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Hanani triple packings and optimal q-ary codes of constant weight three

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Abstract The exact sizes of optimal q-ary codes of length n, constant weight w and distance $d = 2w - 1$ have only been determined for $q \in \{2, 3\}$, and for $w|(q-1)n$ and n sufficiently large. We completely determine the exact size of optimal q-ary codes of constant weight three and minimum distance five for all q by establishing a connection with Hanani triple packings, and settling their existence.

Keywords Constant-weight codes · Hanani triple packings · Hanani triple systems · Resolvable designs

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1 Introduction

Constant-weight codes (CWCs) play an important role in coding theory (see [26, Chap. 17] for example). While a vast amount of knowledge exists for binary CWCs [1,4,26], the study of q-ary constant-weight codes with $q > 2$ has intensified only recently [2,3,5–10,13–18,20,22–25,27–29,31,32,35–42], due to several important applications requiring nonbinary alphabets, such as power line communications, bandwidth-efficient channels, and DNA computing.

Let $A_q(n, d, w)$ denote the maximum size of a q-ary code of length n, minimum (Hamming) distance d and constant weight w. Such a code is said to be *optimal* if it achieves this size. Most work on the determination of $A_q(n, d, w)$ focused on some specified small q, usually for $q \leq 4$, for fixed d and w. The only known nontrivial values $A_q(n, d, w)$ determined completely for all n and $q \geq 2$ are $(d, w) = (3, 2)$ and $(4, 3)$ [6,8]. Our concern in this paper is on $A_q(n, 2w - 1, w)$, and more specifically $A_q(n, 5, 3)$. The following results summarise the present state of knowledge concerning $A_q(n, 2w - 1, w)$.

Theorem 1.1 (Östergård and Svanström [27])

$$A_3(n, 2w - 1, w) = \max\left\{M : n \geq \left\lceil \frac{Mw}{2} \right\rceil + \max\left\{\left\lfloor \frac{Mw}{2} \right\rfloor - \binom{M}{2}, 0\right\}\right\}.$$

Theorem 1.2 (Chee and Ling [8]) $A_q(n, 3, 2) = \min\left\{\left\lfloor \frac{(q-1)n}{2} \right\rfloor, \binom{n}{2}\right\}$ *for all n and $q \geq 2$.*

Theorem 1.3 (Chee et al. [7]) $A_q(n, 2w - 1, w) = \frac{(q-1)n}{w}$ *if either*

(i) $w | (q - 1)n$ *and* $n \geq 2w(w(q - 1) - 1)^2 + 1$*, or*
(ii) $w | n$ *and* $n \geq w((w - 1)(q - 2) + 1)$.

In particular, when $w \geq 3$ and $q \geq 4$, the value of $A_q(n, 2w - 1, w)$ is only known when n is large enough (Theorem 1.3).

Our contribution in this paper is the complete determination of $A_q(n, 5, 3)$. The solution is constructive and is based on the theory of combinatorial designs. In particular, we generalize the concept of Hanani triple systems to Hanani triple packings and strong Hanani triple packings. These designs are shown to have intimate relationships to q-ary codes of constant weight three and distance five. We settle the existence problem completely for Hanani triple packings and with a small number of possible exceptions for strong Hanani triple packings. An application of these results gives:

Main Theorem $A_q(n, 5, 3) = \min\left\{\left\lfloor \frac{(q-1)n}{3} \right\rfloor, D(n, 3)\right\}$, *where*

$$D(n, 3) = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1, & if\ n \equiv 5 \bmod 6; \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor, & otherwise. \end{cases}$$

2 Preliminaries

2.1 q-Ary constant-weight codes

Let n be a positive integer. The set $\{1, 2, \ldots, n\}$ is denoted by I_n, and the ring $\mathbb{Z}/n\mathbb{Z}$ is denoted by $\mathbb{Z}_n$. For finite sets R and X, R^X denotes the set of vectors of length $|X|$, where each component of a vector $\mathsf{u} \in R^X$ has value in R and is indexed by an element of X, that is, $\mathsf{u} = (\mathsf{u}_x)_{x \in X}$, and $\mathsf{u}_x \in R$ for each $x \in X$.

A *q-ary code of length n* is a set $\mathcal{C} \subseteq \mathbb{Z}_q^X$, for some X of size n. The elements of $\mathcal{C}$ are called *codewords*. The *(Hamming) weight* of a vector $\mathsf{u} \in \mathbb{Z}_q^X$ is defined as $\|\mathsf{u}\| = |\{x \in X : \mathsf{u}_x \neq 0\}|$. The metric induced by this weight is the *(Hamming) distance*, d_H, so that $d_H(\mathsf{u}, \mathsf{v}) = \|\mathsf{u} - \mathsf{v}\|$, for $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_q^X$. The *support* of a vector $\mathsf{u} \in \mathbb{Z}_q^X$ is $\text{supp}(\mathsf{u}) = \{x \in X : \mathsf{u}_x \neq 0\}$.

A code $\mathcal{C}$ is said to have *distance d* if $d_H(\mathsf{u}, \mathsf{v}) \geq d$ for all distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$. If $\|\mathsf{u}\| = w$ for every $\mathsf{u} \in \mathcal{C}$, then $\mathcal{C}$ is said to be of *constant weight w*. A q-ary code of length n, distance d, and constant weight w is denoted an $(n, d, w)_q$-code. The number of codewords in an $(n, d, w)_q$-code is called its *size*. The maximum size of an $(n, d, w)_q$-code is denoted $A_q(n, d, w)$, and an $(n, d, w)_q$-code achieving this size is said to be *optimal*.

The following Johnson-type bound for q-ary CWCs was established by Svanström [32].

Proposition 2.1 (Johnson Bound)

$$A_q(n, d, w) \leq \left\lfloor \frac{n(q-1)}{w} A_q(n-1, d, w-1) \right\rfloor.$$

The Johnson bound implies the following upper bound.

Corollary 2.2 $A_q(n, 2w-1, w) \leq \left\lfloor \frac{n(q-1)}{w} \right\rfloor$.

In particular, we have $A_q(n, 5, 3) \leq \left\lfloor \frac{(q-1)n}{3} \right\rfloor$.

2.2 Designs

A *set system* is a pair $(X, \mathcal{B})$ such that X is a finite set of *points* and $\mathcal{B}$ is a set of subsets of X, called *blocks*. The *order* of the set system is $|X|$, the number of points. For a nonnegative integer k, a set system $(X, \mathcal{B})$ is said to be *k-uniform* if $|B| = k$ for all $B \in \mathcal{B}$.

Let $v \geq k$. A *(v, k)-packing* is a k-uniform set system $(X, \mathcal{B})$ of order v, such that each 2-subset of X occurs in at most one block in $\mathcal{B}$. The *packing number $D(v, k)$* is the maximum number of blocks in any (v, k)-packing. A (v, k)-packing $(X, \mathcal{B})$ is said to be *optimal* if $|\mathcal{B}| = D(v, k)$. The values of $D(v, k)$ have been determined for all v when $k \in \{3, 4\}$ [30]. In particular, we have

$$D(v, 3) = \begin{cases} \left\lfloor \frac{v}{3} \left\lfloor \frac{v-1}{2} \right\rfloor \right\rfloor - 1, & \text{if } v \equiv 5 \bmod 6; \\ \left\lfloor \frac{v}{3} \left\lfloor \frac{v-1}{2} \right\rfloor \right\rfloor, & \text{otherwise.} \end{cases}$$

Let $(X, \mathcal{B})$ be a set system and $\mathcal{G} = \{G_1, G_2, \ldots, G_u\}$ be a partition of X into subsets, called *groups*. The triple $(X, \mathcal{G}, \mathcal{B})$ is a group divisible design (GDD) when

(i) each 2-subset of X not contained in a group appears in exactly one block, and
(ii) $|B \cap G| \leq 1$ for all $B \in \mathcal{B}$ and $G \in \mathcal{G}$.

Denote a GDD $(X, \mathcal{G}, \mathcal{B})$ by k-GDD if $|B| = k$ for all $B \in \mathcal{B}$. The *type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. An "exponential" notation is usually used to describe the type: a type $g_1^{u_1} g_2^{u_2} \ldots g_t^{u_t}$ denotes u_i occurrences of g_i, $1 \leq i \leq t$.

A partial parallel class (PPC) of a k-uniform set system of order v is a collection of disjoint blocks, and is *maximum* if it contains $\lfloor v/k \rfloor$ blocks and *non-maximum* otherwise. If a PPC covers each point exactly once, we call it a parallel class (PC). A set system is *resolvable* if its blocks can be partitioned into PCs. A resolvable k-GDD is denoted by k-RGDD. A 3-RGDD of type 1^v is known as a Kirkman triple system, and is denoted by KTS(v).

Given a k-uniform set system $(X, \mathcal{B})$ and a PPC $P \subset \mathcal{B}$, we use $\overline{P}$ to denote the set of missing points of P in X, that is, $\overline{P} = X \setminus \cup_{B \in P} B$.

We require the following results.

Proposition 2.3 (Ge and Miao [21]) *A* 3*-RGDD of type* h^u *exists if and only if* $u \geq 3$, $h(u-1)$ *is even,* $hu \equiv 0 \bmod 3$*, and* $(h, u) \notin \{(2, 3), (2, 6), (6, 3)\}$.

