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The algorithmic complexity of colour switching

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Abstract

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A new graph colouring related problem called the colour switching problem is introduced. By giving a reduction from the colour switching problem to the weighted matching problem on bipartite graphs, we show that the colour switching problem can be solved in $O(n^{3/2} \log n)$ time. The colour switching problem has many useful applications in computer science and engineering. We outline an application to the channel allocation problem in mobile cellular radio systems.

Keywords: Graph colouring, combinatorial optimization, design of algorithms, channel allocation

1. Introduction

Let G be a (finite, simple, undirected) graph with vertex set $V(G)$ and edge set $E(G)$. For a positive integer λ, a *λ-colouring* of G is a function $f: V(G) \to C$, where C is a set of λ elements called *colours*. A λ-colouring $f: V(G) \to C$ of a graph G is said to be *proper* if for all $c \in C$, the set of vertices $f^{-1}(c)$ induces an edge-free subgraph of G. The smallest λ for which G has a proper λ-colouring is called the *chromatic number* of G, and is denoted by $\chi(G)$. The problem of deciding if $\chi(G) = K$, for a given graph G and positive integer K, is NP-complete [7]. The *graph colouring problem* is to find a $\chi(G)$-colouring for a given graph G. The graph colouring problem is NP-hard.

Despite the convincing evidence for their computational intractability, graph colouring problems have found useful applications in widely different areas of computer science, engineering, and management science. In many of these applications, we have a set T of tasks to be performed, and a set R of resources. Each of the tasks in T requires a resource in R before it can be carried out. However, there are constraints which prevent certain tasks in T to be allocated the same resource. The problem is to find a way of assigning resources to the tasks so that the minimum number of resources is used. This problem can be modeled as a graph colouring problem by constructing a graph G with $V(G) = T$, with an edge between vertices t_i and t_j if task t_i cannot be allocated the same resource as task t_j. Allocating

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the minimum number of resources to the tasks is easily seen to be equivalent to finding a proper $\chi(G)$-colouring for G. Since at present all algorithms solving the graph colouring problem require exponential time in the worst case, we cannot expect them to be useful in large scale problems. We thus settle for fast (polynomial-time) heuristics that produce proper λ-colourings, hopefully with λ close to $\chi(G)$.

In this paper, we introduce a new problem related to the graph colouring problem. Let $f: V(G) \to C$ and $f': V(G) \to C'$ be two λ-colourings of a graph G. We say that f and f' are *equivalent* if and only if there exists a bijection $\pi: C \to C'$ such that $\pi(f(v)) = f'(v)$ for $v \in V(G)$. Suppose that G is coloured with a λ-colouring f, and that a λ'-colouring f' of G with $\lambda' < \lambda$ is given. The *colour switching problem* is to switch the colours on the minimum number of vertices of G so that a λ'-colouring equivalent to f' results. More formally, we may state the colour switching problem as follow:

Colour switching
Instance: A graph G coloured with a λ-colouring f, and a λ'-colouring f' of G with $\lambda' < \lambda$.
Problem: Find $f^*: V(G) \to C^*$ minimizing

$$|\{v \in V(G): f^*(v) \neq f(v)\}|$$

such that f^* is a λ'-colouring of G that is equivalent to f'.

The colour switching problem has many useful applications. Suppose that in our previous example, we have found an assignment of resources to tasks using a graph colouring heuristic. If later on at some point in time, we have a better graph colouring heuristic that produces an assignment of resources to tasks using less resources, we may want to reassign the resources to the tasks according to this new assignment. In the situation where removing a resource from a task and reassigning a new resource to it incurs a high cost, we would want to minimize the number of reassignments necessary. This is precisely the colour switching problem. We outline in Section 3 an application of the colour switching problem to channel allocation in mobile cellular radio systems.

We prove in the next section that the colour switching problem can be solved in polynomial time.

2. Maximum weight bipartite matching and colour switching

A *matching* M in a graph G is a subset of edges in $E(G)$ such that no two edges of M have a common endvertex. Given a graph G and a weight function $w: E(G) \to \mathbb{Z}_{>0}$, the *weight* of a matching M in G, denoted $w(M)$, is the quantity $\sum_{e \in M} w(e)$. The *weighted matching problem* is that of finding a matching of maximum weight. The weighted matching problem is solvable in polynomial time, as was first shown by Edmonds [2]. For bipartite graphs, polynomial-time algorithms for solving the weighted matching problem were known as early as 1955 [8]. The fastest algorithm known at present for the weighted matching problem on bipartite graphs is the algorithm of Gabow and Tarjan [3] that runs in $O(m\sqrt{n}\log(nN))$ time, where $m = |E(G)|$, $n = |V(G)|$, and N is an upper bound on $w(e)$, $e \in E(G)$. We give in this section an efficient reduction from the colour switching problem to the weighted matching problem on bipartite graphs.

