Graphs and Combinatorics (2002) 18:421–445

Graphs and Combinatorics

Uniform Group Divisible Designs with Block Sizes Three and n*

Yeow Meng Chee[1] and Alan C.H. Ling[2]

[1] 12 Jambol Place, Singapore 119339. e-mail: ymchee@alumni.uwaterloo.ca
[2] Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan 49931-1295, USA. e-mail: aling@mtu.edu

Abstract. The existence of group divisible designs of type g^t with block sizes three and n, $4 \leq n \leq 10$, is completely settled for all values of g and t.

1. Introduction

A *set system* is a pair $(X, \mathscr{A})$, where X is a finite set of *points* and $\mathscr{A}$ is a set of subsets of X, called *blocks*. The set $\mathscr{K}$ is a *set of block sizes* for $(X, \mathscr{A})$ if $|A| \in \mathscr{K}$ for every $A \in \mathscr{A}$. For a set $\mathscr{K}$ of block sizes for $(X, \mathscr{A})$, if an element $k \in \mathscr{K}$ is superscripted with a $*$, it indicates that there is exactly one block of size k in $(X, \mathscr{A})$. We note that if $\mathscr{K}$ is a set of block sizes for $(X, \mathscr{A})$, then any set containing $\mathscr{K}$ is also a set of block sizes for $(X, \mathscr{A})$.

Let $(X, \mathscr{A})$ be a set system with set of block sizes $\mathscr{K}$. If $(X, \mathscr{A})$ has the property that every 2-subset of X appears in precisely one block, it is a *pairwise balanced design* (PBD), and is denoted by $\mathscr{K}$-PBD$(|X|)$. A $\{k\}$-PBD(v) is a *Steiner 2-design* $\mathrm{S}(2, k, v)$. An important idea in the study of PBDs is that of closure. Let $B(\mathscr{K})$ denote the set of positive integers v for which there exists a $\mathscr{K}$-PBD(v). The set $B(\mathscr{K})$ is called the *PBD-closure* of the set $\mathscr{K}$. A *partial design* is a set system $(X, \mathscr{A})$ for which every 2-subset of X is contained in *at most* one block.

Let $(X, \mathscr{A})$ be a set system, and let $\mathscr{G} = \{G_1, \ldots, G_s\}$ be a partition of X into subsets, called *groups*. The triple $(X, \mathscr{G}, \mathscr{A})$ is a *group divisible design* (GDD) when every 2-subset of X not contained in a group appears in exactly one block and $|A \cap G| \leq 1$ for all $A \in \mathscr{A}$ and $G \in \mathscr{G}$. We denote a GDD $(X, \mathscr{G}, \mathscr{A})$ by $\mathscr{K}$-GDD if $\mathscr{K}$ is the set of block sizes for $(X, \mathscr{A})$. The *group-type*, or simple *type*, of a GDD $(X, \mathscr{G}, \mathscr{A})$ is the multiset $[|G| \mid G \in \mathscr{G}]$. When more convenient, we use the exponential notation to describe the type of a GDD: A GDD of type ${g_1}^{t_1} \cdots {g_s}^{t_2}$ is a

* This work was done in 1995 while the authors were graduate students at the University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

GDD where there are exactly t_i groups of size g_i, for $1 \le i \le s$. A GDD is *uniform* if all its groups have the same size, that is, if it is of type g^t. A $\{k\}$-GDD of type m^k is called a *transversal design*, and is denoted by $\mathrm{TD}(k, m)$.

GDDs and PBDs are intimately related. First, a PBD is a GDD with groups of size one. On the other hand, a $\mathscr{K}$-GDD of type $[g_1, \ldots, g_s]$ can be viewed as a $\mathscr{K} \cup \{g_1, \ldots, g_s\}$-PBD$(\sum_{i=1}^{s} g_i)$ by considering the groups of the GDD to be blocks of the PBD also. Such a GDD can also be used to create a $\mathscr{K} \cup \{g_1 + 1, \ldots, g_s + 1\}$-PBD$(1 + \sum_{i=1}^{s} g_i)$ by adjoining a new point to each group, and considering the resulting subsets as blocks. Conversely, a GDD can be obtained from a PBD by deleting a point.