A k*-frame* is a k-GDD $(X, \mathcal{G}, \mathcal{B})$, such that $\mathcal{B}$ can be partitioned into a collection of PPCs, where the complement of each PPC is exactly a group.

Proposition 2.4 (Ge and Miao [21], Wei and Ge [34]) *There exist* 3*-frames of the following types:*

(i) h^u*, when* $u \geq 4$, $h \equiv 0 \bmod 2$ *and* $h(u-1) \equiv 0 \bmod 3$*,*
(ii) $12^u m^1$*, when* $u \geq 4$ *and* $m \in \{6, 18\}$.

2.3 Connection between codes and packings

Chee et al. [7] showed that the following two conditions are necessary and sufficient for a q-ary code $\mathcal{C}$ of constant weight w to have distance $2w - 1$:

(C1) for any distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$, $|\text{supp}(\mathsf{u}) \cap \text{supp}(\mathsf{v})| \leq 1$, and
(C2) for any distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$, if $x \in \text{supp}(\mathsf{u}) \cap \text{supp}(\mathsf{v})$, then $\mathsf{u}_x \neq \mathsf{v}_x$.

These easily imply the following result.

Corollary 2.5 *Let* $\mathcal{C} \subseteq \mathbb{Z}_q^X$ *be an* $(n, 2w-1, w)_q$*-code, and* $\mathcal{B} = \{\text{supp}(\mathsf{u}) : \mathsf{u} \in \mathcal{C}\}$*. Then* $(X, \mathcal{B})$ *is an* (n, w)*-packing.*

By Corollary 2.5, $A_q(n, 2w-1, w)$ cannot be larger than the packing number $D(n, w)$. In fact, we show that $A_q(n, 2w-1, w) = D(n, w)$ for all sufficiently large q. First, we introduce the following definition.

Definition 2.6 Let $(X, \mathcal{B})$ be a set system, and let $P \subseteq \mathcal{B}$. For a positive integer i, define

$$\mathcal{C}(P, i) = \{\mathsf{u}^B \in \mathbb{Z}_{i+1}^X : B \in P\},$$

where

$$\mathsf{u}_x^B = \begin{cases} i, & \text{if } x \in B; \\ 0, & \text{if } x \notin B. \end{cases}$$

Proposition 2.7 $A_q(n, 2w-1, w) = D(n, w)$ *for all* $q \geq \left\lfloor \frac{n-1}{w-1} \right\rfloor + 1$.

Proof Let $(X, \mathcal{B})$ be an optimal (n, w)-packing with $D(n, w)$ blocks, then each point occurs in at most $\left\lfloor \frac{n-1}{w-1} \right\rfloor$ blocks. For the code $\mathcal{C}(\mathcal{B}, 1)$, there are at most $\left\lfloor \frac{n-1}{w-1} \right\rfloor$ 1's at each coordinate. We replace these 1's with $1, 2, 3, \ldots$ to make the nonzero elements at each coordinate all distinct. The result is an $(n, 2w-1, w)_{\left\lfloor \frac{n-1}{w-1} \right\rfloor + 1}$-code with $D(n, w)$ codewords. It is also an $(n, 2w-1, w)_q$-code for each $q \geq \left\lfloor \frac{n-1}{w-1} \right\rfloor + 1$. By Corollary 2.5, these codes are optimal. □

Corollary 2.8 $A_q(n, 5, 3) = D(n, 3)$ *for all* $q \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 1$.

Table 1 The number of blocks in an HTP(n)

n	$D(n, 3) = b \cdot h + \star$
$6t$	$6t^2 - 2t = 2t(3t - 1) + 0$
$6t + 1$	$6t^2 + t = 2t \cdot 3t + t$
$6t + 2$	$6t^2 + 2t = 2t \cdot 3t + 2t$
$6t + 3$	$6t^2 + 5t + 1 = (2t + 1)(3t + 1) + 0$
$6t + 4$	$6t^2 + 6t + 1 = (2t + 1)(3t + 1) + t$
$6t + 5$	$6t^2 + 9t + 2 = (2t + 1)(3t + 2) + 2t$

3 Hanani triple packings and optimal $(n, 5, 3)_q$-codes

In this section, we construct optimal $(n, 5, 3)_q$-codes from Hanani triple packings.

Throughout this section, let $h = \left\lfloor \frac{n-1}{2} \right\rfloor$, $b = \left\lfloor \frac{n}{3} \right\rfloor$, $t = \left\lfloor \frac{n}{6} \right\rfloor$, and $c \equiv n \bmod 3$, where $0 \le c \le 2$. Then $n = 3b + c$, and

$$A_q(n, 5, 3) \le \left\lfloor \frac{(q-1)(3b+c)}{3} \right\rfloor = (q-1)b + \left\lfloor \frac{(q-1)c}{3} \right\rfloor.$$

3.1 Hanani triple packings

A Hanani triple packing (HTP) of order n, denoted $HTP(n)$, is an optimal $(n, 3)$-packing whose block set can be partitioned into PPCs with all but at most one being maximum. Hanani triple packings are a generalization of some well known objects in combinatorial design theory, such as Hanani triple systems [12,33] and Kirkman triple systems [11].

The number of blocks of an HTP(n) is provided in Table 1.

We mainly use HTP(n)'s to construct optimal $(n, 5, 3)_q$-codes for $q \le h + 1$. For these q, a stronger condition is needed.

Definition 3.1 Let $n \not\equiv 0 \bmod 3$, and consider an HTP(n) with PPCs $P_1, P_2, \ldots, P_{h+1}$, where $P_1, P_2, \ldots, P_h$ are maximum PPCs. For each $1 \le i \le h$, let $a_{i,1}, a_{i,2}, \ldots, a_{i,c}$ be the elements in $\overline{P_i}$. The HTP(n) is called *strong* if it satisfies the property that for each $1 \le s \le t$,

(i) $\{a_{3s-2,j}, a_{3s-1,j}, a_{3s,j}\}$ is a block in P_{h+1} for each $1 \le j \le c$, and
(ii) if $c = 2$, then any 2-subset of $\{a_{3s-2,1}, a_{3s-2,2}, a_{3s-1,1}\}$ is not contained in any blocks of $(\bigcup_{i=1}^{3s-1} P_i) \cup (\bigcup_{i=1}^{s-1} \bigcup_{j=1}^{c} \{\{a_{3i-2,j}, a_{3i-1,j}, a_{3i,j}\}\})$.

When $n \equiv 0 \bmod 3$, every HTP(n) is called strong.

Example 3.2 A strong HTP(8). Let $X = \mathbb{Z}_6 \cup \{\infty_0, \infty_1\}$. The PPCs of the HTP(8) are given below. The elements of $\overline{P_i}$ are listed here (and elsewhere in this paper) in the order $a_{i,1}, a_{i,2}, \ldots, a_{i,c}$.

$$P_1 = \{\{1, 5, \infty_0\}, \{2, 4, \infty_1\}\}, \overline{P_1} = \{0, 3\};$$
$$P_2 = \{\{2, 3, \infty_0\}, \{0, 5, \infty_1\}\}, \overline{P_2} = \{1, 4\};$$
$$P_3 = \{\{0, 4, \infty_0\}, \{1, 3, \infty_1\}\}, \overline{P_3} = \{2, 5\};$$
$$P_4 = \{\{0, 1, 2\}, \{3, 4, 5\}\}.$$

Example 3.3 A strong HTP(10). Let $X = \mathbb{Z}_6 \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. The PPCs of the HTP(10) are given below.

$$P_1 = \{\{\infty_2, 0, 1\}, \{\infty_3, 2, 3\}, \{\infty_0, 4, 5\}\}, \overline{P_1} = \{\infty_1\};$$
$$P_2 = \{\{\infty_1, 2, 4\}, \{\infty_3, 0, 5\}, \{\infty_0, 1, 3\}\}, \overline{P_2} = \{\infty_2\};$$
$$P_3 = \{\{\infty_1, 1, 5\}, \{\infty_2, 3, 4\}, \{\infty_0, 0, 2\}\}, \overline{P_3} = \{\infty_3\};$$
$$P_4 = \{\{\infty_1, 0, 3\}, \{\infty_2, 2, 5\}, \{\infty_3, 1, 4\}\}, \overline{P_4} = \{\infty_0\};$$
$$P_5 = \{\{\infty_1, \infty_2, \infty_3\}\}.$$

Example 3.4 A strong HTP(17). Let $X = \mathbb{Z}_{12} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. The PPCs of the HTP(17) are given below.

$$P_1 = \{\{2, 8, \infty_0\}, \{6, 5, \infty_1\}, \{4, 3, \infty_2\}, \{9, 11, \infty_3\}, \{10, 7, \infty_4\}\}, \overline{P_1} = \{0, 1\};$$
$$P_2 = \{\{9, 3, \infty_0\}, \{1, 7, \infty_1\}, \{0, 10, \infty_2\}, \{6, 8, \infty_3\}, \{11, 2, \infty_4\}\}, \overline{P_2} = \{4, 5\};$$
$$P_3 = \{\{7, 6, \infty_0\}, \{3, 10, \infty_1\}, \{11, 5, \infty_2\}, \{0, 2, \infty_3\}, \{1, 4, \infty_4\}\}, \overline{P_3} = \{8, 9\};$$
$$P_4 = \{\{11, 10, \infty_0\}, \{8, 9, \infty_1\}, \{1, 6, \infty_2\}, \{7, 4, \infty_3\}, \{0, 5, \infty_4\}\}, \overline{P_4} = \{2, 3\};$$
$$P_5 = \{\{5, 4, \infty_0\}, \{0, 11, \infty_1\}, \{2, 9, \infty_2\}, \{1, 10, \infty_3\}, \{3, 8, \infty_4\}\}, \overline{P_5} = \{6, 7\};$$
$$P_6 = \{\{1, 0, \infty_0\}, \{2, 4, \infty_1\}, \{8, 7, \infty_2\}, \{3, 5, \infty_3\}, \{6, 9, \infty_4\}\}, \overline{P_6} = \{10, 11\};$$
$$P_7 = \{\{0, 9, 7\}, \{11, 4, 6\}, \{2, 3, 1\}, \{8, 5, 10\}, \{\infty_0, \infty_1, \infty_2\}\}, \overline{P_7} = \{\infty_3, \infty_4\};$$
$$P_8 = \{\{0, 6, 3\}, \{4, 10, 9\}, \{1, 8, 11\}, \{2, 5, 7\}, \{\infty_0, \infty_3, \infty_4\}\}, \overline{P_8} = \{\infty_1, \infty_2\};$$
$$P_9 = \{\{0, 4, 8\} + i : i = 0, 1, 2, 3\}.$$

3.2 Connection between Hanani triple packings and optimal codes

A strong HTP(n) can be used to construct optimal $(n, 5, 3)_q$-codes for all $q \geq 2$.