Let a graph G coloured with a λ-colouring $f: V(G) \to C$, and a λ'-colouring $f': V(G) \to C'$ of G with $\lambda' < \lambda$ be an instance of the colour switching problem. Without loss of generality, assume that $C \cap C' = \emptyset$. We construct a bipartite graph H as follows. Let $V(H) = C \cup C'$. An edge e appears between two vertices c and c' if and only if there exists a vertex $v \in V(G)$ such that $f(v) = c$ and $f'(v) = c'$. The weight on the edge e is defined as:

$$w(e) = |\{v \in V(G): f(v) = c \text{ and } f'(v) = c'\}|. \tag{1}$$

This construction of H can be carried out in $O(n)$ time.

Lemma 1. *Let $f^*: V(G) \to C^*$ be any λ'-colouring of G that is equivalent to f'. Then*

$$|\{v \in V(G): f^*(v) = f(v)\}| \leqslant w(M^*),$$

where M^* is a matching in H of maximum weight.

Proof. Suppose that $f^*: V(G) \to C^*$ is a λ'-colouring of G that is equivalent to f'. Let $\text{fix}(G) = \{v \in V(G): f^*(v) = f(v)\}$ and let M be the multiset

$$\{(f^*(v), f'(v)): v \in \text{fix}(G)\}.$$

Let $\text{supp}(M)$ denote the set of distinct elements of M. Clearly, $\text{supp}(M) \subseteq E(H)$. Let $e_1 = (f^*(v_1), f'(v_1))$ and $e_2 = (f^*(v_2), f'(v_2))$ be two distinct edges in $\text{supp}(M)$. Suppose that $f^*(v_1) = f^*(v_2)$. Then since f^* and f' are equivalent, there exists a bijection $\pi: C^* \to C'$ such that $\pi(f^*(v)) = f'(v)$ for all $v \in V(G)$. In particular, we have $f'(v_1) = \pi(f^*(v_1)) = \pi(f^*(v_2)) = f'(v_2)$, and this contradicts the assumption the e_1 and e_2 are distinct. Hence, $f^*(v_1) \neq f^*(v_2)$. A similar proof yields the result that $f'(v_1) \neq f'(v_2)$. Consequently, $\text{supp}(M)$ is a matching in H. Since the multiplicity of each edge $e \in \text{supp}(M)$ in M is at most $w(e)$ defined in (1), we have

$$|\text{fix}(G)| \leqslant w(M^*)$$

and the lemma follows. □

We now prove that the upperbound in Lemma 1 can in fact be achieved.

Lemma 2. *There exists a λ'-colouring $f^*: V(G) \to C^*$ of G that is equivalent to f', and*

$$|\{v \in V(G): f^*(v) = f(v)\}| = w(M^*),$$

where M^* is a matching in H of maximum weight.

Proof. Let M^* be a matching in H of maximum weight, and let $F = \{v \in V(G): (f(v), f'(v)) \in M^*\}$. If $v \in V(G) \setminus F$, then there exists a vertex $P(v) \in F$ such that either $f'(v) = f'(P(v))$ or $f(v) = f(P(v))$, for otherwise M^* would not be a maximum weight matching in H. Let $D = \{v \in V(G) \setminus F: \exists P(v) \in F \text{ for which } f'(v) = f'(P(v))\}$. For any two vertices u, $v \in V(G) \setminus (F \cup D)$, we say that $u \sim v$ if and only if $f'(u) = f'(v)$. It is clear that $\sim$ is an equivalence relation, and hence partitions $V(G) \setminus (F \cup D)$ into equivalence classes $V_1, V_2, \ldots, V_k$. We define the function $f^*: V(G) \to C^*$ as follows:

for each $v \in F$
  $f^*(v) := f(v)$;
for each $v \in D$
  $f^*(v) := f(P(v))$;
$C^* := \{f^*(v): v \in F \cup D\}$;
for $i = 1, 2, \ldots, k$ {
  pick any $c \notin C^*$;
  for each $v \in V_i$
    $f^*(v) := c$;
  $C^* := C^* \cup \{c\}$;
}

It is not difficult to see that this definition of f^* ensures that f^* is equivalent to f'.