GDDs play an important role in the construction of many different classes of combinatorial designs. Consequently, there has been much interest in the construction of rich classes of GDDs. The existence of uniform $\{3\}$-GDDs is completely settled by Hanani [8].

Theorem 1.1 (Hanani). *There exists a* $\{3\}$-*GDD of type* g^t *if and only if* $t \ge 3$, $g^2\binom{t}{2} \equiv 0 \pmod 3$, *and* $g(t-1) \equiv 0 \pmod 2$.

This result is extended by Colbourn, Hoffman and Rees [5] who proved the following.

Theorem 1.2 (Colbourn, Hoffman and Rees). *Let* g, t, *and* u *be nonnegative integers. There exists a* $\{3\}$-*GDD of type* $g^t u^1$ *if and only if the following conditions are all satisfied*:

(i) *if* $g > 0$, *then* $t \ge 3$, *or* $t = 2$ *and* $u = g$, *or* $t = 1$ *and* $u = 0$, or $t = 0$;
(ii) $u \le g(t-1)$ *or* $gt = 0$;
(iii) $g(t-1) + u \equiv 0 \pmod 2$ *or* $gt = 0$;
(iv) $gt \equiv 0 \pmod 2$ *or* $u = 0$;
(v) $g^2\binom{t}{2} + gtu \equiv 0 \pmod 3$.

More recently, Colbourn, Cusack and Kreher [4] provided necessary and sufficient conditions for the existence of $\{3\}$-GDDs of type $g^t 1^r$.

Theorem 1.3 (Colbourn, Cusack and Kreher). *Let* g, t, *and* r *be positive integers. Then there exists a* $\{3\}$-*GDD of type* $g^t 1^r$ *if and only if the following conditions are all satisfied*:

(i) $g \equiv 1 \pmod 2$;
(ii) $t + r \equiv 1 \pmod 2$;
(iii) *if* $t = 1$, *then* $r \ge g + 1$;
(iv) if $t = 2$, *then* $r \ge g$;
(v) $g^2\binom{t}{2} + gtr + \binom{r}{2} \equiv 0 \pmod 3$.

In this paper, we establish the existence of large classes of GDDs by extending Hanani's result in another direction. More specifically, we settle completely the existence question for (uniform) $\{3, n\}$-GDDs of type g^t, for $4 \le n \le 10$, and all values of g and t. The result is trivial when $gt = 0$ or $t = 1$ since there can be no blocks. When $g = 1$, such a $\{3, n\}$-GDD is just a $\{3, n\}$-PBD(t), whose existence for all $n \leq 10$ has been determined [7]. So we assume both g and t to be at least two.

Main Theorem. *Let* $4 \leq n \leq 10$, $g \geq 2$ *and* $t \geq 2$. *There exists a* $\{3, n\}$*-GDD of type* g^t *if and only if the following conditions are all satisfied*:

(i) $t \geq 3$;
(ii) *if* $g^2\binom{t}{2} \not\equiv 0 \pmod 3$ or $g(t-1) \equiv 1 \pmod 2$, *then* $t \geq n$;
(iii) $gt \in B(\{3, n, g\})$;
(iv) $gt + 1 \in B(\{3, n, g+1\})$;
(v) *if* $n \equiv 0$ *or* $1 \pmod 3$, *then* $g^2\binom{t}{2} \equiv 0 \pmod 3$;
(vi) *if* $n \equiv 1 \pmod 2$, *then* $g(t-1) \equiv 0 \pmod 2$;
(vii) *if* $n \equiv 2 \pmod 6$, $g^2\binom{t}{2} \not\equiv 0 \pmod 3$, $g(t-1) \equiv 0 \pmod 2$, *then* $gt \geq (n+1)(3n+2)/6$;
(viii) *if* $n = 8$, *then* $(g, t) \neq (5, 8)$.