Proposition 3.5 *If there exists a strong* HTP(n)*, then*

$$A_q(n, 5, 3) = \min\left\{\left\lfloor \frac{(q-1)n}{3} \right\rfloor, D(n, 3)\right\}$$

for all $q \geq 2$.

Proof When $q \leq h + 1$, we consider three cases:

(i) If $n \equiv 0 \bmod 3$, there are h maximum PPCs, $P_1, P_2, \ldots, P_h$, in a strong HTP(n), and each has size b. For $2 \leq q \leq h + 1$, let $\mathcal{C}_q = \bigcup_{i=1}^{q-1} \mathcal{C}(P_i, i)$. $\mathcal{C}_q$ satisfies Conditions (C1) and (C2) and is hence an $(n, 5, 3)_q$-code. Optimality of this code follows from the Johnson bound.

(ii) If $n \equiv 1 \bmod 3$, there are h maximum PPCs $P_1, P_2, \ldots, P_h$, and a PPC P_{h+1} with t blocks in a strong HTP(n). Let $\overline{P_i} = \{x_i\}$, for $1 \leq i \leq h$. For each $1 \leq s \leq h$, $s \equiv 0 \bmod 3$, define the vector u^s with support $\{x_{s-2}, x_{s-1}, x_s\}$, where $\mathsf{u}^s_{x_i} = i$ for $i \in \{s-2, s-1, s\}$. Finally, let $\mathcal{C}_2 = \mathcal{C}(P_1, 1)$ and recursively define

$$\mathcal{C}_q = \begin{cases} \mathcal{C}_{q-1} \cup \mathcal{C}(P_{q-1}, q-1), & \text{if } q \not\equiv 1 \bmod 3; \\ \mathcal{C}_{q-1} \cup \mathcal{C}(P_{q-1}, q-1) \cup \{\mathsf{u}^{q-1}\}, & \text{otherwise.} \end{cases}$$

Then each $\mathcal{C}_q$ is an optimal $(n, 5, 3)_q$-code.

(iii) If $n \equiv 2 \bmod 3$, there are $h+1$ PPCs $P_1, P_2, \ldots, P_{h+1}$, in a strong HTP($n$). All PPCs are maximum when $n \equiv 2 \bmod 6$, and with the exception of P_{h+1} (which is non-maximum with $2t$ blocks) when $n \equiv 5 \bmod 6$. Let $\overline{P_i} = \{a_{i,1}, a_{i,2}\}$, for $1 \le i \le h$. For each $1 \le s \le h$, define the vector u^s with support

$$\mathrm{supp}(\mathsf{u}^s) = \begin{cases} \{a_{s,1}, a_{s,2}, a_{s+1,1}\}, & \text{if } s \equiv 1 \bmod 3; \\ \{a_{s-1,1}, a_{s,1}, a_{s+1,1}\}, & \text{if } s \equiv 2 \bmod 3; \\ \{a_{s-2,2}, a_{s-1,2}, a_{s,2}\}, & \text{if } s \equiv 0 \bmod 3, \end{cases}$$

where $\mathsf{u}^s_{a_{i,j}} = i$, for each $a_{i,j}$ in the support of u^s. Let $\mathcal{C}_2 = \mathcal{C}(P_1, 1)$. For each q, $3 \le q \le h+1$, except when $n \equiv 5 \bmod 6$ and $q = h+1$, define recursively

$$\mathcal{C}_q = \begin{cases} \mathcal{C}_{q-1} \cup \mathcal{C}(P_{q-1}, q-1) \cup \{\mathsf{u}^{q-2}\}, & \text{if } q \equiv 0 \bmod 3; \\ (\mathcal{C}_{q-1} \backslash \{\mathsf{u}^{q-3}\}) \cup \mathcal{C}(P_{q-1}, q-1) \cup \{\mathsf{u}^{q-2}, \mathsf{u}^{q-1}\}, & \text{if } q \equiv 1 \bmod 3; \\ \mathcal{C}_{q-1} \cup \mathcal{C}(P_{q-1}, q-1), & \text{if } q \equiv 2 \bmod 3. \end{cases}$$

When $n \equiv 5 \bmod 6$ and $q = h+1$, define recursively $\mathcal{C}_q = \mathcal{C}_{q-1} \cup \mathcal{C}(P_{q-1}, q-1)$. Then each $\mathcal{C}_q$ is an optimal $(n, 5, 3)_q$-code.

For $q \ge h+1$, the conclusion follows from Corollary 2.8. □

Before closing this section, we give three examples as applications of Proposition 3.5 using Examples 3.2–3.4. We list the codes for $q \le h+1$.

Example 3.6 For $n = 8$, $\mathcal{C}_2 = \mathcal{C}(P_1, 1) = \{01000110, 00101001\}$; $\mathcal{C}_3 = \mathcal{C}_2 \cup \mathcal{C}(P_2, 2) \cup \{\mathsf{u}^1\}$, where u^1 is the vector with $\mathsf{u}^1_0 = 1$, $\mathsf{u}^1_3 = 1$, $\mathsf{u}^1_1 = 2$ and $\mathsf{u}^1_x = 0$ for all other $x \in X$, i.e., $\mathcal{C}_3 = \mathcal{C}_2 \cup \{00220020, 20000202, 12010000\}$; $\mathcal{C}_4 = (\mathcal{C}_3 \backslash \{\mathsf{u}^1\}) \cup \mathcal{C}(P_3, 3) \cup \{\mathsf{u}^2, \mathsf{u}^3\}$, where $\mathcal{C}(P_3, 3) = \{30003030, 03030003\}$ and $\mathsf{u}^2 = 12300000$, $\mathsf{u}^3 = 00012300$. Then $\mathcal{C}_q$ is an optimal $(8, 5, 3)_q$-code for $q \in \{2, 3, 4\}$.

In the following two examples, we omit listing the codewords in $\mathcal{C}(P_i, i)$ since they are obvious.

Example 3.7 For $n = 10$, $\mathcal{C}_2 = \mathcal{C}(P_1, 1)$; $\mathcal{C}_3 = \mathcal{C}_2 \cup \mathcal{C}(P_2, 2)$; $\mathcal{C}_4 = \mathcal{C}_3 \cup \mathcal{C}(P_3, 3) \cup \{\mathsf{u}^3\}$, where $\mathsf{u}^3 = 0000000123$; $\mathcal{C}_5 = \mathcal{C}_4 \cup \mathcal{C}(P_4, 4)$. Then $\mathcal{C}_q$ is an optimal $(10, 5, 3)_q$-code for $q \in \{2, 3, 4, 5\}$.

Example 3.8 For $n = 17$, $\mathcal{C}_2 = \mathcal{C}(P_1, 1)$; $\mathcal{C}_3 = \mathcal{C}_2 \cup \mathcal{C}(P_2, 2) \cup \{\mathsf{u}^1\}$, where $\mathsf{u}^1 = 11002000000000000$; $\mathcal{C}_4 = (\mathcal{C}_3 \backslash \{\mathsf{u}^1\}) \cup \mathcal{C}(P_3, 3) \cup \{\mathsf{u}^2, \mathsf{u}^3\}$, where $\mathsf{u}^2 = 10002000300000000$ and $\mathsf{u}^3 = 01000200030000000$; $\mathcal{C}_5 = \mathcal{C}_4 \cup \mathcal{C}(P_4, 4)$; $\mathcal{C}_6 = \mathcal{C}_5 \cup \mathcal{C}(P_5, 5) \cup \{\mathsf{u}^4\}$, where $\mathsf{u}^4 = 00440050000000000$; $\mathcal{C}_7 = (\mathcal{C}_6 \backslash \{\mathsf{u}^4\}) \cup \mathcal{C}(P_6, 6) \cup \{\mathsf{u}^5, \mathsf{u}^6\}$, where $\mathsf{u}^5 = 00400050006000000$, $\mathsf{u}^6 = 00040005000600000$; $\mathcal{C}_8 = \mathcal{C}_7 \cup \mathcal{C}(P_7, 7)$; $\mathcal{C}_9 = \mathcal{C}_8 \cup \mathcal{C}(P_8, 8)$. Then $\mathcal{C}_q$ is an optimal $(17, 5, 3)_q$-code for $2 \le q \le 9$.

