

## THE $\alpha$ -ARBORICITY OF COMPLETE UNIFORM HYPERGRAPHS\*

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**Abstract.**  $\alpha$ -acyclicity is an important notion in database theory. The  $\alpha$ -arboricity of a hypergraph  $\mathcal{H}$  is the minimum number of  $\alpha$ -acyclic hypergraphs that partition the edge set of  $\mathcal{H}$ . The  $\alpha$ -arboricity of the complete 3-uniform hypergraph is determined completely.

**Key words.**  $\alpha$ -arboricity, acyclic hypergraph, decomposition, Steiner system

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**1. Introduction.** There is a natural bijection between database schemas and hypergraphs, where each attribute of a database schema  $\mathbf{D}$  corresponds to a vertex in a hypergraph  $\mathcal{H}$ , and each relation  $R$  of attributes in  $\mathbf{D}$  corresponds to an edge in  $\mathcal{H}$ . Many properties of databases have therefore been studied in the context of hypergraphs. One such property of databases is the important notion of  $\alpha$ -acyclicity. Besides being a desirable property in the design of databases [2], [3], [8], [9], [10], many NP-hard problems concerning databases can be solved in polynomial time when restricted to instances for which the corresponding hypergraphs are  $\alpha$ -acyclic [3], [16], [19]. Examples of such problems include determining global consistency, evaluating conjunctive queries, and computing joins or projections of joins.

When faced with such computationally intractable problems on a general database schema, it is natural to decompose it into  $\alpha$ -acyclic instances on which efficient algorithms can be applied. This has motivated some recent studies on the  $\alpha$ -arboricity of hypergraphs, the minimum number of  $\alpha$ -acyclic hypergraphs into which the edges of a given hypergraph can be partitioned [4], [14], [17].

In this paper, we give a general construction for partitioning complete uniform hypergraphs into  $\alpha$ -acyclic hypergraphs based on Steiner systems, and we completely determine the  $\alpha$ -arboricity of complete 3-uniform hypergraphs.

**2. Preliminaries.** We assume familiarity with basic concepts and notions in graph theory.

Let  $n$  be a positive integer. The set  $\{1, \dots, n\}$  is denoted by  $[n]$ . Disjoint union of sets is denoted by  $\sqcup$ . We use  $\sqcup$  in place of  $\cup$  when we want to emphasize the disjointness of the sets involved in a union.

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For  $X$  a finite set and  $k$  a nonnegative integer, the set of all  $k$ -subsets of  $X$  is denoted  $\binom{X}{k}$ ; that is,  $\binom{X}{k} = \{K \subseteq X : |K| = k\}$ . A *hypergraph* is a pair  $\mathcal{H} = (X, \mathcal{A})$ , where  $X$  is a finite set and  $\mathcal{A} \subseteq 2^X$ . The elements of  $X$  are called *vertices* and the elements of  $\mathcal{A}$  are called *edges*. The *order* of  $\mathcal{H}$  is the number of vertices in  $X$ , and the *size* of  $\mathcal{H}$  is the number of edges in  $\mathcal{A}$ . If  $\mathcal{A} \subseteq \binom{X}{k}$ , then  $\mathcal{H}$  is said to be  *$k$ -uniform*. A 2-uniform hypergraph is just the usual notion of a *graph*. The *complete  $k$ -uniform hypergraph*  $(X, \binom{X}{k})$  of order  $n$  is denoted  $K_n^{(k)}$ . A hypergraph is *empty* if it has no edges. The degree of a vertex  $v$  is the number of edges containing  $v$ .

A *Steiner system*  $S(t, k, n)$  is a  $k$ -uniform hypergraph  $(X, \mathcal{A})$  such that every  $T \in \binom{X}{t}$  is contained in exactly one edge in  $\mathcal{A}$ .

A *group divisible design  $k$ -GDD* is a triple  $(X, \mathcal{G}, \mathcal{A})$ , where  $(X, \mathcal{A})$  is a  $k$ -uniform hypergraph,  $\mathcal{G} = \{G_1, \dots, G_t\}$  is a partition of  $X$  into parts  $G_i, i \in [t]$ , called *groups*, such that every  $T \in \binom{X}{2}$  not contained in a group is contained in exactly one edge in  $\mathcal{A}$ , and every  $T \in \binom{X}{2}$  contained in a group is not contained in any edge in  $\mathcal{A}$ . The *type* of a  $k$ -GDD  $(X, \mathcal{G}, \mathcal{A})$  is the multiset  $[|G_1|, \dots, |G_t|]$ . The exponential notation  $g_1^{t_1} \dots g_s^{t_s}$  is used to denote the multiset where element  $g_i$  has multiplicity  $t_i, i \in [s]$ .

We require the following result from Colbourn, Hoffman, and Rees [5] on the existence of 3-GDDs.

**THEOREM 2.1.** *Let  $g, t$ , and  $u$  be nonnegative integers. There exists a 3-GDD of type  $g^t u^1$  if and only if the following conditions are all satisfied:*

- (i) if  $g > 0$ , then  $t \geq 3$ , or  $t = 2$  and  $u = g$ , or  $t = 1$  and  $u = 0$ , or  $t = 0$ ;
- (ii)  $u \leq g(t - 1)$  or  $gt = 0$ ;
- (iii)  $g(t - 1) + u \equiv 0 \pmod{2}$  or  $gt = 0$ ;
- (iv)  $gt \equiv 0 \pmod{2}$  or  $u = 0$ ;
- (v)  $g^2 \binom{t}{2} + gtu \equiv 0 \pmod{3}$ .

**2.1. Graphs of hypergraphs.** Given a hypergraph  $\mathcal{H}$ , we may define the following graphs on  $\mathcal{H}$ .

