Six New Constant Weight Binary Codes

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Abstract

We give six improved bounds on $A(n,d,w)$, the maximum cardinality of a binary code of length n with minimum distance d and constant weight w.

1 Introduction

A *binary code* of *length* n is any set $\mathcal{C} \subseteq \{0,1\}^n$. The elements of $\mathcal{C}$ are called *codewords*. $\mathcal{C}$ is said to have *minimum distance* d and *constant weight* w if the Hamming distance between any two distinct codewords is at least d and $\|\mathbf{u}\|^2 = w$ for all $\mathbf{u} \in \mathcal{C}$. For simplicity, we refer to a binary code of length n, minimum distance d, and constant weight w as an (n,d,w)-*code*. We also assume without loss of generality that d is even. Define $A(n,d,w)$ to be the maximum cardinality of an (n,d,w)-code, that is,

$$A(n,d,w) = \max\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n,d,w)-\text{code}\}.$$

The function $A(n, d, w)$ is fundamental in the theory of error-correcting codes [2]. Unfortunately, the exact determination of $A(n, d, w)$ is difficult. Most efforts have therefore focused on establishing good bounds for $A(n, d, w)$. The function $A(n, d, w)$ is also widely studied in combinatorial design theory, under the guise of packing designs [3].

In this paper, we give some improved lower bounds on $A(n, d, w)$.

2 Results

2.1 A Cyclic $(30, 8, 5)$-Code

The set of all distinct cyclic shifts of the two vectors

$$100000100000100000100000100000$$

$$110100000010001000000000000000$$

is a $(30, 8, 5)$-code with 36 codewords.

2.2 Length-Reduction Heuristic

We represent a binary code $\mathcal{C}$ by a $\{0,1\}$-matrix $M(\mathcal{C})$ whose columns are the codewords of $\mathcal{C}$. Let $\mathcal{M}_{n,m}(d, w)$ be the set of all $n \times m$ $\{0,1\}$-matrices M with constant column sum w, such that the Hamming distance between any two distinct columns $\mathbf{u}$ and $\mathbf{v}$ of M is at least d. So for an (n, d, w)-code $\mathcal{C}$, we have $M(\mathcal{C}) \in \mathcal{M}_{n,|\mathcal{C}|}(d, w)$. For a positive integer i, $M \in \{0,1\}^{n \times m}$, and $\mathbf{u} \in \{0,1\}^n$, we denote by $M_i(\mathbf{u})$ the matrix obtained by replacing the ith column of M by $\mathbf{u}$. We also denote by $\tilde{M}$ the matrix obtained from M by deleting its last row.

The `length-reduction` heuristic works as follows. The inputs are n, m, d, and w, where $n \geq w \geq d/2$. We begin with $M \in \mathcal{M}_{N,m}(d, w)$, for some N. At each stage of the heuristic, we generate a random integer i and a random element $\mathbf{u} \in \{0,1\}^n$ whose last component is zero. If $M_i(\mathbf{u}) \in \mathcal{M}_{n,m}(d, w)$, then we replace M by $M_i(\mathbf{u})$. It could happen at this point that the last row of M is a zero vector. If this is the case, we replace M by $\tilde{M}$, and repeat the process. We stop when M has only n rows.

Let

$$I_{m,w} = \begin{bmatrix} \mathbf{1}_w & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_w & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_w \end{bmatrix},$$

where $\mathbf{1}_w$ is the w-dimensional column vector of all ones. Clearly, $I_{m,w} \in \mathcal{M}_{mw,m}(d,w)$ for any $d \leq 2w$. For our experiments, the initial choice of M is $I_{m,w}$.

`length-reduction` **heuristic**(n, m, d, w)

Step 1: $M = I_{m,w}$ and $N = mw$.

Step 2: Repeat Step 3 to Step 5 until $N = n$.

Step 3: Randomly choose $i \in \{1, 2, \ldots, m\}$ and $\mathbf{u} \in \{0,1\}^N$.

Step 4: If $M_i(\mathbf{u}) \in \mathcal{M}_{N,m}(d,w)$, then $M = M_i(\mathbf{u})$.