4 Existence of strong Hanani triple packings

We establish the existence of strong Hanani triple packings in this section. Note that the existence of a strong HTP(n) for $n \le 5$ is trivial.

4.1 The case $n \equiv 0 \bmod 3$

When $n \equiv 3 \bmod 6$, a KTS(n) is a (strong) HTP(n). When $n \equiv 0 \bmod 6$, a 3-RGDD of type $2^{n/2}$ is a (strong) HTP(n). Proposition 2.3 then implies the following.

Proposition 4.1 *There exists a strong* HTP(n) *for all* $n \equiv 0 \bmod 3$, *except when* $n \in \{6, 12\}$.

4.2 The case $n \equiv 1 \bmod 6$

When $n = 6t+1$, an HTP(n) is a 3-GDD of type 1^{6t+1}, whose set of blocks can be partitioned into $3t$ maximum PPCs, and a non-maximum PPC with t blocks. Such a design is called a *Hanani triple system* and it has been shown by Vanstone et al. [33] that Hanani triple systems of order n exist for all $n \equiv 1 \bmod 6$, except when $n \in \{7, 13\}$. We proof that every Hanani triple system is strong.

Proposition 4.2 *Every Hanani triple system is strong.*

Proof Let $(X, \mathcal{B})$ be a Hanani triple system of order $6t + 1$, with $\mathcal{B}$ being partitioned into $3t$ maximum PPCs $P_1, P_2, \ldots, P_{3t}$, and a non-maximum PPC P_{3t+1} with t blocks. If $x \in X$ is contained in the blocks of P_{3t+1}, then x is missed by exactly one P_i, $1 \le i \le 3t$, since each point of X occurs in exactly $3t$ blocks in $\mathcal{B}$. Thus, we can arrange the order of P_i, $1 \le i \le 3t$, in such a way that for each $1 \le s \le t$, $\overline{P_{3s-2}} \cup \overline{P_{3s-1}} \cup \overline{P_{3s}}$ is a block in P_{3t+1}. □

Corollary 4.3 *There exists a strong* HTP(n) *for all* $n \equiv 1 \bmod 6$*, except when* $n \in \{7, 13\}$.

4.3 The case $n \equiv 2 \bmod 6$

Example 4.4 A strong HTP(20) can be constructed on the point set $\mathbb{Z}_{18} \cup \{\infty_0, \infty_1\}$ as follows. The maximum PPCs P_1, P_4, P_7 and the corresponding sets $\overline{P_i}$ are given by

$$P_1 = \{\{14, 7, \infty_0\}, \{10, 15, \infty_1\}, \{5, 16, 8\}, \{1, 3, 11\}, \{2, 12, 4\}, \{13, 6, 17\}\}, \overline{P_1} = \{0, 9\};$$
$$P_4 = \{\{16, 9, \infty_0\}, \{17, 1, \infty_1\}, \{14, 13, 10\}, \{0, 4, 5\}, \{7, 3, 6\}, \{8, 12, 15\}\}, \overline{P_4} = \{2, 11\};$$
$$P_7 = \{\{6, 5, \infty_0\}, \{12, 14, \infty_1\}, \{8, 9, 11\}, \{4, 17, 3\}, \{0, 16, 13\}, \{7, 2, 15\}\}, \overline{P_7} = \{1, 10\}.$$

For $i \in \{2, 3, 5, 6, 8, 9\}$, the maximum PPC $P_i = (P_{i-1} + 6) \bmod 18$, and $\overline{P_i}$ is obtained the same way. The non-maximum PPC is $P_{10} = \{\{0, 6, 12\} + i : i = 0, 1, \ldots, 5\}$.

Proposition 4.5 *There exists a strong HTP*(n) *for all* $n \equiv 2 \bmod 6$*, except possibly when* $n = 14$.

Proof When $n \in \{8, 20\}$, a strong HTP(n) exists by Examples 3.2 and 4.4.

When $n \ge 26$, write $n = 6t + 2$ and note that a 3-frame of type 2^{3t+1} is an HTP$(6t + 2)$. Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-frame of type 6^t, which exists by Proposition 2.4, with $X = \mathbb{Z}_6 \times I_t$, $\mathcal{G} = \{\mathbb{Z}_6 \times \{i\} : 1 \le i \le t\}$, and $\mathcal{B}$ being partitioned into PPCs $P_1, P_2, \ldots, P_{3t}$. Assume, for each $1 \le i \le t$, that $P_{3i-2}, P_{3i-1}, P_{3i}$ are the PPCs missing the points in the group $\mathbb{Z}_6 \times \{i\}$. Now adjoin two new points ∞_0 and ∞_1 to X and for each $1 \le i \le t$, construct a strong HTP(8) on $(\mathbb{Z}_6 \times \{i\}) \cup \{\infty_0, \infty_1\}$ with maximum PPCs P_1^i, P_2^i, P_3^i and P_4^i, such that $\overline{P_4^i} = \{\infty_0, \infty_1\}$. Let $P'_{j+3(i-1)} = P_{j+3(i-1)} \cup P_j^i$ for $1 \le i \le t$ and $1 \le j \le 3$. Further, let $P'_{3t+1} = \cup_{i=1}^{t} P_4^i$. Then $(X \cup \{\infty_0, \infty_1\}, \cup_{i=1}^{3t+1} P'_i)$ is a strong HTP(n). □

4.4 The case $n \equiv 4 \bmod 6$

Lemma 4.6 *There exists a strong* HTP(n) *for* $n \in \{16, 22, 28, 34, 40, 46\}$.

Proof For $n \in \{16, 22, 28, 34, 40, 46\}$, write $n = 6t + 4$. We construct a strong HTP$(6t + 4)$ on point set $\mathbb{Z}_{6t} \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. Then maximum PPCs P_1, P_2, P_3 for each t are given in the table below.

	i	P_i	$\overline{P_i}$
$t=2$	1	$\{9,2,\infty_0\}\{3,10,\infty_1\}\{7,8,\infty_2\}\{4,5,\infty_3\}\{6,11,1\}$	0
	2	$\{10,7,\infty_0\}\{1,0,\infty_1\}\{11,4,\infty_2\}\{8,6,\infty_3\}\{2,3,5\}$	9
	3	$\{0,11,\infty_0\}\{5,8,\infty_1\}\{9,6,\infty_2\}\{3,1,\infty_3\}\{2,4,7\}$	10
$t=3$	1	$\{5,13,\infty_0\}\{2,7,\infty_1\}\{3,16,\infty_2\}\{14,4,\infty_3\}\{15,11,12\}\{8,6,1\}\{9,10,17\}$	0
	2	$\{14,15,\infty_0\}\{6,3,\infty_1\}\{5,8,\infty_2\}\{12,17,\infty_3\}\{2,1,9\}\{4,0,7\}\{13,16,11\}$	10
	3	$\{10,12,\infty_0\}\{17,16,\infty_1\}\{6,7,\infty_2\}\{13,9,\infty_3\}\{1,3,5\}\{15,2,4\}\{11,8,0\}$	14
$t=4$	1	$\{16,7,\infty_0\}\{18,21,\infty_1\}\{6,8,\infty_2\}\{4,3,\infty_3\}\{9,13,23\}\{12,17,19\}\{15,1,14\}$ $\{10,5,11\}\{2,20,22\}$	0
	2	$\{9,18,\infty_0\}\{5,16,\infty_1\}\{4,11,\infty_2\}\{1,12,\infty_3\}\{15,22,8\}\{13,20,17\}\{0,19,10\}$ $\{23,14,3\}\{7,2,6\}$	21
	3	$\{14,5,\infty_0\}\{19,20,\infty_1\}\{17,0,6\}\{1,3,\infty_2\}\{8,11,\infty_3\}\{10,7,13\}\{21,12,2\}$ $\{22,18,16\}\{9,15,4\}$	23
$t=5$	1	$\{20,28,\infty_0\}\{17,3,\infty_1\}\{25,8,\infty_2\}\{19,24,\infty_3\}\{21,23,4\}\{11,12,29\}\{10,16,15\}$ $\{22,6,18\}\{0,1,13\}\{7,5,14\}\{26,9,27\}$	2
	2	$\{23,1,\infty_0\}\{20,22,\infty_1\}\{6,17,\infty_2\}\{4,15,\infty_3\}\{3,29,24\}\{25,26,18\}\{21,0,27\}$ $\{7,11,10\}\{9,2,13\}\{16,19,28\}\{14,8,12\}$	5
	3	$\{21,18,\infty_0\}\{13,24,\infty_1\}\{22,15,\infty_2\}\{29,2,\infty_3\}\{17,10,23\}\{1,9,7\}\{0,28,6\}$ $\{26,5,12\}\{4,20,8\}\{16,11,25\}\{19,14,3\}$	27
$t=6$	1	$\{27,7,\infty_0\}\{35,18,\infty_1\}\{26,10,\infty_2\}\{14,17,\infty_3\}\{20,9,29\}\{34,4,11\}\{16,6,5\}$ $\{32,31,2\}\{8,12,21\}\{33,22,3\}\{28,25,19\}\{13,23,15\}\{1,24,30\}$	0
	2	$\{17,26,\infty_0\}\{14,1,\infty_1\}\{3,25,\infty_2\}\{27,28,\infty_3\}\{12,10,2\}\{5,11,7\}\{22,6,9\}$ $\{16,30,31\}\{32,29,15\}\{35,20,34\}\{33,18,23\}\{0,8,19\}\{24,13,21\}$	4
	3	$\{4,12,\infty_0\}\{22,15,\infty_1\}\{6,17,\infty_2\}\{31,0,\infty_3\}\{29,10,13\}\{20,5,18\}\{7,3,35\}$ $\{28,26,33\}\{24,8,9\}\{19,34,2\}\{16,25,11\}\{21,23,30\}\{1,32,27\}$	14
$t=7$	1	$\{29,40,\infty_0\}\{36,38,\infty_1\}\{25,22,\infty_2\}\{19,12,\infty_3\}\{0,37,18\}\{34,1,10\}\{39,7,17\}$ $\{16,32,13\}\{5,35,8\}\{15,23,41\}\{11,4,6\}\{2,31,14\}\{33,9,26\}\{20,28,21\}\{27,24,30\}$	3
	2	$\{33,14,\infty_0\}\{39,41,\infty_1\}\{27,32,\infty_2\}\{3,16,\infty_3\}\{24,17,4\}\{5,36,10\}\{25,21,19\}$ $\{28,13,11\}\{40,8,30\}\{1,9,0\}\{20,22,2\}\{7,23,29\}\{35,18,6\}\{26,15,31\}\{37,38,12\}$	34
	3	$\{7,18,\infty_0\}\{10,25,\infty_1\}\{36,17,\infty_2\}\{8,11,\infty_3\}\{22,41,26\}\{13,2,37\}\{14,5,6\}$ $\{27,39,12\}\{38,0,32\}\{23,31,19\}\{1,21,30\}\{28,16,24\}\{34,15,40\}\{20,33,29\}\{4,9,3\}$	35