**DEFINITION 2.2.** *Let  $\mathcal{H} = (X, \mathcal{A})$  be a hypergraph. The line graph of  $\mathcal{H}$  is the graph  $L(\mathcal{H}) = (V, \mathcal{E})$ , where  $V = \mathcal{A}$  and  $\mathcal{E} = \{\{A, B\} \subseteq \binom{V}{2} : A \cap B \neq \emptyset\}$ .*

**DEFINITION 2.3.** *Let  $\mathcal{H} = (X, \mathcal{A})$  be a hypergraph. The primal graph or 2-section of  $\mathcal{H}$  is the graph  $G(\mathcal{H}) = (X, \mathcal{E})$  such that  $\{x, y\} \in \mathcal{E}$  if and only if  $\{x, y\} \subset A$  for some  $A \in \mathcal{A}$ .*

A hypergraph  $\mathcal{H}$  is *conformal* if for every clique  $K$  in  $G(\mathcal{H})$ , there is an edge in  $\mathcal{H}$  that contains  $K$ . A hypergraph  $\mathcal{H}$  is *chordal* if  $G(\mathcal{H})$  is chordal, that is, every cycle of length at least four in  $G(\mathcal{H})$  contains two nonconsecutive vertices that are adjacent.

**2.2. Acyclic hypergraphs.** Graham [11], and independently, Yu and Ozsoyoglu [20], [21], defined an acyclicity property (which has come to be known as  $\alpha$ -acyclicity) for hypergraphs in the context of database theory, via a transformation now known as the *GYO reduction*. Given a hypergraph  $\mathcal{H} = (X, \mathcal{A})$ , the GYO reduction applies the following operations repeatedly to  $\mathcal{H}$  until none can be applied:

- (i) If a vertex  $x \in X$  has degree 1, then delete  $x$  from the edge containing it.
- (ii) If  $A, B \in \mathcal{A}$  are distinct edges such that  $A \subseteq B$ , then delete  $A$  from  $\mathcal{A}$ .
- (iii) If  $A \in \mathcal{A}$  is empty, that is,  $A = \emptyset$ , then delete  $A$  from  $\mathcal{A}$ .

**DEFINITION 2.4.** *A hypergraph  $\mathcal{H}$  is  $\alpha$ -acyclic if GYO reduction on  $\mathcal{H}$  results in an empty hypergraph.*

The notion of  $\alpha$ -acyclicity is closely related to conformality and chordality for hypergraphs. Beeri et al. [3] showed what follows.

**THEOREM 2.5.**  $\mathcal{H}$  is  $\alpha$ -acyclic if and only if  $\mathcal{H}$  is conformal and chordal.

Let  $\mathcal{H} = (X, \mathcal{A})$  be a hypergraph. Assign to every edge  $\{A, B\}$  of  $L(\mathcal{H})$  the weight  $|A \cap B|$ . We denote this weighted line graph of  $\mathcal{H}$  by  $L'(\mathcal{H})$ . The maximum weight of a forest in  $L'(\mathcal{H})$  is denoted  $w(\mathcal{H})$ . Acharya and Las Vergnas [1] introduced the hypergraph invariant

$$\mu(\mathcal{H}) = \sum_{A \in \mathcal{A}} |A| - \left| \bigcup_{A \in \mathcal{A}} A \right| - w(\mathcal{H}),$$

called the *cyclomatic number* of  $\mathcal{H}$ , and they used it to characterize conformal and chordal hypergraphs.

**THEOREM 2.6.** (Acharya and Las Vergnas [1]). *A hypergraph  $\mathcal{H}$  satisfies  $\mu(\mathcal{H}) = 0$  if and only if  $\mathcal{H}$  is conformal and chordal.*

Theorems 2.5 and 2.6 immediately imply the following.

**COROLLARY 2.7.** *A hypergraph  $\mathcal{H}$  is  $\alpha$ -acyclic if and only if  $\mu(\mathcal{H}) = 0$ .*

Li and Wang [15] were unaware of these connections and rediscovered Corollary 2.7 recently. An easy consequence is that a maximum  $\alpha$ -acyclic  $k$ -uniform hypergraph of order  $n$  has size  $n - k + 1$  [18]. Let  $L_{k-1}(\mathcal{H})$  denote the spanning subgraph of  $L'(\mathcal{H})$  containing only those edges of  $L'(\mathcal{H})$  of weight  $k - 1$ . We derive the following characterizations of maximum  $\alpha$ -acyclic  $k$ -uniform hypergraphs.

**COROLLARY 2.8.** *A  $k$ -uniform hypergraph  $\mathcal{H} = (X, \mathcal{A})$  of order  $n$  and size  $n - k + 1$  is  $\alpha$ -acyclic if and only if  $L(\mathcal{H})$  contains a spanning tree, each edge of which has weight  $k - 1$  (in other words,  $L_{k-1}(\mathcal{H})$  is connected).*

*Proof.* By Corollary 2.7, we have

$$\begin{aligned} w(\mathcal{H}) &= \sum_{A \in \mathcal{A}} |A| - \left| \bigcup_{A \in \mathcal{A}} A \right| \\ &= (n - k + 1)k - n \\ &= (n - k)(k - 1). \end{aligned}$$

Since every edge in  $L'(\mathcal{H})$  has weight at most  $k - 1$ , and a forest of  $L'(\mathcal{H})$  contains at most  $n - k$  edges (and contains exactly  $n - k$  edges if and only if the forest is a spanning tree), the corollary follows.  $\square$

An  $\alpha$ -acyclic decomposition of a hypergraph  $\mathcal{H} = (X, \mathcal{A})$  is a set of  $\alpha$ -acyclic hypergraphs  $\{(X, \mathcal{A}_i)\}_{i=1}^c$  such that  $\mathcal{A}_1, \dots, \mathcal{A}_c$  partition  $\mathcal{A}$ ; that is,  $\mathcal{A} = \sqcup_{i=1}^c \mathcal{A}_i$ . The size of the  $\alpha$ -acyclic decomposition is  $c$ .

