preserving and distance-increasing mappings from ternary vectors to permutations," *IEEE Trans. Inform. Theory*, vol. 54, no. 3, pp. 1334–1339, 2008.

[20] Y. M. Chee, S. H. Dau, A. C. H. Ling, and S. Ling, "Linear size optimal q-ary constant-weight codes and constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 56, no. 1, pp. 140–151, 2010.

[21] S. Huczynska, "Equidistant frequency permutation arrays and related constant composition codes," *Des. Codes Cryptogr.*, vol. 54, no. 2, pp. 109–120, 2010.

[22] P. J. Dukes, "Coding with injections," *Des. Codes Cryptogr.*, DOI 10.1007/s10623-011-9547-4, online first, 2 August 2011.

[23] F. Gao and G. Ge, "Optimal ternary constant-composition codes of weight four and distance five," *IEEE Trans. Inform. Theory*, vol. 57, no. 6, pp. 3742–3757, 2011.

[24] D. J. J. Versfeld, A. J. H. Vinck, and H. C. Ferreira, "Reed-solomon coding to enhance the reliability of M-FSK in a power line environment," in *ISPLC 2005: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications*, 2005, pp. 100–104.

[25] D. J. J. Versfeld, A. J. H. Vinck, J. N. Ridley, and H. C. Ferreira, "Constructing coset codes with optimal same-symbol weight for detecting narrowband interference in M-FSK systems," *IEEE Trans. Inform. Theory*, vol. 56, no. 12, pp. 6347–6353, 2010.

[26] Y. M. Chee, H. M. Kiah, P. Purkayastha, and C. Wang, "Importance of symbol equity in coded modulation for power line communications," in *ISIT 2012: Proceedings of the IEEE International Symposium on Information Theory*, 2012.

[27] I. F. Blake and R. C. Mullin, *An Introduction to Algebraic and Combinatorial Coding Theory*. New York: Academic Press, 1976.

[28] P. Frankl and M. Deza, "On the maximum number of permutations with given maximal or minimal distance," *J. Comb. Theory Ser. A*, vol. 22, no. 3, pp. 352–360, 1977.

[29] E. R. Lamken, "Generalized balanced tournament designs," *Trans. Amer. Math. Soc.*, vol. 318, no. 2, pp. 473–490, 1990.

[30] N. V. Semakov and V. A. Zinoviev, "Equidistant q-ary codes with maximal distance and resolvable balanced incomplete block designs," *Problemy Peredači Informacii*, vol. 4, no. 2, pp. 3–10, 1968.

[31] E. R. Lamken, "Existence results for generalized balanced tournament designs with block size 3," *Des. Codes Cryptogr.*, vol. 3, no. 1, pp. 33–61, 1992.

[32] ——, "Constructions for generalized balanced tournament designs," *Discrete Math.*, vol. 131, no. 1–3, pp. 127–151, 1994.

[33] J. Yin, J. Yan, and C. Wang, "Generalized balanced tournament designs and related codes," *Des. Codes Cryptogr.*, vol. 46, no. 2, pp. 211–230, 2008.

[34] R. M. Wilson, "An existence theory for pairwise balanced designs. I. Composition theorems and morphisms," *J. Combin. Theory Ser. A*, vol. 13, pp. 220–245, 1972.

[35] R. J. R. Abel, C. J. Colbourn, and J. H. Dinitz, "Mutually orthogonal Latin squares (MOLS)," in *The CRC Handbook of Combinatorial Designs*, 2nd ed., C. J. Colbourn and J. H. Dinitz, Eds. Boca Raton: CRC Press, 2007, pp. 160–193.