For each $1 \le i \le 3$ and $1 \le s \le t-1$, the maximum PPC P_{i+3s} is obtained from P_i by adding $6s$ under $\mathbb{Z}_{6t}$. Let $P_{3t+1} = \{\{0, 2t, 4t\} + i : 0 \le i \le 2t-1\} \cup \{\{\infty_1, \infty_2, \infty_3\}\}$. The complement of each maximum PPC contains only one point, and $\overline{P_{3i-2}} \cup \overline{P_{3i-1}} \cup \overline{P_{3i}}$, $1 \le i \le t$, form the t blocks of the last non-maximum PPC P_{3t+2}. □

Proposition 4.7 *There exists a strong* HTP(n) *for all* $n \equiv 4 \bmod 6$.

Proof When $n \le 46$, a strong HTP(n) exists by Example 3.3 and Lemma 4.6. When $n \ge 52$, write $n = 6t+4$ and consider the following cases.

For $t = 2s$: Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-frame of type 12^s, which exists by Proposition 2.4, with $X = \mathbb{Z}_{12} \times I_s$, $\mathcal{G} = \{\mathbb{Z}_{12} \times \{i\} : 1 \le i \le s\}$ and $\mathcal{B}$ being partitioned into PPCs P_i, $1 \le i \le 6s$. Assume that for each $1 \le i \le s$, P_j, $6i-5 \le j \le 6i$, are the six PPCs missing the points in the group $\mathbb{Z}_{12} \times \{i\}$. Let $Y = \{\infty_0, \infty_1, \infty_2, \infty_3\}$. For each $1 \le i \le s$, construct a strong HTP(16) on $(\mathbb{Z}_{12} \times \{i\}) \cup Y$, with seven maximum PPCs P^i_j, $1 \le j \le 7$, and a non-maximum PPC P^i_8, such that $\{\infty_1, \infty_2, \infty_3\}$ is a block in P^i_7 and $\overline{P^i_7} = \{\infty_0\}$. Let $P'_{j+6(i-1)} = P_{j+6(i-1)} \cup P^i_j$ for $1 \le i \le s$ and $1 \le j \le 6$. Finally, let $P'_{6s+1} = \cup_{i=1}^{s} P^i_7$ and $P'_{6s+2} = \cup_{i=1}^{s} P^i_8$. Then $(X \cup Y, \cup_{i=1}^{6s+2} P'_i)$ is a strong HTP(n).

For $t = 2s + 1$: Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-frame of type $12^s 6^1$, which exists by Proposition 2.4, with $X = (\mathbb{Z}_{12} \times I_s) \cup (\mathbb{Z}_6 \times \{s+1\})$, $\mathcal{G} = \{\mathbb{Z}_{12} \times \{i\} : 1 \le i \le s\} \cup \{\mathbb{Z}_6 \times \{s+1\}\}$ and $\mathcal{B}$ being partitioned into PPCs of $4s-2$ blocks P_i, $1 \le i \le 6s$, and PPCs of $4s$ blocks P_i, $6s+1 \le i \le 6s+3$. Assume that for each $1 \le i \le s$, P_j, $6i-5 \le j \le 6i$, are PPCs missing the points in the group $\mathbb{Z}_{12} \times \{i\}$, and P_j, $6s+1 \le j \le 6s+3$, are PPCs missing the points in the group $\mathbb{Z}_6 \times \{s+1\}$. Let $Y = \{\infty_0, \infty_1, \infty_2, \infty_3\}$. For each $1 \le i \le s$, construct a strong HTP(16) on $(\mathbb{Z}_{12} \times \{i\}) \cup Y$, with seven maximum PPCs P^i_j, $1 \le j \le 7$, and a non-maximum PPC P^i_8, such that $\{\infty_1, \infty_2, \infty_3\}$ is a block in P^i_7 and $\overline{P^i_7} = \{\infty_0\}$. Finally, construct a strong HTP(10) on $(\mathbb{Z}_6 \times \{s+1\}) \cup Y$, with four maximum PPCs P^{s+1}_j, $1 \le j \le 4$, and a non-maximum PPC $P^{s+1}_5 = \{\{\infty_1, \infty_2, \infty_3\}\}$. Let $P'_{j+6(i-1)} = P_{j+6(i-1)} \cup P^i_j$ for $1 \le i \le s$ and $1 \le j \le 6$, $P'_{6s+j} = P_{6s+j} \cup P^{s+1}_j$ for $1 \le j \le 3$, $P'_{6s+4} = (\cup_{i=1}^{s}(P^i_7 \setminus \{\{\infty_1, \infty_2, \infty_3\}\})) \cup P^{s+1}_4$ and $P'_{6s+5} = (\cup_{i=1}^{s} P^i_8) \cup P^{s+1}_5$. Then $(X \cup Y, \cup_{i=1}^{6s+5} P'_i)$ is a strong HTP(n). □

4.5 The case $n \equiv 5 \bmod 6$

Example 4.8 A strong HTP(23) can be constructed on the point set $\mathbb{Z}_{18} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$ as follows. The maximum PPCs P_1, P_4, P_7 and the corresponding sets $\overline{P_i}$ are given by

$$P_1 = \{\{0, 1, \infty_0\}, \{16, 8, \infty_1\}, \{4, 5, \infty_2\}, \{11, 13, \infty_3\}, \{7, 12, \infty_4\}, \{14, 17, 3\}, \{15, 2, 6\}\},$$
$$\overline{P_1} = \{9, 10\};$$
$$P_4 = \{\{14, 11, \infty_0\}, \{7, 3, \infty_1\}, \{9, 12, \infty_2\}, \{6, 8, \infty_3\}, \{10, 15, \infty_4\}, \{4, 17, 0\}, \{5, 13, 16\}\},$$
$$\overline{P_4} = \{1, 2\};$$
$$P_7 = \{\{3, 4, \infty_0\}, \{0, 5, \infty_1\}, \{8, 13, \infty_2\}, \{16, 9, \infty_3\}, \{11, 2, \infty_4\}, \{12, 10, 1\}, \{7, 14, 15\}\},$$
$$\overline{P_7} = \{6, 17\}.$$

For each $i \in \{2, 3, 5, 6, 8, 9\}$, the maximum PPC P_i is obtained from P_{i-1} by adding 6 under $\mathbb{Z}_{18}$, and $\overline{P_i}$ is obtained the same way. Let $P_{10} = \{\{14, 10, 13\} + 6i, \{17, 9, 6\} + 6i : i = 0, 1, 2\} \cup \{\{\infty_0, \infty_1, \infty_2\}\}$ and $P_{11} = \{\{8, 10, 0\} + 6i, \{1, 3, 5\} + 6i : i = 0, 1, 2\} \cup \{\{\infty_0, \infty_3, \infty_4\}\}$. Finally, $P_{12} = \{\{0, 6, 12\} + i : i = 0, 1, \dots, 5\}$ is the last non-maximum PPC.

Proposition 4.9 *There exists a strong* HTP(n) *for all* $n \equiv 5 \bmod 6$*, except when* $n = 11$*, and possibly when* $n \in \{29, 35, 41, 47, 59\}$.

Proof An exhaustive computer search shows that an HTP(11) does not exist. When $n \in \{17, 23\}$, a strong HTP(n) has been constructed in Examples 3.4 and 4.8. When $n \ge 53$, $n \ne 59$, write $n = 6t + 5$ and consider the following cases.