**DEFINITION 2.9.** *The  $\alpha$ -arboricity of a hypergraph  $\mathcal{H}$ , denoted  $\alpha\text{arb}(\mathcal{H})$ , is the minimum size of an  $\alpha$ -acyclic decomposition of  $\mathcal{H}$ .*

**3. Previous work.** Trivially,  $\alpha\text{arb}(K_n^{(1)}) = \alpha\text{arb}(K_n^{(n)}) = 1$ , since both  $K_n^{(1)}$  and  $K_n^{(n)}$  are  $\alpha$ -acyclic. It is also known that  $\alpha\text{arb}(K_n^{(2)}) = \alpha\text{arb}(K_n^{(n-1)}) = \lceil n/2 \rceil$  (see, for example, [4]). Li [14] also showed that  $\alpha\text{arb}(K_n^{(n-2)}) = \lceil n(n-1)/6 \rceil$ . In general, Li [14] showed that

$$(3.1) \quad \left\lceil \frac{1}{k} \binom{n}{k-1} \right\rceil \leq \alpha \text{arb}(K_n^{(k)}) \leq \frac{1}{2} \binom{n+1}{k-1}.$$

The upper and lower bounds in (3.1) differ by approximately a factor of  $k/2$ . Wang [17] conjectured the lower bound to be the true value of  $\alpha \text{arb}(K_n^{(k)})$ .

CONJECTURE 3.1.  $\alpha \text{arb}(K_n^{(k)}) = \lceil \frac{1}{k} \binom{n}{k-1} \rceil$ .

Recently, Chee et al. [4] showed that Conjecture 3.1 holds when  $k = n - 3$ , so that Conjecture 3.1 is now known to hold for all  $n$  when  $k = 1, 2, n - 3, n - 2, n - 1, n$ . Chee et al. [4] also showed that Conjecture 3.1 holds whenever there exists a Steiner system  $S(n - k, n - k + 1, n)$  and that Conjecture 3.1 holds in an asymptotic sense when  $k$  is large enough. More precisely, the following was obtained.

THEOREM 3.2. (Chee et al. [4]). *Let  $\delta$  be a positive constant. Then for  $k = n - O(\log^{1-\delta} n)$ , we have*

$$\alpha \text{arb}(K_n^{(k)}) = (1 + o(1)) \frac{1}{k} \binom{n}{k-1},$$

where the  $o(1)$  is in  $n$ .

**4. Decompositions based on Steiner systems.** First, note that the cardinality of the Steiner system  $S(k - 1, k, n)$  is precisely  $\frac{1}{k} \binom{n}{k-1}$ , i.e., when such a system exists, the lower bound given by (3.1). Therefore, the idea of our construction consists in using the blocks of a  $S(k - 1, k, n)$  as *centers* of our partitions of  $K_n^{(k)}$  into  $\alpha$ -acyclic hypergraphs. Each of these hypergraphs is based on a *center*  $C$  (in this case a block from the Steiner system) to which are added  $n - 3$  edges, each of which intersect the *center* on  $k - 1$  vertices (we name these hypergraphs *star-shaped*). The reader may find it helpful to consult Figure 4.1, which illustrates the following proof for  $n = 7$  and  $k = 3$ , using the Steiner triple system  $S(2, 3, 7)$  ( $\mathbb{Z}_7, \mathcal{A}$ ) with  $\mathcal{A} = \{\{i, i + 1, i + 3\} : i \in \mathbb{Z}_7\}$ .

THEOREM 4.1. *If there exists an  $S(k - 1, k, n)$ , then  $\alpha \text{arb}(K_n^{(k)}) = \frac{1}{k} \binom{n}{k-1}$ .*

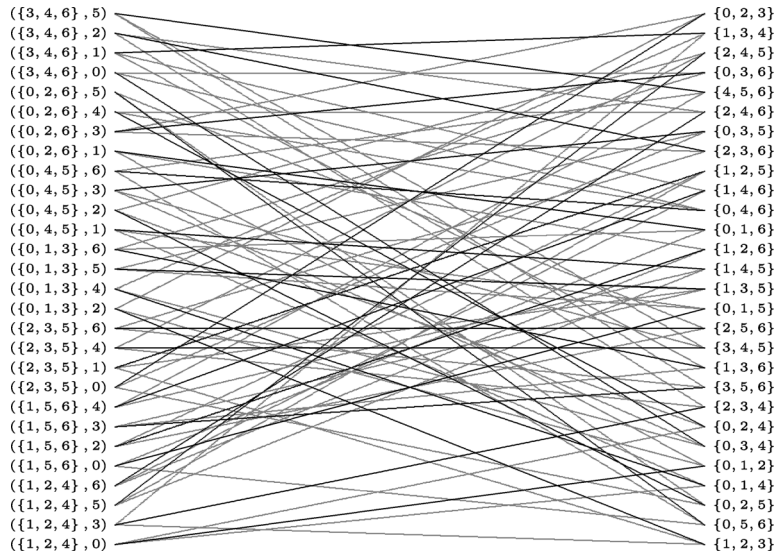


FIG. 4.1. Case  $n = 7, k = 3$ .

*Proof.* Let  $k$  and  $n$  be positive integers,  $2 \leq k \leq n$ . Let  $(X, \mathcal{A})$  be an  $S(k-1, k, n)$ . Define a bipartite graph  $G$  with bipartition  $V(G) = P \sqcup Q$ , where  $P = \{(A, x) : A \in \mathcal{A} \text{ and } x \in X \setminus A\}$  and  $Q = \binom{X}{k} \setminus \mathcal{A}$  so that vertex  $(A, x) \in P$  is adjacent to vertex  $K \in Q$  if and only if  $K \subset A \cup \{x\}$ . Thus, the neighborhood of vertex  $(A, x) \in P$  is the set  $N(A, x) = \{(A \cup \{x\}) \setminus \{u\} : u \in A\}$ , and the neighborhood of vertex  $K \in Q$  is the set  $N(K) = \{(A, x) : x \in K, A \in \mathcal{A} \text{ and } K \setminus \{x\} \subset A\}$ . Evidently,  $|N(A, x)| = k$  for all  $(A, x) \in P$ . To see that  $|N(K)| = k$  for all  $K \in Q$ , note that each of the  $k(k-1)$ -subsets of  $K$  is contained in exactly one  $A \in \mathcal{A}$ , since  $(X, \mathcal{A})$  is an  $S(k-1, k, n)$ . It follows that  $|N(A, x)| = |N(K)| = k$  and  $G$  is  $k$ -regular. Hence,  $G$  has a perfect matching  $M$ .