The remainder of this paper proves this theorem. We first examine necessity of the conditions, which is quite straightforward.

Condition (i) follows from the fact that in a $\{3, n\}$-GDD, every block intersects at least three groups. Condition (ii) is an extension of this observation. When $g^2\binom{t}{2} \not\equiv 0 \pmod 3$ or $g(t-1) \equiv 1 \pmod 2$, there cannot exist a $\{3\}$-GDD of type g^t. Therefore, there must be at least one block of size n, and hence there are at least n groups.

Conditions (iii) and (iv) follow from earlier remarks concerning the relationship between GDDs and PBDs.

For condition (v), observe that the number of 2-subsets contained in each block is a multiple of three, so the number of 2-subsets not contained in a group must be a multiple of three. For condition (vi), observe that every point must be in an even number of 2-subsets that are contained in the blocks. Hence $g(t-1) \equiv 0 \pmod 2$.

For (vii), we must have at least one block of size n, and each point is in an even number of blocks of size n. The following lemma shows that the fewest number of blocks of size n that such a configuration can have is $n+1$.

Lemma 1.1. *Let* $(X, \mathscr{A})$ *be a partial design for which every point is contained in an even number of blocks and there is a block of size* n, *then* $|\mathscr{A}| \geq n+1$.

Proof. Let A be a block of size n. For each $x \in A$, let $\mathscr{B}_x$ denote the set of all blocks, other than A, that contain x. Since every point is contained in an even number of blocks, $\mathscr{B}_x$ is nonempty for each $x \in A$. Now, $\mathscr{B}_x \cap \mathscr{B}_{x'} = \emptyset$ for distinct points $x, x' \in A$, for otherwise there would exist a block in addition to A that contains both x and x', contradicting the fact that $(X, \mathscr{A})$ is a partial design. Hence, $\cup_{x \in A} \mathscr{B}_x$ contains at least n distinct blocks, which together with A, give $n+1$ blocks. □

However, the number of blocks of size n must not be divisible by three. Hence, we must have at least $n+2$ blocks of size n, forming a partial design. The following result of Mendelsohn and Rees [11] is useful.

Theorem 1.4 (Mendelsohn and Rees). *Let $(X, \mathscr{A})$ be a partial design with block size k, in which there are b blocks. Then letting $r = \lfloor (b-1)/k \rfloor$, we have*

$$|X| \geq \frac{b(2rk - b + 1)}{r(r+1)}. \tag{1}$$

It is not difficult to verify that the quantity on the right hand side of inequality (1) is an increasing function of b. Using Theorem 1.4, we see that the partial design formed by the blocks of size n must contain at least $\lceil (n+2)(3n-1)/6 \rceil$ points. Hence, $gt \geq (n+1)(3n+2)/6$.

We now treat the nonexistence of a $\{3, 8\}$-GDD of type 5^8. In a $\{3, 8\}$-GDD $(X, \mathscr{A})$ of type 5^8, there are $b_8 \equiv 1 \pmod 3$ blocks of size eight, and each point lies on an odd number r_8 of blocks of size eight. Since there are five points in each group, and r_8 is odd, the number b_8 of blocks of size eight must be odd. Therefore, $b_8 \equiv 1 \pmod 6$, and $b_8 \geq 7$. If there are more than ten blocks of size eight, then by Theorem 1.4, there must be at least 41 points. So we only have to consider the case when $b_8 = 7$.

Suppose there are seven blocks of size eight.Then there are eight points, each of which lies on precisely three blocks of size eight. Consider the dual incidence structure of these seven blocks; let $\mathscr{Y}$ be the set of blocks of size eight and for each point $x \in X$ that lies in more than one block of size eight, let B_x be the set of blocks containing x. Since any two blocks intersect in at most one point, the dual structure $(\mathscr{Y}, \cup_{x \in X}\{B_x\})$ is a partial design, with seven points and eight blocks of size three. This is impossible. If there are ten blocks of size eight, then there are 20 points, each of which lies on exactly three blocks of size eight. The dual incidence structure of these ten blocks is a partial design with ten points and 20 blocks of size three. This is again impossible.