For $t = 2s$: Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-frame of type 12^s, which exists by Proposition 2.4, with $X = \mathbb{Z}_{12} \times I_s$, $\mathcal{G} = \{\mathbb{Z}_{12} \times \{i\} : 1 \le i \le s\}$ and $\mathcal{B}$ being partitioned into PPCs P_i, $1 \le i \le 6s$. Assume that for each $1 \le i \le s$, P_j, $6i-5 \le j \le 6i$, are the six PPCs missing the points in the group $\mathbb{Z}_{12} \times \{i\}$. Let $Y = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. For each $1 \le i \le s$, construct a strong HTP(17) on $(\mathbb{Z}_{12} \times \{i\}) \cup Y$, with eight maximum PPCs P^i_j, $1 \le j \le 8$, and a non-maximum PPC P^i_9, such that $\{\infty_0, \infty_1, \infty_2\} \in P^i_7$, $\overline{P^i_7} = \{\infty_3, \infty_4\}$, $\{\infty_0, \infty_3, \infty_4\} \in P^i_8$ and $\overline{P^i_8} = \{\infty_1, \infty_2\}$. Let $P'_{j+6(i-1)} = P_{j+6(i-1)} \cup P^i_j$ for $1 \le i \le s$ and $1 \le j \le 6$, $P'_{6s+j} = \cup_{i=1}^{s} P^i_{6+j}$ for $1 \le j \le 3$. Then $(X \cup Y, \cup_{i=1}^{6s+3} P'_i)$ is a strong HTP(n).

For $t = 2(s+1)+1$: Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-frame of type $12^s 18^1$, which exists by Proposition 2.4, with $X = (\mathbb{Z}_{12} \times I_s) \cup (\mathbb{Z}_{18} \times \{s+1\})$, $\mathcal{G} = \{\mathbb{Z}_{12} \times \{i\} : 1 \le i \le s\} \cup \{\mathbb{Z}_{18} \times \{s+1\}\}$ and $\mathcal{B}$ being partitioned into PPCs of $4s+2$ blocks P_i, $1 \le i \le 6s$, and PPCs of $4s$ blocks P_i, $6s+1 \le i \le 6s+9$. Assume that for each $1 \le i \le s$, P_j, $6i-5 \le j \le 6i$, are PPCs missing the points in the group $\mathbb{Z}_{12} \times \{i\}$, and P_j for $6s+1 \le j \le 6s+9$ are PPCs missing points in the group $\mathbb{Z}_{18} \times \{s+1\}$. Let $Y = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. For each $1 \le i \le s$, construct a strong HTP(17) on $(\mathbb{Z}_{12} \times \{i\}) \cup Y$, with eight maximum PPCs P_j^i, $1 \le j \le 8$, and a non-maximum PPC P_9^i, such that $\{\infty_0, \infty_1, \infty_2\} \in P_7^i$, $\overline{P_7^i} = \{\infty_3, \infty_4\}$, $\{\infty_0, \infty_3, \infty_4\} \in P_8^i$ and $\overline{P_8^i} = \{\infty_1, \infty_2\}$. Finally, construct a strong HTP(23) on $(\mathbb{Z}_{18} \times \{s+1\}) \cup Y$, with eleven maximum PPCs P_j^{s+1}, $1 \le j \le 11$ and a non-maximum PPC P_{12}^{s+1} such that $\{\infty_0, \infty_1, \infty_2\} \in P_{10}^{s+1}$, $\overline{P_{10}^{s+1}} = \{\infty_3, \infty_4\}$, $\{\infty_0, \infty_3, \infty_4\} \in P_{11}^{s+1}$ and $\overline{P_{11}^{s+1}} = \{\infty_1, \infty_2\}$. Let $P'_{j+6(i-1)} = P_{j+6(i-1)} \cup P_j^i$ for $1 \le i \le s$ and $1 \le j \le 6$, $P'_{6s+j} = P_{6s+j} \cup P_j^{s+1}$ for $1 \le j \le 9$, $P'_{6s+j} = (\cup_{i=1}^s P_{j-3}^i) \cup P_j^{s+1}$ for $10 \le j \le 11$ and $P'_{6s+12} = (\cup_{i=0}^s P_9^i) \cup C_{12}^{s+1}$. Then $(X \cup Y, \cup_{i=1}^{6s+12} P'_i)$ is a strong HTP(n). □

4.6 Summary

Propositions 4.1, 4.5, 4.7, 4.9 and Corollary 4.3 combine to give the following result on the existence of strong Hanani triple systems.

Theorem 4.10 *There exists a strong* HTP(n) *for every positive integer n except when* $n \in \{6, 7, 11, 12, 13\}$ *and possibly when* $n \in \{14, 29, 35, 41, 47, 59\}$.

5 Existence of Hanani triple packings

For completeness, we determine the existence of Hanani triple packings in this section. Since a strong Hanani triple packing is also a Hanani triple packing, it follows from Theorem 4.10 that we need only to consider $n \in \{14, 29, 35, 41, 47, 59\}$. It turns out that Hanani triple packings for these remaining orders all exist.

Lemma 5.1 *There exists an* HTP(29).

Proof An HTP(29) is constructed on $\mathbb{Z}_{24} \cup \{\infty_0, \ldots, \infty_4\}$ with 14 maximum PPCs P_i, $1 \le i \le 14$ and one non-maximum PPC P_{15}. The PPCs P_i, $1 \le i \le 4$, and P_{15} are given by

$$\begin{aligned}
P_1 = \ & \{\{\infty_0, 2, 3\}, \{\infty_1, 4, 5\}, \{\infty_2, 6, 7\}, \{\infty_3, 8, 9\}, \{\infty_4, 10, 12\}, \\
& \{11, 13, 14\}, \{15, 16, 19\}, \{17, 20, 22\}, \{18, 21, 23\}\}; \\
P_2 = \ & \{\{\infty_0, 0, 4\}, \{\infty_1, 1, 7\}, \{\infty_2, 5, 8\}, \{\infty_3, 6, 10\}, \{\infty_4, 9, 11\}, \\
& \{12, 15, 20\}, \{13, 17, 21\}, \{14, 19, 23\}, \{16, 18, 22\}\}; \\
P_3 = \ & \{\{\infty_0, 1, 13\}, \{\infty_1, 10, 22\}, \{\infty_2, 11, 18\}, \{\infty_3, 12, 23\}, \\
& \{\infty_4, 15, 21\}, \{0, 7, 19\}, \{2, 9, 16\}, \{3, 14, 20\}, \{6, 8, 17\}\}; \\
P_4 = \ & \{\{\infty_0, 15, 22\}, \{\infty_1, 3, 16\}, \{\infty_2, 1, 20\}, \{\infty_3, 5, 19\},
\end{aligned}$$

$$\{\infty_4, 0, 14\}, \{2, 11, 12\}, \{4, 10, 21\}, \{8, 13, 23\}, \{9, 17, 18\}\};$$
$$P_{15} = \{\{\{0, 8, 20\}, \{1, 11, 19\}, \{2, 15, 18\}, \{3, 12, 21\}, \{4, 14, 17\}, \{5, 10, 16\}, \{6, 13, 22\}, \{7, 9, 23\}\}.$$

For $5 \leq i \leq 12$, P_i is obtained from P_{i-4} by adding 8 under $\mathbb{Z}_{24}$. Finally, let $P_{13} = \{B + 8 : B \in P_{15}\} \cup \{\{\infty_0, \infty_1, \infty_2\}\}$ and $P_{14} = \{B + 16 : B \in P_{15}\} \cup \{\{\infty_0, \infty_3, \infty_4\}\}$. □

Lemma 5.2 *There exists an* HTP(41).

Proof An HTP(41) is constructed on $\mathbb{Z}_{36} \cup \{\infty_0, \ldots, \infty_4\}$ with 20 maximum PPCs P_i, $1 \leq i \leq 20$ and one non-maximum PPC P_{21}. The PPCs P_i, $1 \leq i \leq 2$, are given by

$$P_1 = \{\{\infty_0, 2, 3\}, \{\infty_1, 4, 5\}, \{\infty_2, 6, 8\}, \{\infty_3, 7, 9\}, \{\infty_4, 10, 13\}, \{11, 12, 14\}, \{15, 19, 24\}, \{16, 20, 23\}, \{17, 27, 33\}, \{18, 29, 31\}, \{21, 25, 32\}, \{22, 28, 34\}, \{26, 30, 35\}\};$$
$$P_2 = \{\{\infty_0, 1, 16\}, \{\infty_1, 7, 18\}, \{0, 5, 24\}, \{6, 23, 35\}, \{\infty_2, 13, 31\}, \{\infty_3, 4, 22\}, \{\infty_4, 15, 32\}, \{8, 19, 34\}, \{9, 14, 33\}, \{10, 17, 26\}, \{11, 21, 27\}\{12, 25, 28\}, \{20, 29, 30\}\}.$$

For $3 \leq i \leq 18$, P_i is obtained from P_{i-2} by adding 4 under $\mathbb{Z}_{36}$. Let $\mathcal{D} = \{\{0, 15, 28\}, \{1, 14, 29\}, \{6, 20, 34\}, \{7, 21, 35\}\}$. Then $P_{21} = \{B + 12i : B \in \mathcal{D}, i = 0, 1, 2\}$, $P_{19} = \{B + 4 : B \in P_{21}\} \cup \{\{\infty_0, \infty_1, \infty_2\}\}$ and $P_{20} = \{B + 8 : B \in P_{21}\} \cup \{\{\infty_0, \infty_3, \infty_4\}\}$. □

Lemma 5.3 *There exists an* HTP(n) *for* $n \in \{35, 47, 59\}$.