Now, for each  $A \in \mathcal{A}$ , let us define the  $k$ -uniform hypergraph  $\mathcal{H}_A = (X, \mathcal{B}_A)$ , where  $\mathcal{B}_A = \{A\} \cup \{K \in Q : \{(A, x), K\} \in M \text{ for some } x \in X \setminus A\}$ . It is easy to check that  $\binom{X}{k} = \sqcup_{A \in \mathcal{A}} \mathcal{B}_A$ . We claim that, in fact, the set of hypergraphs  $\{\mathcal{H}_A\}_{A \in \mathcal{A}}$  is an  $\alpha$ -acyclic decomposition of  $(X, \binom{X}{k})$ . To see this, note that  $\mathcal{H}_A$  has order  $n$  and size  $n - k + 1$ , and observe that each edge in  $\mathcal{B}_A \setminus \{A\}$  intersects  $A$  in exactly  $k - 1$  vertices. Hence,  $L_{k-1}(\mathcal{H}_A)$  is connected. It follows from Corollary 2.8 that  $\mathcal{H}_A$  is  $\alpha$ -acyclic. The size of the  $\alpha$ -acyclic decomposition  $\{\mathcal{H}_A\}_{A \in \mathcal{A}}$  is the size of an  $S(k-1, k, n)$ , which is precisely  $\frac{1}{k} \binom{n}{k-1}$ .  $\square$

**COROLLARY 4.2.** *We have  $\alpha \text{arb}(K_n^{(k)}) = \frac{1}{k} \binom{n}{k-1}$  whenever any one of the following conditions holds:*

- (i)  $k = 2$  and  $n \equiv 0 \pmod{2}$ , or
- (ii)  $k = 3$  and  $n \equiv 1, 3 \pmod{6}$ , or
- (iii)  $k = 4$  and  $n \equiv 2, 4 \pmod{6}$ , or
- (iv)  $k = 5$  and  $n \in \{11, 23, 35, 47, 71, 83, 107, 131\}$ , or
- (v)  $k = 6$  and  $n \in \{12, 24, 36, 48, 72, 84, 108, 132\}$ .

*Proof.* For (i), note that an  $S(1, 2, n)$  is a perfect matching in the complete graph  $K_n$ , and hence exists if and only if  $n$  is even. For (ii), an  $S(2, 3, n)$  is a Steiner triple system and exists if and only if  $n \equiv 1$  or  $3 \pmod{6}$  (see, for example, [7]). For (iii), an  $S(3, 4, n)$  is a Steiner quadruple system, existence for which was settled by Hanani [13], who showed that  $n \equiv 2$  or  $4 \pmod{6}$  is necessary and sufficient. For (iv)–(v), see [12], [6] for existence results.  $\square$

**5.  $\alpha$ -arboricity of  $K_n^{(3)}$ .** We determine  $\alpha \text{arb}(K_n^{(3)})$  completely in this section. Corollary 4.2 determined  $\alpha \text{arb}(K_n^{(3)})$  for all  $n \equiv 1, 3 \pmod{6}$ , so we focus on the remaining cases of  $n \equiv 0, 2, 4, 5 \pmod{6}$  here.

**5.1. The case  $n \equiv 0, 4 \pmod{6}$ .** In this subsection,  $n \equiv 0, 4 \pmod{6}$ ,  $n \geq 4$ .

Let  $X = Y \sqcup Z$ , where  $|Y| = n - 3$  and  $Z = \{\infty_1, \infty_2, \infty_3\}$ . Let  $(Y, \mathcal{A})$  be an  $S(2, 3, n - 3)$ .

Our proof here is similar to the one given previously. Our classes, however, are now of two different kinds: not only do we need our former *star-shaped* hypergraphs whose *centers* belong to a Steiner triple system on  $Y$ , but also classes whose *centers* are two triples  $\{y, \infty_1, \infty_2\}$  and  $\{y, \infty_1, \infty_3\}$  (intersecting on  $y, \infty_1$ ) for all  $y \in Y$ . As previously, any edge of our  $\alpha$ -acyclic hypergraphs intersects at least one edge of its center on exactly two vertices. The decomposition is completed by another *star-shaped* class containing the triples  $\{y, \infty_2, \infty_3\}$ , where  $y \in X \setminus \{\infty_2, \infty_3\}$ .

We define the bipartite graph  $\Gamma$  with bipartition  $V(\Gamma) = P \sqcup Q$ , where

$$P = \left( \bigcup_{A \in \mathcal{A}} \{(A, x) : x \in X \setminus A\} \right) \cup \left( \bigcup_{y \in Y} \{(\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}, z) : z \in Y \setminus \{y\}\} \right),$$

$$Q = \binom{X}{3} \setminus (\mathcal{A} \cup \{\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}, \{y, \infty_2, \infty_3\} : y \in Y\} \cup \{Z\})$$

with adjacency of vertices in  $\Gamma$  as follows:

- (i) Vertex  $(\{a, b, c\}, x) \in P$  is adjacent to vertices  $\{a, b, x\}, \{a, c, x\}, \{b, c, x\} \in Q$ .
- (ii) Vertex  $(\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}, z) \in P$  is adjacent to vertices  $\{y, z, \infty_h\} \in Q, h \in [3]$ .

Every vertex in  $P$  being of degree 3, let us prove the same holds for the vertices of  $Q$ . For any pair of vertices  $u, v \in Y$ , we name  $A_{uv}$  the unique triple of  $\mathcal{A}$  containing both  $u$  and  $v$ .