This completes the proof of necessity for the conditions in the Main Theorem. The necessary conditions for the existence of $\{3, n\}$-GDDs of type g^t for $n \equiv 1$ or $3 \pmod 6$ is identical to the necessary conditions for the existence of $\{3\}$-GDDs of type g^t. Since all $\{3\}$-GDDs of type g^t satisfying these necessary conditions exist by Theorem 1.1, we do not need to consider the cases $n = 7$ and 9 here.

2. Recursive Constructions

We prove sufficiency for the conditions in the Main Theorem by developing a set of recursive constructions which we present in this section. First, we require some definitions.

A *parallel class* in a GDD is a set of disjoint blocks that contain each point of the GDD exactly once. A GDD is *resolvable* if all of its blocks can be partitioned into parallel classes.

Let $(X, \mathscr{A})$ be a set system with set of block sizes $\mathscr{K}$, let $\mathscr{G} = \{G_1, \ldots, G_s\}$ be a partition of X into groups, and let $\mathscr{H} = \{H_1, \ldots, H_s\}$ be a set of pairwise disjoint subsets of X, called *holes*, with the property that $H_i \subseteq G_i$, for $1 \leq i \leq s$. The quadruple $(X, \mathscr{H}, \mathscr{G}, \mathscr{A})$ is an *incomplete group divisible design* (IGDD), denoted

$\mathscr{K}$-IGDD, if any 2-subset of X not contained in a group or $\cup_{i=1}^{s} H_i$ appears in precisely one block. The *type* of an IGDD is the multiset $[(|G_1|,|H_1|),\ldots,(|G_s|,|H_s|)]$. As usual, when it is more convenient, we use the exponential notation to represent the type of an IGDD. An *incomplete transversal design* (ITD), $\mathrm{TD}(k,m){-}\mathrm{TD}(k,h)$ is a $\{k\}$-IGDD of type $(m,h)^k$. By considering each point on a fixed block of a $\mathrm{TD}(k,m)$ as a hole of size one, we see that the existence of a $\mathrm{TD}(k,m)$ implies the existence of a $\mathrm{TD}(k,m){-}\mathrm{TD}(k,1)$.

We now present the tools of this paper. The main recursion we use is Wilson's Fundamental Construction (WFC) for GDDs [14].

Theorem 2.1 (WFC). *Let $(X,\mathscr{G},\mathscr{A})$ be a (master) GDD, where $\mathscr{G}=\{G_1,\ldots,G_s\}$. Let $\omega: X\to\mathbb{Z}_{\geq 0}$ be a weight function. Suppose that for each block $A=\{x_1,\ldots,x_k\}\in\mathscr{A}$, there exists an (ingredient) $\mathscr{K}$-GDD of type $[\omega(x_1),\ldots,\omega(x_k)]$. Then there exists a $\mathscr{K}$-GDD of type $[\sum_{x\in G_1}\omega(x),\ldots,\sum_{x\in G_s}\omega(x)]$.*

The following Wilson-style theorem for IGDDs follows easily from the proof for Theorem 2.1 [14].

Theorem 2.2. Let *$(X,\mathscr{H},\mathscr{G},\mathscr{A})$ be a (master) IGDD, where $\mathscr{G}=\{G_1,\ldots,G_s\}$ and $\mathscr{H}=\{H_1,\ldots,H_s\}$. Let $\omega: X\to\mathbb{Z}_{\geq 0}$ be a weight function. Suppose that for each block $A=\{x_1,\ldots,x_k\}\in\mathscr{A}$, there exists an (ingredient) $\mathscr{K}$-GDD of type $[\omega(x_1),\ldots,\omega(x_k)]$. Then there exists a $\mathscr{K}$-IGDD of type $[(\sum_{x\in G_1}\omega(x),\sum_{x\in H_1}\omega(x)),\ldots,(\sum_{x\in G_s}\omega(x),\sum_{x\in H_s}\omega(x))]$.*