Suppose $v \in D$. If $f(v) = f(P(v))$, this would imply that $v \in F$ since $(f(v), f'(v)) = (f(P(v)), f'(P(v))) \in M^*$. This contradiction shows that $f(v) \neq f(P(v))$ and hence $f^*(v) \neq f(v)$. Suppose now that $v \in V(G) \setminus (F \cup D)$, that is, $f^*(v) \notin \{f^*(v): v \in F \cup D\}$. If $f^*(v) = f(v)$, then $f^*(v) = f(P(v))$ for some $P(v) \in F$. It follows that $f^*(P(v)) = f(P(v))$ and consequently we have $f^*(v) = f(P(v)) \in \{f^*(v): v \in F \cup D\}$, a contradiction. Therefore,

$$\begin{aligned} |\{v \in V(G): f^*(v) = f(v)\}| &= |F| \\ &= w(M^*). \end{aligned}$$ □

Theorem 3. *The colour switching problem can be solved in time*

$$O\left(\sqrt{\lambda + \lambda'} \min\{n, \lambda\lambda'\} \log(n(\lambda + \lambda')) + n\right),$$

where $n = |V(G)|$.

Proof. Given an instance of the colour switching problem, we construct H in $O(n)$ time. By Lemma 1 and Lemma 2, solving the colour switching problem is equivalent to finding a maximum weight matching in H. Since H has $\lambda + \lambda'$ vertices, at most $\min\{n, \lambda\lambda'\}$ edges, and the weight on each edge is also at most n, the algorithm of

Gabow and Tarjan [3] finds a matching M^* in H of maximum weight in

$$O(\sqrt{\lambda + \lambda'} \min\{n, \lambda\lambda'\} \log(n(\lambda + \lambda')))$$

time. The proper λ'-colouring f^* given in the proof of Lemma 2 is constructable from M^* in $O(n)$ time. □

Corollary 4. *The colour switching problem can be solved in* $O(n^{3/2} \log n)$ *time.*

Proof. Every graph on n vertices can be coloured using at most n colours. Hence $\lambda + \lambda' < 2n$. □

We note that the colour switching problem can be solved in $O(n)$ optimal time if λ and λ' are fixed.

Going back to our resource allocation example given in Section 1, it is frequently the case in real-world scenarios that the cost of removing a resource from a task and reassigning a new resource to it is dependent on the task itself. This suggests a natural generalization of the colour switching problem:

Weighted colour switching
Instance: A graph G coloured with a λ-colouring f, a λ'-colouring f' of G with $\lambda' < \lambda$, and a function $cost: V(G) \to \mathbb{Z}_{>0}$.
Problem: Find $f^*: V(G) \to C^*$ minimizing

$$\sum_{v \in V(G): f^*(v) \neq f(v)} cost(v)$$

such that f^* is a λ'-colouring of G that is equivalent to f'.

The same technique we used to solve the colour switching problem in polynomial time is applicable to the weighted colour switching problem as well. The only difference is that the following definition of w is used instead of the one given by (1).

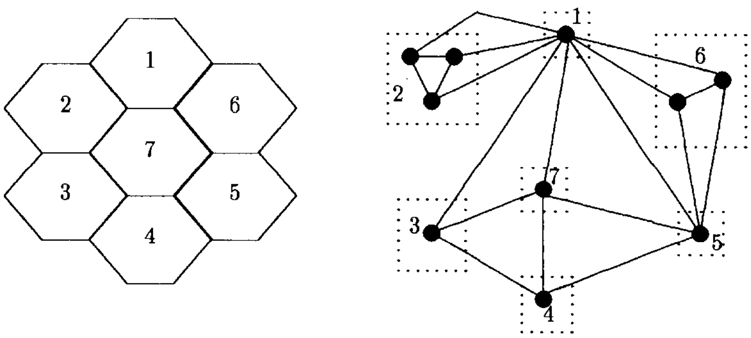
$$w(e) = \sum_{v \in V(G): f(v) = c \text{ and } f'(v) = c'} cost(v).$$

If N is an upperbound on $cost(v)$ for all $v \in V(G)$, the running time of our algorithm is that stated in the theorem below:

Theorem 5. *We can solve in time* $O(n^{3/2} \log(nN))$ *the weighted colour switching problem. If* λ *and* λ' *are fixed, the weighted colour switching problem can be solved in optimal time* $O(n + \log N)$. □

3. Application of colour switching to frequency allocation

In a cellular radio system, the geographical service area to be covered is partitioned into cells (usually hexagonal in shape), each of which is served by a transmitter of moderate power. The idea of using transmitters of moderate power is so that signals can only be propagated over short distances, and therefore frequencies can be reused in other cells. This reuse of frequencies



(i) Cells (ii) Graph G

Fig. 1. Construction of G.

enhances service capability, service performance, and electromagnetic spectrum utilization.