- (i)  $\{a, b, c\} \subseteq Y$  is adjacent to  $(A_{ab}, c), (A_{bc}, a),$  and  $(A_{ac}, b)$ .
- (ii)  $\{a, b, \infty_h\} \in Q$  is adjacent to  $(A_{ab}, \infty_h), (\{b, \infty_1, \infty_2\}, \{b, \infty_1, \infty_3\}, a),$  and  $(\{a, \infty_1, \infty_2\}, \{a, \infty_1, \infty_3\}, b)$ .

Hence,  $\Gamma$  is 3-regular and consequently has a perfect matching  $M$ .

For each  $A \in \mathcal{A}$ , let us define the 3-uniform hypergraph  $\mathcal{H}_A = (X, \mathcal{B}_A)$ , where

$$\mathcal{B}_A = \{A\} \cup \{T \in Q : \{(A, x), T\} \in M \text{ for some } x \in X \setminus A\}.$$

Then  $\mathcal{H}_A$  is of order  $n$  and size  $n - 2$ . Each edge in  $\mathcal{B}_A \setminus \{A\}$  intersects  $A$  in exactly two vertices. Hence,  $L_2(\mathcal{H}_A)$  is connected. It follows from Corollary 2.8 that  $\mathcal{H}_A$  is  $\alpha$ -acyclic.

In addition, for each  $y \in Y$ , define the 3-uniform hypergraph  $\mathcal{H}_y = (X, \mathcal{B}_y)$ , where

$$\mathcal{B}_y = \{\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}\} \cup \{T \in Q : (\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}, z), T\} \in M \text{ for some } z \in Y \setminus \{y\}\}.$$

Then  $\mathcal{H}_y$  is of order  $n$  and size  $n - 2$ . In  $L_2(\mathcal{H}_y)$ , the vertex  $\{y, \infty_1, \infty_2\}$  is adjacent to  $\{y, \infty_1, \infty_3\}$ , and each vertex in  $\mathcal{B}_y \setminus \{\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}\}$  is adjacent to one of the vertices  $\{y, \infty_1, \infty_2\}$  or  $\{y, \infty_1, \infty_3\}$ . Hence,  $L_2(\mathcal{H}_y)$  is connected. It follows from Corollary 2.8 that  $\mathcal{H}_y$  is  $\alpha$ -acyclic.

Finally, define the 3-uniform hypergraph  $\mathcal{H} = (X, \mathcal{B})$ , where  $\mathcal{B} = \{\{y, \infty_2, \infty_3\} : y \in X \setminus \{\infty_2, \infty_3\}\}$ . Note that  $\mathcal{H}$  is  $\alpha$ -acyclic, since it GYO-reduces to an empty hypergraph.

Now, we have

$$\binom{X}{3} = \left( \bigsqcup_{A \in \mathcal{A}} \mathcal{B}_A \right) \bigsqcup \left( \bigsqcup_{y \in Y} \mathcal{B}_y \right) \bigsqcup \mathcal{B}$$

so that  $\{\mathcal{H}_A\}_{A \in \mathcal{A}} \cup \{\mathcal{H}_y\}_{y \in Y} \cup \{\mathcal{H}\}$  is an  $\alpha$ -acyclic decomposition of  $K_n^{(3)}$ . The size of this decomposition is

$$\frac{(n-3)(n-4)}{6} + (n-3) + 1 = \frac{n(n-1)}{6},$$

which matches the lower bound in (3.1). This gives the following result.

PROPOSITION 5.1.  $\alpha_{arb}(K_n^{(3)}) = n(n-1)/6$  for all  $n \equiv 0, 4 \pmod{6}$ .

**5.2. The case  $n \equiv 5 \pmod{6}$ .** In this subsection,  $n \equiv 5 \pmod{6}$ ,  $n \geq 5$ . Write  $n = 6k + 5$ . Let  $X = Y \sqcup \{\infty_1, \infty_2\}$ , where  $|Y| = 6k + 3$ , and let  $(Y, \mathcal{G}, \mathcal{A})$  be a 3-GDD of type  $3^{2k+1}$ , which exists by Theorem 2.1. Our construction is still based on *star-shaped* hypergraphs *centered* on the triples of the 3-GDD, but this time we will need to define *centers* consisting of three triples, pairwise intersecting on two elements. Also, for numerical reasons,  $2k + 1$  of our classes are of order only  $n - 2$  and size  $n - 4$ .

Suppose  $\mathcal{G} = \{G_1, \dots, G_{2k+1}\}$ , where  $G_i = \{g_{i,1}, g_{i,2}, g_{i,3}\}$ ,  $i \in [2k + 1]$ . To keep our expressions succinct, we let

$$T_{i,j,j'}^h = \{g_{i,j}, g_{i,j'}, \infty_h\}$$

for  $i \in [2k + 1]$ ,  $1 \leq j < j' \leq 3$  and  $h \in [2]$  and

$$G_{i,j} = \{g_{i,j}, \infty_1, \infty_2\}$$

for  $i \in [2k + 1]$  and  $j \in [3]$ .

Define the bipartite graph  $\Gamma$  with bipartition  $V(\Gamma) = P \sqcup Q$ , where

$$\begin{aligned} P &= \left( \bigcup_{A \in \mathcal{A}} \{(A, x) : x \in X \setminus A\} \right) \cup \left( \bigcup_{G \in \mathcal{G}} \{(G, x) : x \in Y \setminus G\} \right) \\ &\quad \cup \left( \bigcup_{i=1}^{2k+1} \{(T_{i,1,2}^1, T_{i,1,3}^1, G_{i,1}, x) : x \in Y \setminus G_i\} \right) \\ &\quad \cup \left( \bigcup_{i=1}^{2k+1} \{(T_{i,1,2}^2, T_{i,2,3}^2, G_{i,2}, x) : x \in Y \setminus G_i\} \right), \\ Q &= \binom{X}{3} \setminus \left( \mathcal{A} \cup \mathcal{G} \cup \bigcup_{\substack{i,h \\ j < j'}} \{T_{i,j,j'}^h, T_{i,j,j'}^h, T_{i,j,j'}^h\} \cup \bigcup_{i,j} G_{i,j} \right) \end{aligned}$$

with adjacency of vertices in  $\Gamma$  as follows:

- (i) Vertex  $(\{a, b, c\}, x) \in P$  is adjacent to vertices  $\{a, b, x\}, \{a, c, x\}, \{b, c, x\} \in Q$ .
- (ii) Vertex  $(T_{i,1,2}^1, T_{i,1,3}^1, G_{i,1}, x) \in P$  is adjacent to vertices  $\{g_{i,\ell}, \infty_1, x\} \in Q$ ,  $\ell \in [3]$ .
- (iii) Vertex  $(T_{i,1,2}^2, T_{i,2,3}^2, G_{i,2}, x) \in P$  is adjacent to vertices  $\{g_{i,\ell}, \infty_2, x\} \in Q$ ,  $\ell \in [3]$ .