IGDDs are useful because of a construction known as "filling in holes". If we have an $\mathscr{L}$-IGDD $(X,\mathscr{H},\mathscr{G},\mathscr{A})$ of type $[(g_1,h_1),\ldots,(g_s,h_s)]$ and a $\mathscr{K}$-GDD $(\cup_{H\in\mathscr{H}}H,\mathscr{H},\mathscr{B})$ of type $[h_1,\ldots,h_s]$, then $(X,\mathscr{G},\mathscr{A}\cup\mathscr{B})$ is a $(\mathscr{K}\cup\mathscr{L})$-GDD of type $[g_1,\ldots,g_s]$.

We also employ a further construction, similar in spirit to [5, Lemma 1.13, pp.78].

Lemma 2.1. *Let $(X,\mathscr{G},\mathscr{B})$ be a* $\{3\}$-GDD *of type $[g_1,\ldots,g_s]$. Let $t\geq 3$. If there exist (ingredients)* $\{3,n\}$-GDDs *of type g_i^t for all $i=1,\ldots,s$, then there exists a $\{3,n\}$-GDD of type $|X|^t$.*

Proof. We form the required GDD on points $X\times\mathbb{Z}_t$. For each block $\{x,y,z\}$, we place on $\{x,y,z\}\times\mathbb{Z}_t$ a $\mathrm{TD}(3,t)$ missing a parallel class (whose existence is equivalent to that of idempotent Latin squares), so that the groups align on $\{x\}\times\mathbb{Z}_t$, $\{y\}\times\mathbb{Z}_t$, and $\{z\}\times\mathbb{Z}_t$, and the missing parallel class aligns on $\{x,y,z\}\times\{i\}$ for $i\in\mathbb{Z}_t$.

Then for each group G, we place on $G\times\mathbb{Z}_t$ a $\{3,n\}$-GDD of type $|G|t$, so that the groups align on $G\times\{i\}$ for $i\in\mathbb{Z}_t$. □

Another useful construction is the following.

Lemma 2.2. *If there exists a $\{3\}$-GDD of type g^t with a parallel class, then there exists a $\{3,g\}$-GDD of type $3^{gt/3}$.*

Proof. Take the groups as blocks, and the blocks in a parallel class as groups. □

One source of $\{3\}$-GDDs with parallel classes is the class of resolvable uniform $\{3\}$-GDDs, whose existence has been settled by Rees [12].

Theorem 2.3 (Rees). *There exists a resolvable* $\{3\}$-*GDD of type* g^t *if and only if* $g(t-1) \equiv 0 \pmod 2$, $gt \equiv 0 \pmod 3$, *and* $(g, t) \notin \{(2,3), (2,6), (6,3)\}$.

The following results on the existence of transversal designs (see, for example, [1]), and on PBD-closures due to Gronau, Mullin, and Pietsch [7], are used without explicit reference throughout §4.

Theorem 2.4. *Let* $TD(k)$ *denote the set of positive integers* m *such that there exists a* $TD(k, m)$. *Then, we have*

(i) $TD(3) = \mathbb{Z}_{>0}$;
(ii) $TD(4) = \mathbb{Z}_{>0} \setminus \{2, 6\}$;
(iii) $TD(5) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 6, 10\}$;
(iv) $TD(6) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 6, 10, 14, 18, 22\}$;
(v) $TD(8) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 5, 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 30, 33, 34, 35, 36, 38, 39, 42, 44, 46, 48, 51, 52, 54, 55, 58, 60, 62, 66, 68, 74, 75\}$.