Proof Let $n = 6t + 5$, $t \in \{5, 7, 9\}$. An HTP(n) is constructed on $\mathbb{Z}_{6t} \cup \{\infty_0, \ldots, \infty_4\}$ with $3t + 2$ maximum PPCs P_i, $1 \leq i \leq 3t + 2$ and one non-maximum PPC P_{3t+3}. For each n, P_1 is given as follows:

$$n = 35: \{\{\infty_0, 2, 5\}, \{\infty_1, 3, 6\}, \{\infty_2, 4, 13\}, \{\infty_3, 7, 20\}, \{\infty_4, 16, 21\}, \{8, 18, 24\}, \{9, 15, 25\}, \{10, 17, 29\}, \{11, 19, 28\}, \{12, 23, 27\}, \{14, 22, 26\}\};$$
$$n = 47: \{\{\infty_0, 2, 5\}, \{\infty_1, 3, 6\}, \{\infty_2, 4, 9\}, \{\infty_3, 7, 16\}, \{\infty_4, 8, 17\}, \{10, 21, 34\}, \{11, 27, 38\}, \{12, 24, 37\}, \{13, 28, 32\}, \{14, 31, 35\}, \{15, 23, 33\}, \{18, 26, 40\}, \{19, 25, 39\}, \{20, 30, 36\}, \{22, 29, 41\}\};$$
$$n = 59: \{\{\infty_0, 9, 32\}, \{\infty_1, 8, 37\}, \{\infty_2, 35, 16\}, \{\infty_3, 36, 19\}, \{42, 2, 6\}, \{\infty_4, 31, 40\}, \{44, 53, 50\}, \{38, 30, 51\}, \{29, 21, 33\}, \{17, 3, 46\}, \{52, 25, 20\}, \{12, 28, 45\}, \{24, 14, 48\}, \{43, 49, 27\}, \{0, 15, 26\}, \{41, 18, 5\}, \{22, 10, 7\}, \{4, 39, 11\}, \{23, 13, 47\}\}.$$

For $2 \leq i \leq 3t$, P_i is obtained from P_1 by adding $2(i - 1)$ under $\mathbb{Z}_{3t}$. Let $\mathcal{D} = \{\{0, 1, 2\}, \{3, 5, 10\}\}$. Then $P_{3t+3} = \{B + 6i : B \in \mathcal{D}, i = 0, 1, \ldots, t-1\}$, $P_{3t+1} = \{B + 2 : B \in P_{3t+3}\} \cup \{\{\infty_0, \infty_1, \infty_2\}\}$ and $P_{3t+2} = \{B + 4 : B \in P_{3t+3}\} \cup \{\{\infty_0, \infty_3, \infty_4\}\}$. □

Theorem 5.4 *There exists an* HTP(n) *for all positive integers* n*, except when* $n \in \{6, 7, 11, 12, 13\}$.

Proof Theorem 4.10 and Lemmas 5.1–5.3 settle all $n \neq 14$. For $n = 14$, a 3-frame of type 2^7, which exists by Proposition 2.4, is an HTP(14). □

6 Determination of $A_q(n, 5, 3)$

Theorem 4.10, together with Proposition 3.5, determines $A_q(n, 5, 3)$ for all $q \geq 2$ when $n \notin \{6, 7, 11, 12, 13, 14, 29, 35, 41, 47, 59\}$. The purpose of this section is to determine $A_q(n, 5, 3)$ for all the remaining values of n. By Corollary 2.8, we need only consider the case when $2 \leq q \leq \lfloor \frac{n-1}{2} \rfloor$.

Lemma 6.1 $A_q(n, 5, 3) = \left\lfloor \frac{(q-1)n}{3} \right\rfloor$ *for* $n \in \{6, 7, 11, 12, 13, 14\}$ *and* $2 \leq q \leq \lfloor \frac{n-1}{2} \rfloor$.

Proof For $n = 6$, $A_2(6, 5, 3) = 2$ is trivial. For $n = 7$, $A_2(7, 5, 3) = 2$ is trivial. $A_3(7, 5, 3) = 4$, since $\mathcal{C}_3 = \{1110000, 2000110, 0020021, 0200202\}$ is an optimal $(7, 5, 3)_3$ code.

For $n = 11$, take an $(11, 3)$-packing over $\mathbb{Z}_{11}$ whose blocks are partitioned into the following four maximum PPCs and $\{\{2, 5, 9\}, \{4, 2, 10\}\}$.

$$P_1 = \{\{1, 8, 0\}, \{3, 6, 9\}, \{5, 7, 10\}\}, \overline{P_1} = \{2, 4\};$$
$$P_2 = \{\{1, 6, 7\}, \{3, 8, 10\}, \{4, 9, 0\}\}, \overline{P_2} = \{5, 2\};$$
$$P_3 = \{\{1, 4, 5\}, \{2, 6, 8\}, \{3, 7, 0\}\}, \overline{P_3} = \{9, 10\};$$
$$P_4 = \{\{1, 2, 3\}, \{4, 7, 8\}, \{5, 6, 0\}\}.$$

For $n = 12$, take a $(12, 3)$-packing over $\mathbb{Z}_{12}$, whose blocks are partitioned into four PCs $P_i = \{\{B + 6j\} : B \in P_i', j = 0, 1\}$, $1 \leq i \leq 4$, where P_i's are given by

$$P_1' = \{\{0, 8, 7\}, \{9, 10, 11\}\};$$
$$P_2' = \{\{0, 1, 9\}, \{4, 8, 11\}\};$$
$$P_3' = \{\{5, 7, 9\}, \{6, 8, 10\}\};$$
$$P_4' = \{\{1, 5, 8\}, \{0, 3, 10\}\}.$$

For $n = 13$, take a $(13, 3)$-packing over $\mathbb{Z}_{13}$, whose blocks are partitioned into five maximum PPCs P_i, $1 \leq i \leq 5$ and a non-maximum PPC $P_6 = \{\{0, 1, 2\}\}$.

$$P_1 = \{\{5, 8, 9\}, \{1, 3, 6\}, \{2, 4, 7\}, \{12, 11, 10\}\}, \overline{P_1} = \{0\};$$
$$P_2 = \{\{0, 3, 7\}, \{4, 12, 8\}, \{11, 6, 5\}, \{2, 9, 10\}\}, \overline{P_2} = \{1\};$$
$$P_3 = \{\{3, 9, 12\}, \{1, 5, 7\}, \{8, 10, 6\}, \{11, 4, 0\}\}, \overline{P_3} = \{2\};$$
$$P_4 = \{\{12, 6, 7\}, \{2, 8, 3\}, \{10, 0, 5\}, \{9, 1, 11\}\};$$
$$P_5 = \{\{9, 0, 6\}, \{11, 7, 8\}, \{5, 2, 12\}, \{1, 4, 10\}\}.$$

For $n = 14$, take a $(14, 3)$-packing over $\mathbb{Z}_{14}$ whose blocks are partitioned into the following five maximum PPCs and a non-maximum PPC $P_6 = \{\{0, 4, 8\}, \{6, 10, 2\}\}$.

$$P_1 = \{\{4, 11, 12\}, \{10, 5, 13\}, \{2, 9, 7\}, \{3, 8, 1\}\}, \overline{P_1} = \{0, 6\};$$
$$P_2 = \{\{0, 9, 12\}, \{1, 2, 13\}, \{5, 3, 6\}, \{11, 7, 8\}\}, \overline{P_2} = \{4, 10\};$$
$$P_3 = \{\{1, 6, 12\}, \{0, 7, 13\}, \{4, 5, 9\}, \{3, 11, 10\}\}, \overline{P_3} = \{8, 2\};$$
$$P_4 = \{\{10, 8, 12\}, \{3, 4, 13\}, \{6, 11, 9\}, \{0, 5, 2\}\}, \overline{P_4} = \{1, 7\};$$
$$P_5 = \{\{2, 3, 12\}, \{8, 9, 13\}, \{0, 1, 10\}, \{4, 6, 7\}\}, \overline{P_5} = \{5, 11\}.$$

We can check that the PPCs of $(n, 3)$-packings for $n \in \{11, 12, 13, 14\}$ satisfy the two properties of strong Hanani triple packings. Thus, we can use methods similar to that in the proof of Proposition 3.5 to construct optimal $(n, 5, 3)_q$-codes for $2 \leq q \leq \lfloor \frac{n-1}{2} \rfloor$. □

Lemma 6.2 $A_q(n, 5, 3) = \left\lfloor \frac{(q-1)n}{3} \right\rfloor$ *for all* $n \equiv 5 \bmod 6$, $n \geq 17$ *and* $q = \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof Let $n = 6t + 5$, where $t \geq 2$. Take a 3-GDD $(X, \mathcal{G}, \mathcal{B})$ of type $3^{2t}5^1$ [19, Theorem 4.2], where $X = \mathbb{Z}_{3t} \cup \{\infty_0, \ldots, \infty_4\}$ and $\{\infty_0, \ldots, \infty_4\}$ is the long group. Then $\mathcal{B} \cup \{\{\infty_0, \infty_1, \infty_2\}\}$ is an $(n, 3)$-packing of size $6t^2 + 7t + 1 = \left\lfloor \frac{(q-1)n}{3} \right\rfloor$. Then the optimal codes can be obtained using the same technique in the proof of Proposition 2.7, since each point occurs at most $3t + 1$ times in the packing. □

Lemma 6.3 $A_q(n, 5, 3) = \left\lfloor \frac{(q-1)n}{3} \right\rfloor$ *for* $n \in \{29, 35, 47\}$ *and* $2 \leq q \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof Let $n = 6t + 5$, where $t \in \{4, 5, 7\}$. We construct an $(n, 3)$-packing over $\mathbb{Z}_{6t} \cup \{\infty_0, \ldots, \infty_4\}$, where the block set consists of $3t$ maximum PPCs P_i, $1 \leq i \leq 3t$, and a non-maximum PPC $P_{3t+1} = \{\{0, 2t, 4t\} + i : i = 0, 1, \ldots, 2t - 1\}$.