Every vertex in  $P$  being of degree 3, let us prove the same holds for the vertices of  $Q$ . For all  $u, v \in Y$ , we name  $A_{uv}$  the unique triple of  $\mathcal{A} \cup \mathcal{G}$  containing both  $u$  and  $v$ .

- (i)  $\{a, b, c\} \subseteq Y$  is adjacent to  $(A_{ab}, c)$ ,  $(A_{bc}, a)$ , and  $(A_{ac}, b)$ .
- (ii)  $\{a, b, \infty_1\} \in Q$ , where  $a \in G_i$  and  $b \in G_{i'}$  with  $i \neq i'$ , is adjacent to  $(A_{ab}, \infty_1)$ ,  $(T_{i,1,2}^1, T_{i,1,2}^1, G_{i,1}, b)$ , and  $(T_{i',1,2}^1, T_{i',1,3}^1, G_{i',1}, a)$ .
- (iii)  $\{a, b, \infty_2\} \in Q$ , where  $a \in G_i$  and  $b \in G_{i'}$  with  $i \neq i'$ , is adjacent to  $(A_{ab}, \infty_2)$ ,  $(T_{i,1,2}^2, T_{i,2,3}^2, G_{i,2}, b)$ , and  $(T_{i',1,2}^2, T_{i',2,3}^2, G_{i',2}, a)$ .

Hence,  $\Gamma$  is 3-regular and consequently has a perfect matching  $M$ .

For each  $A \in \mathcal{A}$ , let us define the 3-uniform hypergraph  $\mathcal{H}_A = (X, \mathcal{B}_A)$ , where

$$\mathcal{B}_A = \{A\} \cup \{T \in Q : \{(A, x), T\} \in M \text{ for some } x \in X \setminus A\}.$$

Then  $\mathcal{H}_A$  is of order  $n$  and size  $n - 2$ . Each edge in  $\mathcal{B}_A \setminus \{A\}$  intersects  $A$  in exactly two vertices. Hence,  $L_2(\mathcal{H}_A)$  is connected. It follows from Corollary 2.8 that  $\mathcal{H}_A$  is  $\alpha$ -acyclic.

In addition, for each  $G \in \mathcal{G}$ , define the 3-uniform hypergraph  $\mathcal{H}_G = (Y, \mathcal{B}_G)$ , where  $\mathcal{B}_G = \{G\} \cup \{T \in Q: \{(G, x), T\} \in M \text{ for some } x \in Y \setminus G\}$ . Then  $\mathcal{H}_G$  is of order  $n - 2$  and size  $n - 4$ . By the same reason as for  $\mathcal{H}_A$ ,  $\mathcal{H}_G$  is  $\alpha$ -acyclic.

Furthermore, for each  $i \in [2k + 1]$ , define the 3-uniform hypergraphs  $\mathcal{H}_i = (X, \mathcal{B}_i)$  and  $\mathcal{H}'_i = (X, \mathcal{B}'_i)$ , where

$$\begin{aligned} \mathcal{B}_i &= \{T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}\} \cup \{T \in Q: \{(T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}, x), T\} \in M \text{ for some} \\ &\quad x \in Y \setminus G_i\}, \\ \mathcal{B}'_i &= \{T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}\} \cup \{T \in Q: \{(T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}, x), T\} \in M \text{ for some} \\ &\quad x \in Y \setminus G_i\}. \end{aligned}$$

Then  $\mathcal{H}_i$  and  $\mathcal{H}'_i$  are each of order  $n$  and size  $n - 2$ . In  $L_2(\mathcal{H}_i)$  (respectively,  $L_2(\mathcal{H}'_i)$ ), the vertex  $T^1_{i,1,2}$  (respectively,  $T^2_{i,1,2}$ ) is adjacent to vertices  $T^1_{i,1,3}$  and  $G_{i,1}$  (respectively,  $T^2_{i,2,3}$  and  $G_{i,2}$ ), and each vertex in  $\mathcal{B}_i \setminus \{T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}\}$  (respectively,  $\mathcal{B}'_i \setminus \{T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}\}$ ) is adjacent to at least one of the vertices  $T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}$  (respectively,  $T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}$ ). Hence,  $L_2(\mathcal{H}_i)$  (respectively,  $L_2(\mathcal{H}'_i)$ ) is connected. It follows from Corollary 2.8 that  $\mathcal{H}_i$  (respectively,  $\mathcal{H}'_i$ ) is  $\alpha$ -acyclic.

Finally, define the 3-uniform hypergraph  $\mathcal{H} = (X, \mathcal{B})$ , where

$$\mathcal{B} = \bigcup_{i=1}^{2k+1} \{T^2_{i,1,3}, T^1_{i,2,3}, G_{i,3}\}.$$

It is easy to see that  $\mathcal{H}$  is  $\alpha$ -acyclic.