Step 5: If the last row of M is the zero vector, then $M = \tilde{M}$ and set $N = N - 1$.

It is easy to see that when the heuristic terminates, we have M as the matrix of an (n,d,w)-code of cardinality m.

Most algorithms and heuristics for constructing constant weight binary codes attempts to pack as many codewords into an (n,d,w)-code as possible, given n, d, and w. Here, the `length-reduction` heuristic takes the alternative approach of minimizing the length n of an (n,d,w)-code, given d, w, and its cardinality.

2.3 New Bounds

The `length-reduction` heuristic has been used to produce five new lower bounds on $A(n,d,w)$.

Theorem 1. $A(18,6,5) \geq 69$, $A(27,8,5) \geq 31$, $A(29,8,5) \geq 34$, $A(33,8,5) \geq 44$ and $A(34,8,5) \geq 47$.

Proof. The matrices in Appendix A represent the necessary (n,d,w)-codes for providing the lower bounds on $A(n,d,w)$. □

3 Conclusion

The following table summarizes the results obtained in this paper.

n	d	w	best lower bound on $A(n, d, w)$ previously known [1, 4, 5]	lower bound on $A(n, d, w)$ obtained in this paper
18	6	5	68	69
27	8	5	30	31
29	8	5	33	34
30	8	5	33	36
33	8	5	43	44
34	8	5	43	47

References

[1] A. E. Brouwer, J. B. Shearer, N. J. A. Sloane, and W. D. Smith, A new table of constant weight codes, *IEEE Trans. Inform. Theory* **36** (1990) 1334–1380.

[2] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam 1977.

[3] W. H. Mills and R. C. Mullin, Coverings and packings, in: *Contemporary Design Theory: A Collection of Surveys* (J. H. Dinitz, D. R. Stinson; eds.) John Wiley & Sons, New York, 1992, 371–399.

[4] E. M. Rains and N. J. A. Sloane, Table of constant weight binary codes, `URL=http://www.research.att.com/~njas/codes/Andw/`.

[5] J. Yin and A. M. Assaf, Constructions of optimal packing designs, *J. Combin. Designs* **6** (1998) 245–260.

Appendix

A Constant Weight Codes

The columns in the following matrices give the codewords for each code.

```
111111111111111111110000000000000000000000000000000000000000000000000
111110000000000000001111111111111100000000000000000000000000000000000
100001111000000000001111000000000001111111111100000000000000000000000
100000000010100000001000000010010010010100101001000000001001100001110
100000000001011001000000010010010101001000010101001000000100110001000
010001000111000000000100011100000001110000000001111111000000000000000
010000100000000001010001000010001010010001000110001000001010100100001
010000001000010100010001000000101010001010000000110100000010010011001
001001000000000001101000101000010010000010001101000111000100110000010
001000100100110000000100000001110000001110000000100000111110000000000
001000010001000100100100001000001001001000000110001001010101001000011
000101000000010110000001001000000000000000110001011000100001111100000
000100010010010010001000000000011000000000101100001101110010000010000
000100001100000000110001001001000100101000000100010000010010101010100
000010100010000101000000110001001000100000011000000001100100000111010
000010010100001000010010001001100001000010100000010001001100001010101
000010001001100010000100010100010000000100100100000000010000010100111
000000000000000100010000000000000010001000000000011010000001000101100
```

$(18, 6, 5)$-code with 69 codewords

```
1111110000000000000000000000000
1000001111100000000000000000000
1000000000010000100000000010110
1000000000001000010000111000000
1000000000000100000101000100001
0100001000011110000000000000000
0100000100000000000000001101010
0100000010000000001000010000101
0100000000100001010001000010000
0010001000000000000110010000010
0010000100000001100000000000000
0010000010000100010000000001000
0010000000100001000000110000100
0001001000000000000000100011001
0001000100010000001110000000000
0001000000100010100000010100000
0001000000000101000010001000100
0000100100001000000001100000000
0000100010000010000100001010000
0000100001010001000000010001000
0000100000100100001000100000010
0000011000000000101001001000000
0000010010010000000010100100000
0000010000101000000100000001100
0000010000000011000000000000011
0000000001001000000100000110000
0000000001000000110010000000001
```