Theorem 2.5. *We have the following PBD-closures*:

(i) $B(\{3, 4\}) = \{v \equiv 0, 1 \pmod 3\} \setminus \{6\}$;
(ii) $B(\{3, 5\}) = \{v \equiv 1 \pmod 2\}$;
(iii) $B(\{3, 6\}) = \{v \equiv 0, 1 \pmod 2\} \setminus \{4, 10, 12, 22\}$;
(iv) $B(\{3, 8\}) = \mathbb{Z}_{>0} \setminus \{2, 4, 5, 6, 10, 11, 12, 14, 16, 17, 18, 20, 23, 26, 28, 29, 30, 34, 35, 36, 38\}$;
(v) $B(\{3, 10\}) = \{v \equiv 0, 1 \pmod 3\} \setminus \{4, 6, 12, 16, 18, 22, 24, 34, 36, 42\}$;
(vi) $B(\{3, 5, 8\}) = \mathbb{Z}_{>0} \setminus \{2, 4, 6, 10, 12, 14, 16, 18, 20, 26, 28, 30, 34\}$;
(vii) $B(\{3, 4, 5, 8\}) = \mathbb{Z}_{>0} \setminus \{2, 6\}$;
(viii) $B(\{3, 5, 6, 8, 10\}) = \mathbb{Z}_{>0} \setminus \{2, 4, 12, 14, 20\}$.

3. Some $\{3\}$-GDDs of Miscellaneous Types

Let v and k be positive integers such that $v \geq k$. We call v k-*good* if there exists a $\{3\}$-GDD of type $[g_1, \ldots, g_s]$ such that $\sum_{i=1}^{s} g_i = v$, $g_i \geq k$, and $g_i \equiv k \pmod 2$ for $1 \leq i \leq s$. It is not hard to see that if v is k-good, then $v \equiv k \pmod 2$. The *spectrum* of k-good integers, denoted $\mathrm{Spec}(k)$, is the set of integers that are k-good. In this section, we completely determine the spectrum of k-good integers for $k \in \{2, 3, 4, 5\}$. These results are used in the next section for determining the existence of some uniform $\{3, n\}$-GDDs.

Lemma 3.1. *Spec*$(5) = \{v \equiv 1 \pmod 2 \mid v \geq 27\} \cup \{15\}$.

Proof. Let $m \geq 2$. Take a TD$(5, 2m+1)$ and assign weight three to each of the points in four of its groups. Assign weights in $\{1, 3, 5, 7\}$ to each of the points in the remaining group. Apply WFC to obtain a $\{3\}$-GDD of type $(6m+3)^4u^1$, where u is odd, and $2m+1 \leq u \leq 7(2m+1)$. The required ingredients exist by Theorem 1.2. This proves that for any odd $v \geq 65$, v is 5-good. For $v \leq 59$, membership of v in Spec(5) can be determined from [3]. For $v = 61$ and 63, note that there exist $\{3\}$-GDDs of types 9^67^1 and 9^7 (Theorem 1.2). □

Lemma 3.2. *Spec*$(4) = \{v \equiv 0 \pmod 2 \mid v \geq 16\} \cup \{12\}$.

Proof. Let $m \geq 3$, $m \neq 6$. Take a TD$(4, m)$ and assign weight two to each point in three of its groups. Assign weights in $\{0, 2, 4\}$ to each of the points in the remaining group so that the sum of the weights of the points in this group is at least four. Apply WFC to obtain a $\{3\}$-GDD of type $(2m)^3u$, where u is even, and $4 \leq u \leq 4m$. The required ingredients exist by Theorem 1.2. This proves that for any even $v \geq 22$, v is 4-good. For $v \leq 20$, membership of v in Spec(4) can be determined from [3]. □

Lemma 3.3. *Spec*$(3) = \{v \equiv 1 \pmod 2 \mid v \geq 15\} \cup \{9\}$.

Proof. Trivially, we have Spec$(5) \subseteq$ Spec(3). For $v \leq 25$, membership of v in Spec(3) can be determined from [3]. □

Lemma 3.4. *Spec*$(2) = \{v \equiv 0 \pmod 2 \mid v \geq 6\} \cup \{2\}$.

Proof. Trivially, we have Spec(4) $\subseteq$ Spec(2). For $v \leq 14$, membership of v in Spec(2) can be determined from [3]. □

4. Sufficiency

This section is devoted to proving the sufficiency o the conditions in the MainTheorem. In what follows, it is implicitly assumed that $t \geq 3$, wherever it occurs.