Now, we have

$$\binom{X}{3} = \left(\bigsqcup_{A \in \mathcal{A}} \mathcal{B}_A\right) \sqcup \left(\bigsqcup_{G \in \mathcal{G}} \mathcal{B}_G\right) \sqcup \left(\bigsqcup_{i=1}^{2k+1} \mathcal{B}_i\right) \sqcup \left(\bigsqcup_{i=1}^{2k+1} \mathcal{B}'_i\right) \sqcup \mathcal{B}$$

so that  $\{\mathcal{H}_A\}_{A \in \mathcal{A}} \cup \{\mathcal{H}_G\}_{G \in \mathcal{G}} \cup \{\mathcal{H}_i\}_{i \in [2k+1]} \cup \{\mathcal{H}'_i\}_{i \in [2k+1]} \cup \{\mathcal{H}\}$  is an  $\alpha$ -acyclic decomposition of  $K_n^{(3)}$ . The size of this decomposition is

$$3 \binom{2k+1}{2} + (2k+1) + (2k+1) + (2k+1) + 1 = 6k^2 + 9k + 4 = \left\lceil \frac{n(n-1)}{6} \right\rceil,$$

which matches the lower bound in (3.1). This gives the following result.

PROPOSITION 5.2.  $\alpha \text{arb}(K_n^{(3)}) = \lceil n(n-1)/6 \rceil$  for all  $n \equiv 5 \pmod{6}$ .

**5.3. The case  $n \equiv 2 \pmod{6}$ .** We treat the remaining case of  $n \equiv 2 \pmod{6}$ .

LEMMA 5.3.  $\alpha \text{arb}(K_8^{(3)}) = 10$ .

*Proof.* The lower bound in (3.1) showed that  $\alpha \text{arb}(K_8^{(3)}) \geq 10$ . We construct an  $\alpha$ -acyclic decomposition meeting this lower bound.

Consider the  $S(2, 3, 7)$   $(\mathbb{Z}_7, \mathcal{A})$ , with  $\mathcal{A} = \{\{i, i + 1, i + 3\} : i \in \mathbb{Z}_7\}$ . Let  $\{\mathcal{H}_A\}_{A \in \mathcal{A}}$  be the  $\alpha$ -acyclic decomposition of  $(\mathbb{Z}_7, \binom{\mathbb{Z}_7}{3})$  produced by the construction of section 4. We use this to construct an  $\alpha$ -acyclic decomposition of  $K_8^{(3)}$  as follows. Let  $X = \mathbb{Z}_7 \sqcup \{\infty\}$ , and let



$$\begin{aligned} \mathcal{B}_1 &= \{\{i, i + 1, \infty\} : i \in \mathbb{Z}_7 \setminus \{0\}\}, \\ \mathcal{B}_2 &= \{\{i, i + 3, \infty\} : i \in \mathbb{Z}_7 \setminus \{1\}\}, \\ \mathcal{B}_3 &= \{\{i + 1, i + 3, \infty\} : i \in \mathbb{Z}_7 \setminus \{2\}\}, \\ \mathcal{B}_4 &= E(\mathcal{H}_{\{0,1,3\}}) \cup \{\{0, 1, \infty\}\}, \\ \mathcal{B}_5 &= E(\mathcal{H}_{\{1,2,4\}}) \cup \{\{1, 4, \infty\}\}, \\ \mathcal{B}_6 &= E(\mathcal{H}_{\{2,3,5\}}) \cup \{\{3, 5, \infty\}\}. \end{aligned}$$

Then  $\{(X, \mathcal{B}_i)\}_{i \in [6]} \cup \{\mathcal{H}_{\{3,4,6\}}, \mathcal{H}_{\{0,4,5\}}, \mathcal{H}_{\{1,5,6\}}, \mathcal{H}_{\{0,2,6\}}\}$  is an  $\alpha$ -acyclic decomposition of  $(X, \binom{X}{3})$  of size 10.  $\square$

Henceforth, in what follows, let  $n \equiv 2 \pmod{6}$ ,  $n \geq 14$ . Write  $n = 6k + 2$ . Let  $X = Y \sqcup \{\infty\}$ , where  $|Y| = 6k + 1$ , and let  $(Y, \mathcal{G}, \mathcal{A})$  be a 3-GDD of type  $3^{2k}1^1$ , which exists by Theorem 2.1. Here again, we use *star-shaped* hypergraphs *centered* on the triples of  $\mathcal{A}$ , but also classes whose *centers* consist of two triples intersecting in two elements. They will be completed with a last *star-shaped* class of order  $2k + 2$  and size  $2k$  (in order to reach the bound).

Suppose  $\mathcal{G} = \{G_1, \dots, G_{2k}, \{g\}\}$ , where  $G_i = \{g_{i,1}, g_{i,2}, g_{i,3}\}$ ,  $i \in [2k]$ . To keep our expressions succinct, we let

$$\begin{aligned} G'_i &= \{g_{i,1}, g_{i,2}, \infty\}, \\ G''_i &= \{g_{i,1}, g_{i,3}, \infty\}, \\ G'''_i &= \{g_{i,2}, g_{i,3}, \infty\} \end{aligned}$$

for  $i \in [2k]$ . Define the bipartite graph  $\Gamma$  with bipartition  $V(\Gamma) = P \sqcup Q$ , where

$$\begin{aligned} P &= \left( \bigcup_{A \in \mathcal{A}} \{(A, x) : x \in X \setminus A\} \right) \cup \left( \bigcup_{i=1}^{2k} \{(G_i, G''_i, x) : x \in Y \setminus G_i\} \right) \\ &\quad \cup \left( \bigcup_{i=1}^{2k} \{(G'_i, G'''_i, x) : x \in Y \setminus G_i\} \right) \cup \{G_i : i \in [2k]\}, \\ Q &= \binom{X}{3} \setminus (\mathcal{A} \cup \mathcal{G} \cup \{G'_i, G''_i, G'''_i : i \in [2k]\}) \end{aligned}$$

with adjacency of vertices in  $\Gamma$  as follows:

- (i) Vertex  $(\{a, b, c\}, x) \in P$  is adjacent to vertices  $\{a, b, x\}, \{a, c, x\}, \{b, c, x\} \in Q$ .
- (ii) Vertex  $(G_i, G''_i, x) \in P$  is adjacent to vertices  $\{g_{i,1}, g_{i,2}, x\}, \{g_{i,1}, g_{i,3}, x\}, \{g_{i,2}, g_{i,3}, x\} \in Q$ .
- (iii) Vertex  $(G'_i, G'''_i, x) \in P$  is adjacent to vertices  $\{g_{i,1}, \infty, x\}, \{g_{i,2}, \infty, x\}, \{g_{i,3}, \infty, x\} \in Q$ .
- (iv) Vertex  $G_i \in P$  is adjacent to vertices  $\{g_{i,j}, g, \infty\} \in Q$ ,  $j \in [3]$ .