$(27, 8, 5)$-code with 31 codewords

```
1111110000000000000000000000000000
1000001000001111000000000000000000
1000000100000000111100000000000000
1000000000100000000001000100010010
1000000000010000000000100000001101
0100001000000000010000001110000000
0100000100000000000011110000000000
0100000001001000000100000001001000
0100000000000010100000000000100011
0010000100000100000000000100101000
0010000010001000100010001000000000
0010000001000000000100010000010001
0010000000010010000001000011000000
0001001000000000000010000001110000
0001000100001000000000000010000110
0001000010000001010001000000000001
0001000000010100001000011000000000
0000101000000000000100000000001010
0000100010000000000000010101000100
0000100001000001000000101000100000
0000100000100000001010000010000001
0000010010000010001000100000010000
0000010001000100100001000000000100
0000010000101000010000010000100000
0000010000010001000110000100000000
0000001111100000000000000000000000
0000000000100010000100001000000100
0000000000000100010000100001000010
0000000000000001100000000010011000
```

$(29, 8, 5)$-code with 34 codewords

```
11111110000000000000000000000000000000000000
10000001111110000000000000000000000000000000
10000000000000100000000000000000101000100010
10000000000000010001001000000010000110000000
10000000000000000000000010010000010000011100
01000001000000000010000000001000010100000001
01000000100000000000100010000010000011000000
01000000001000000001000000010101001000000000
01000000000001111100000000000000000000000000
00100000100000010000000100000001000000001001
00100000000010000100000011000000100010000000
00100000000001000011110000000000000000000000
00100000000000001000000000000100000101010000
00010001000000100000001110000000000000000000
00010000100000001010000000000000000010000110
00010000000010000000010000001010001000001000
00010000000000000101000000000000000000110001
00001001000001000000000000100000001010010000
00001000010000000010000001000110000000000000
00001000001000001000011000000000010000100000
00001000000100100000000000001001000001000100
00000101000000000000100001000001000000000010
00000100001000000110000000100000000001001000
00000100000100001000000100010010100000000000
00000100000011000000001000000100000000000101
00000010010000100000010000010000000010000001
00000010001000010000100000001000100000010000
00000010000101000000000001000000000100101000
00000010000010000001000100000000010001000010
00000000100000000000001001111000000000000000
00000000010000000100100100000000001100000100
00000000010000000000000000100001110000000000
00000000000100010000010010100100000000000010
```

(33, 8, 5)-code with 44 codewords

```
11111111000000000000000000000000000000000000000
10000000111111000000000000000000000000000000000
10000000000000010000000000100001000000010011000
10000000000000001000011000000000000100001000100
10000000000000000101000010001000000011000000000
01000000100000111000000000000000000000000000000
01000000010000000000000000000011000010000000101
01000000001000000001000001000000000100000101010
01000000000100000000100100000100100000000010000
00100000010000100011110000000000000000000000000
00100000001000010000001000011110000000000000000
00100000000010000000000010000000101000001100000
00100000000000001000000100000000000000010000011
00010000000010000100010000000010000000100010000
00010000000000100000000000000100000000010100100
00010000000000000000100011111000000000000000000
00010000000000000000100000010000011100000000000
00001000100000000001000000010001100000000000000
00001000001000100000000010000000010000000010001
00001000000010000000100001001000000000000000100
00001000000001010000000100000000000101001000000
00000100100000000000010000100100000010000000010
00000100001000000000100000000000000000111000000
00000100000100001010000000001001010000000100000
00000100000001000100001000000000100000000001001
00000010100000000000000100000010010001000001000
00000010010000000000000000101000100100100000000
00000010000100000000010001000000001000010000000
00000010000001000010000000010000000000001010010
00000001010000000100000001000100010000010000000
00000001000100000000000010010000000000100001100
00000001000010010010000000000000000000101000000 1
00000001000001001001000000100010001000000000000
00000000000000010000001000000000100100100000010
```

(34, 8, 5)-code with 47 codewords