For each $n \in \{29, 35, 47\}$, the first maximum PPC P_1 which misses $\{0, 1\}$ is listed below. For each $1 \leq i \leq t - 1$, P_{3i+1} is obtained from P_1 by adding $2i$ under $\mathbb{Z}_{6t}$. For all other maximum PPCs, P_{i+1} is obtained from P_i by adding $2t$ under $\mathbb{Z}_{6t}$. For each $2 \leq i \leq 3t$, $\overline{P_i}$ is obtained the same way.

$$
\begin{aligned}
29:\ & \{\{2, 4, 7\}, \{3, 5, 6\}, \{8, 17, \infty_0\}, \{9, 15, 19\}, \{10, 16, 20\}, \{11, 22, \infty_1\},\\
& \{12, 23, \infty_2\}, \{13, 18, \infty_3\}, \{14, 21, \infty_4\}\}.\\
35:\ & \{\{2, 4, 7\}, \{8, 12, 19\}, \{9, 20, \infty_0\}, \{10, 23, 27\}, \{11, 26, \infty_1\}, \{3, 5, 6\},\\
& \{13, 18, \infty_2\}, \{14, 22, 28\}, \{15, 21, 29\}, \{16, 25, \infty_3\}, \{17, 24, \infty_4\}\}.\\
47:\ & \{\{2, 4, 7\}, \{8, 12, 18\}, \{9, 13, 19\}, \{10, 17, 32\}, \{24, 39, \infty_0\}, \{3, 5, 6\},\\
& \{14, 33, \infty_2\}, \{15, 26, 34\}, \{16, 28, 37\}, \{20, 31, 38\}, \{11, 27, 36\},\\
& \{21, 29, 41\}, \{22, 35, \infty_3\}, \{23, 40, \infty_4\}, \{25, 30, \infty_1\}\}.
\end{aligned}
$$

We can check that the PPCs of $(n, 3)$-packings for $n \in \{29, 35, 47\}$ satisfy the two properties of strong Hanani triple packings. Thus, we can use methods similar to that in the proof of Proposition 3.5 to construct optimal $(n, 5, 3)_q$-codes for $2 \leq q \leq 3t + 1$. When $q = 3t + 2$, the optimal codes are from Proposition 6.2. □

Lemma 6.4 $A_q(41, 5, 3) = \left\lfloor \frac{41(q-1)}{3} \right\rfloor$ *for all integers* q, $2 \leq q \leq 20$.

Proof We construct a $(41, 3)$-packing on $\mathbb{Z}_{36} \cup \{\infty_0, \ldots, \infty_4\}$, where the block set consists of 18 maximum PPCs P_i, $1 \leq i \leq 18$, and a non-maximum PPC $P_{19} = \{\{0, 12, 24\} + i : i = 0, 1, \ldots, 11\}$. The maximum PPCs P_1, P_{10} missing $\{0, 1\}$ and $\{2, 3\}$ are listed below.

For each $i \in \{1, 2\}$, $j \in \{1, 10\}$, P_{3i+j} is obtained from P_j by adding $4i$ under $\mathbb{Z}_{36}$. For each $i \in \{0, 1, \ldots, 5\}$, $j \in \{2, 3\}$, P_{3i+j} is obtained from P_{3i+j-1} by adding 12 under $\mathbb{Z}_{36}$. For each $i \in I_{18} \backslash \{1, 10\}$, $\overline{P_i}$ is obtained the same way.

$$
\begin{aligned}
P_1 =\ & \{\{\infty_0, 2, 4\}, \{\infty_1, 3, 5\}, \{\infty_2, 6, 9\}, \{\infty_3, 7, 8\}, \{\infty_4, 10, 15\},\\
& \{11, 14, 18\}, \{12, 16, 19\}, \{13, 17, 20\}, \{21, 26, 35\}, \{22, 29, 30\},\\
& \{23, 28, 33\}, \{24, 32, 34\}, \{25, 27, 31\}\};\\
P_{10} =\ & \{\{\infty_0, 1, 11\}, \{\infty_1, 0, 14\}, \{\infty_2, 12, 35\}, \{\infty_3, 17, 34\}, \{\infty_4, 4, 21\},\\
& \{5, 18, 28\}, \{6, 19, 27\}, \{7, 16, 25\}, \{8, 23, 29\}, \{9, 20, 31\},\\
& \{10, 24, 30\}, \{13, 22, 33\}, \{15, 26, 32\}\}.
\end{aligned}
$$

We can check that the PPCs of this $(41, 3)$-packing satisfy the two properties of strong Hanani triple packings. Thus we can use methods similar to that in the proof of Proposition 3.5 to construct optimal $(41, 5, 3)_q$-codes for $2 \leq q \leq 19$. When $q = 20$, the optimal code is from Proposition 6.2. □

Lemma 6.5 $A_q(59, 5, 3) = \left\lfloor \frac{59(q-1)}{3} \right\rfloor$ *for all integers* q, $2 \leq q \leq 29$.

Proof Take a 3-frame $(X, \mathcal{G}, \mathcal{B})$ of type $12^4 6^1$ from Lemma 2.4, where $X = (\mathbb{Z}_{12} \times I_4) \cup (\mathbb{Z}_6 \times \{5\})$, $\mathcal{G} = \{\mathbb{Z}_{12} \times \{i\} : 1 \leq i \leq 4\} \cup \{\mathbb{Z}_6 \times \{5\}\}$ and $\mathcal{B}$ is partitioned into PPCs of 14 blocks P_i, $1 \leq i \leq 24$, and PPCs of 16 blocks P_i, $25 \leq i \leq 27$. Assume that for each $1 \leq i \leq 4$, P_j, $6i - 5 \leq j \leq 6i$, are PPCs missing the points in the group $\mathbb{Z}_{12} \times \{i\}$, and P_j, $25 \leq j \leq 27$ are PPCs missing the points in the group $\mathbb{Z}_6 \times \{5\}$. Let $Y = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. For each $1 \leq i \leq 4$, construct a strong HTP(17) on $(\mathbb{Z}_{12} \times \{i\}) \cup Y$, with eight maximum PPCs P_j^i, $1 \leq j \leq 8$, and a non-maximum PPC P_9^i, such that $\{\infty_0, \infty_1, \infty_2\} \in P_7^i$, $\overline{P_7^i} = \{\infty_3, \infty_4\}$, $\{\infty_0, \infty_3, \infty_4\} \in P_8^i$ and $\overline{P_8^i} = \{\infty_1, \infty_2\}$.

Let $P'_{j+6(i-1)} = P_{j+6(i-1)} \cup P_j^i$ for $1 \leq i \leq 4$ and $1 \leq j \leq 6$. Let $P'_{25} = \cup_{i=1}^{s} P_9^i$, which is a non-maximum PPC. Then $(X \cup Y, \cup_{i=1}^{25} P'_i)$ is a $(59, 3)$-packing satisfying the two properties of strong Hanani triple packings, from which we can get optimal $(59, 5, 3)_q$-codes for $2 \leq q \leq 25$.

Now for $2 \leq k \leq 4$, we construct an optimal k-ary code of length 11 over $(\mathbb{Z}_6 \times \{5\}) \cup Y$, denoted by $\mathcal{D}_k$. Add 24 to all the nonzero components to get a code $\mathcal{D}_{k+24}$, such that the nonzero elements come from $\{25, 26, 27\}$. For $26 \leq q \leq 28$, $\mathcal{C}_q = \mathcal{C}_{25} \cup (\cup_{i=25}^{q-1} \mathcal{C}(P_i, i)) \cup \mathcal{D}_q$ is an optimal $(59, 5, 3)_q$-code. For $q = 29$, the optimal code is from Proposition 6.2. □

Combining Corollary 2.8, Proposition 3.5, Theorem 4.10 and the lemmas in this section, we have the following result.

Theorem 6.6 $A_q(n, 5, 3) = \min\left\{\left\lfloor \frac{(q-1)n}{3} \right\rfloor, D(n, 3)\right\}$, *where*

$$D(n, 3) = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1, & \textit{if } n \equiv 5 \bmod 6, \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor, & \textit{otherwise.} \end{cases}$$

7 Conclusion

This paper investigates constructions for optimal $(n, 5, 3)_q$-codes for all integers n and $q \geq 2$ via the study of Hanani triple packings, a generalization of the well known Hanani triple systems. We establish the existence of strong Hanani triple packings, with a small finite number of possible exceptions and determine $A_q(n, 5, 3)$ for all n and $q \geq 2$. Previously, the exact value of $A_q(n, 5, 3)$ is known only for $q \in \{2, 3\}$, and for general q with $3|(q-1)n$ and sufficiently large n.

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