Every vertex in  $P$  being of degree 3, let us prove the same holds for the vertices of  $Q$ . For all  $u, v \in Y$ , we name  $A_{uv}$  the unique triple of  $\mathcal{A}$  containing both  $u$  and  $v$ .

- (i)  $\{a, b, c\} \subseteq Y$ , where  $a, b$ , and  $c$  belong to three different groups, is adjacent to  $(A_{ab}, c), (A_{ac}, b)$ , and  $(A_{bc}, a)$ .

- (ii)  $\{a, b, c\} \subseteq Y$ , where  $a$  and  $b$  belong to the same group  $G_i$  and  $c \notin G_i$ , is adjacent to  $(A_{ac}, b)$ ,  $(A_{bc}, a)$ , and  $(G_i, G''_i, c)$ .
- (iii)  $\{g_{i,j}, g'_{i',j'}, \infty\} \in Q$  (hence  $i \neq i'$ ) is adjacent to  $(A_{g_{i,j}g'_{i',j'}}, \infty)$ ,  $(G'_i, G'''_i, g'_{i',j'})$ , and  $(G'_i, G'''_i, g_{i,j})$ .
- (iv)  $\{g_{i,j}, g, \infty\}$  is adjacent to  $(A_{g_{i,j}g}, \infty)$ ,  $G_i$ , and  $(G'_i, G'''_i, g)$ .

Hence,  $\Gamma$  is 3-regular and consequently has a perfect matching  $M$ .

For each  $A \in \mathcal{A}$ , let us define the 3-uniform hypergraph  $\mathcal{H}_A = (X, \mathcal{B}_A)$ , where

$$\mathcal{B}_A = \{A\} \cup \{T \in Q : \{(A, x), T\} \in M \text{ for some } x \in X \setminus A\}.$$

Then  $\mathcal{H}_A$  is of order  $n$  and size  $n - 2$ . Each edge in  $\mathcal{B}_A \setminus \{A\}$  intersects  $A$  in exactly two vertices. Hence,  $L_2(\mathcal{H}_A)$  is connected. It follows from Corollary 2.8 that  $\mathcal{H}_A$  is  $\alpha$ -acyclic.

In addition, for each  $i \in [2k]$ , define the 3-uniform hypergraphs  $\mathcal{H}_i = (X, \mathcal{B}_i)$  and  $\mathcal{H}'_i = (X, \mathcal{B}'_i)$ , where

$$\mathcal{B}_i = \{G_i, G''_i\} \cup \{T \in Q : \{(G_i, G''_i, x), T\} \in M \text{ for some } x \in Y \setminus G_i\},$$

$$\mathcal{B}'_i = \{G'_i, G'''_i\} \cup \{T \in Q : \{(G'_i, G'''_i, x), T\} \in M \text{ for some } x \in Y \setminus G_i\}.$$

Then  $\mathcal{H}_i$  and  $\mathcal{H}'_i$  are each of order  $n$  and size  $n - 2$ . By the same reason as for  $\mathcal{H}_A$ ,  $\mathcal{H}_i$  and  $\mathcal{H}'_i$  are  $\alpha$ -acyclic.

Finally, define the 3-uniform hypergraph  $\mathcal{H} = (X, \mathcal{B})$ , where

$$\mathcal{B} = \bigcup_{i=1}^{2k} \{T \in Q : \{G_i, T\} \in M\}.$$

It is easy to see that  $\mathcal{H}$  is  $\alpha$ -acyclic and has order  $2k + 2$  and size  $2k$ .

Now, we have

$$\binom{X}{3} = \left( \bigsqcup_{A \in \mathcal{A}} \mathcal{B}_A \right) \sqcup \left( \bigsqcup_{i=1}^{2k} \mathcal{B}_i \right) \sqcup \left( \bigsqcup_{i=1}^{2k} \mathcal{B}'_i \right) \sqcup \mathcal{B}$$

so that  $\{\mathcal{H}_A\}_{A \in \mathcal{A}} \cup \{\mathcal{H}_i\}_{i \in [2k]} \cup \{\mathcal{H}'_i\}_{i \in [2k]} \cup \{\mathcal{H}\}$  is an  $\alpha$ -acyclic decomposition of  $K_n^{(3)}$ . The size of this decomposition is

$$\left( 3 \binom{2k}{2} + 2k \right) + 2k + 2k + 1 = 6k^2 + 3k + 1 = \left\lceil \frac{n(n-1)}{6} \right\rceil,$$

which matches the lower bound in (3.1). Together with Lemma 5.3, this gives the following result.

PROPOSITION 5.4.  $\alpha \text{arb}(K_n^{(3)}) = \lceil n(n-1)/6 \rceil$  for all  $n \equiv 2 \pmod{6}$ ,  $n \geq 8$ .

**5.4. Summary.** Corollary 4.2 (i) and Propositions 5.1, 5.2, and 5.4 combine to give the following theorem.

THEOREM 5.5.  $\alpha \text{arb}(K_n^{(3)}) = \lceil n(n-1)/6 \rceil$  for all  $n \geq 3$ .

**6. Conclusion.** The problem of determining the  $\alpha$ -arboricity of hypergraphs is a problem motivated by database theory. In this paper, we continue the study of the  $\alpha$ -arboricity of complete uniform hypergraphs. We give a general construction based on Steiner systems and determine completely the value of  $\alpha \text{arb}(K_n^{(3)})$ . Previously,  $\alpha \text{arb}(K_n^{(k)})$  was only known for  $k = 1, 2, n - 3, n - 2, n - 1, n$ .

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