4.1. Small Ingredients

We first give some small uniform $\{3, n\}$-GDDs that are needed for the recursive constructions. To obtain these GDDs, we use a variant of Stinson's hill-climbing algorithm [13]. Our algorithm is similar to that described in [3], except that in addition to specifying the groups, we have to specify a set of blocks of size n, so that the leave can be partitioned into triangles. We list in Table 1 the parameters of the GDDs that are constructed by this algorithm. We do not list the blocks for these GDDs here as they exhibit no particular structure and are space consuming. Details on how to find the blocks for these GDDs can be found in the Appendix.

Lemma 4.1. *All uniform* $\{3, n\}$*-GDDs of types listed in Table* 1 *exist.*

Table 1. Some small uniform $\{3, n\}$-GDDs

n	Type									
4	3^6	5^6	7^6	9^6	11^6	13^6	19^6			
6	5^{10}	5^{12}	5^{22}	7^{10}	7^{12}	7^{22}	11^{10}	11^{12}	11^{22}	13^{10}
	13^{12}	13^{22}								
8	2^{20}	2^{23}	2^{26}	2^{29}	2^{32}	2^{35}	2^{38}	2^{41}	3^{14}	3^{18}
	3^{20}	3^{26}	3^{28}	3^{30}	3^{34}	4^{11}	4^{14}	4^{17}	4^{20}	4^{26}
	4^{32}	4^{38}	5^{10}	5^{11}	5^{12}	5^{14}	5^{16}	5^{17}	5^{18}	5^{20}
	5^{22}	5^{24}	5^{26}	5^{32}	5^{38}	5^{44}	7^{10}	7^{11}	7^{12}	7^{14}
	7^{16}	7^{17}	7^{18}	7^{20}	7^{23}	7^{26}	7^{28}	7^{29}	7^{30}	7^{34}
	7^{35}	7^{36}	7^{38}	8^{11}	8^{14}	8^{17}	9^{10}	9^{12}	10^{11}	10^{14}
	10^{17}	11^{10}	11^{11}	11^{12}	11^{14}	11^{16}	11^{17}	11^{18}	11^{20}	11^{23}
	11^{26}	11^{28}	11^{29}	11^{30}	11^{34}	11^{35}	11^{36}	11^{38}	12^{11}	12^{14}
	12^{17}	13^{10}	13^{11}	13^{12}	13^{14}	13^{16}	13^{17}	13^{18}	13^{20}	13^{23}
	13^{26}	13^{28}	13^{29}	13^{30}	13^{34}	13^{35}	13^{36}	13^{38}	14^{8}	14^{11}
	14^{14}	14^{17}	17^{10}	17^{12}	19^{10}	19^{12}	20^{8}	23^{10}	23^{12}	25^{10}
	25^{12}									
10	3^{16}	3^{18}	3^{20}	3^{22}	3^{24}	3^{26}	3^{32}	3^{34}	3^{36}	3^{38}
	3^{42}	3^{44}	3^{56}	3^{62}	5^{10}	5^{12}	5^{16}	5^{18}	5^{22}	5^{24}
	5^{34}	5^{36}	5^{42}	7^{10}	7^{12}	7^{16}	7^{18}	7^{22}	7^{24}	7^{34}
	7^{36}	7^{42}	9^{12}	9^{14}	11^{10}	11^{12}	11^{16}	11^{18}	11^{22}	11^{24}
	11^{34}	11^{36}	11^{36}	11^{42}	13^{10}	13^{12}	13^{16}	13^{18}	13^{22}	13^{24}
	13^{34}	13^{36}	13^{42}	15^{14}	17^{12}	19^{12}	21^{14}	23^{12}	25^{12}	33^{14}
	39^{14}									

4.2. The Case $n = 4$

We show that conditions (i) and (v) of the Main Theorem suffice to ensure the existence of $\{3, 4\}$-GDDs of type g^t.