In order to minimize the amount of co-channel interference among users, there is a minimum physical separation imposed between cells that use the same frequencies. The physical separation depends on the terrain, climatic conditions, power and height of transmitter, etc. Since frequency is a scarce resource, it is important to be able to utilize it efficiently, i.e., we want to use as small a number of frequencies as possible for the functioning of the cellular radio system. This problem is known as the *frequency assignment problem* in the literature and has been investigated by many researchers (see [1,4–6,9–12] and the literature quoted therein).

To obtain the minimum number of frequencies needed for the cellular radio system, we construct a graph G with $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$, where k is the total number of cells. Each V_i, $1 \leqslant i \leqslant k$, is a set of d_i vertices, where d_i is the demand of cell i, i.e., d_i is the number of frequencies required by cell i. For each i, $1 \leqslant i \leqslant k$, all vertices in V_i are adjacent, i.e., V_i induces a clique in G. Also, there exists an edge (u, v) between vertices $u \in V_i$ and $v \in V_j$ if cell i and cell j cannot use the same frequency due to co-channel interference. For example, Fig. 1(i) shows the cells covering a certain geographical area to be served and Fig. 1(ii) gives the corresponding graph G. In this example, $d_1 = d_3 = d_4 = d_5 = d_7 = 1$, $d_2 = 3$, and $d_6 = 2$. It is easy to see that the least number of frequencies needed in the cellular radio system is the chromatic number of G. Unfortunately, the problem of finding the chromatic number of G is NP-hard. Hence, no polynomial-time algorithms exist for solving the problem unless P = NP.

In countries with mobile cellular radio systems, there is already an existing frequency assignment to the cells. This assignment is either done manually or with some automated heuristics. Due to advancement in computer hardware and algorithmic techniques, better frequency assignment to cells will surely be found at some point in time. With a better frequency assignment, we would want to re-tune the transmitters in some of the cells so that this better frequency assignment is realized. Re-tuning of transmitters is expensive and time consuming. As a result, the minimization of the amount of re-tuning is necessary. The minimization of the amount of re-tuning is the colour switching problem.

4. Concluding remarks

We have introduced in this paper a problem called the colour switching problem that is related to the graph colouring problem. The colour switching problem has many useful applications in computer science and engineering. We also proposed an algorithm that solves the colour switching problem in polynomial time by reducing it to a weighted matching problem on bipartite graphs. The algorithm is fast and has a worst-case running time of $O(n^{3/2} \log n)$. A generalization of the colour switching problem to that with vertex costs can be solved in $O(n^{3/2} \log(nN))$ time. A sketch on the application of the colour switching problem to channel allocation in mobile cellular radio systems is given.

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References

[1] F. Box, A heuristic-technique for assigning frequencies to mobile radio nets, *IEEE Trans. Vehicular Technology* **27** (1978) 57–64.

[2] J. Edmonds, Maximum matching and a polyhedron with 0,1-vertices, *J. Res. Nat. Bur. Standards B* **69** (1965) 125–130.

[3] H.N. Gabow and R.E. Tarjan, Faster scaling algorithms for network problems, *SIAM J. Comput.* **18** (1989) 1013–1026.

[4] A. Gamst, Some lower bounds for a class of frequency assignment problems, *IEEE Trans. Vehicular Technology* **35** (1986) 8–14.

[5] A Gamst and W. Rave, On frequency assignment in mobile automatic telephone systems, in: *Proc. GLOBECOM '82* (IEEE Press, New York, 1982) 309–315.

[6] W.K. Hale, Frequency assignment: Theory and application, *Proc. IEEE* **68** (1980) 1497–1514.

[7] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller and J.W. Thatcher, eds., *Complexity of Computer Computations* (Plenum, New York, 1972) 85–103.
[8] H.W. Kuhn, The Hungarian method for the assignment problem, *Naval Res. Logistics Quart.* **2** (1955) 83–97.
[9] D. Kunz, Channel assignment for cellular radio using neural networks, *IEEE Trans. Vehicular Technology* **40** (1991) 188–193.
[10] K. Mathur, H.M. Salkin, K. Nishimura and S. Morito, The design of an interative computer software system for the frequency-assignment problem, *IEEE Trans. Electromagnetic Compatibility* **26** (1984) 207–212.
[11] K.N. Sivarajan, R.J. McEliece and J.W. Ketchum, Channel assignment in mobile radio, in: *Proc. 39th IEEE Vehicular Technology Society Conf.* (IEEE Press, New York, 1989) 846–850.
[12] J.A. Zoellner and C.L. Beall, A breakthrough in spectrum conserving frequency assignment technology, *IEEE Trans. Electromagnetic Compatibility* **19** (1977) 313–319.