4.2.1. $t \equiv 2 \pmod 3$
We must have $g \equiv 0 \pmod 3$.

Lemma 4.2. *There exists a* $\{3, 4\}$*-GDD of type* 3^t *for all* $t \equiv 2 \pmod 3$.

Proof. First note that the existence of $\{3, 4\}$-GDDs of types 3^5 and 3^8 can be established by deleting a point from Steiner systems $S(2, 4, 16)$ and $S(2, 4, 25)$, respectively. For all $t \equiv 2 \pmod 3$, $t \in B(\{3, 4, 5, 8\})$. So there exists a $\{3, 4, 5, 8\}$-GDD of type 1^t. Assign weight three to each point of this GDD and apply WFC. The required ingredients exist by Theorem 1.1 and the note above. $\square$

Lemma 4.3. *Let* $t \equiv 2 \pmod 3$. *Then there exists a* $\{3, 4\}$*-GDD of type* g^t *for all* $g \equiv 0 \pmod 3$.

Proof. For any $g \notin \{6, 18\}$, assign weight $g/3$ to each point of a $\{3,4\}$-GDD of type 3^t, whose existence is guaranteed by Lemma 4.2. Apply WFC to obtain a $\{3,4\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4. For the cases $g \in \{6, 18\}$, Theorem 1.1 gives $\{3\}$-GDDs of types 6^t and 18^t for all $t \geq 3$. □

4.2.2. $t \equiv 0$ *or* $1 \pmod 3$
All values of g are admissible here.

Lemma 4.4. *Let* $t \equiv 0$ *or* $1 \pmod 3$, $t \neq 6$. *Then there exists a* $\{3,4\}$-*GDD of type* g^t *for all* $g \geq 2$.

Proof. Since $t \in B(\{3,4\})$, there exists a $\{3,4\}$-GDD of type 1^t. Assign weight g, $g \notin \{2,6\}$, to every point of this GDD and apply WFC to obtain a $\{3,4\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4. When $g \in \{2,6\}$, Theorem 1.1 gives $\{3\}$-GDDs of types 2^t and 6^t. □

Lemma 4.5. *Let* $a \geq 0$ *and* $0 \leq w \leq 6a$. *If there exist a* $TD(6, m+1) - TD(6, a)$ *and a* $\{3,4\}$-*GDD of type* w^6, *then there exists a* $\{3,4\}$-*GDD of type* $(3m + w)^6$.

Proof. Take a $\mathrm{TD}(6, m+1) - \mathrm{TD}(6, a)$ and assign weight three to each point not in the holes. Arbitrarily assign a weight in $\{0, \ldots, 6\}$ to each point in the holes so that the weight for each hole is w (the weight of a hole is the sum of the weights of its points). Apply Theorem 2.2 with ingredients $\{3,4\}$-GDDs of type 3^5u^1, $0 \leq u \leq 6$, which can be constructed by taking a Kirkman triple system of order 15 (which has seven parallel classes), and adjoining u new points, each to a different parallel class. This gives a $\{3,4\}$-IGDD of type $(3m + w, w)^6$. Now fill in the holes with a $\{3,4\}$-GDD of type w^6. □

Lemma 4.6. *There exists a* $\{3,4\}$-*GDD of type* g^6 *for all* $g \geq 2$.

Proof. When $m \notin \{2,3,4,6,10,14,18,22\}$, there exists a $\mathrm{TD}(6, m) - \mathrm{TD}(6, 1)$. By Theorem 1.1 and Lemma 4.1, there exists a $\{3,4\}$-GDD of type w^6 for $w \in \{2,3,4,5,6\}$. Apply Lemma 4.5 to obtain a $\{3,4\}$-GDD of type $(3(m-1) + w)^6$. This gives $\{3,4\}$-GDDs of type g^6 for all $g \geq 14$, $g \notin \{19,31,43,55,67,127\}$. For $g \leq 13$ and $g = 19$, the existence of $\{3,4\}$-GDDs of type g^6 is handled by Theorem 1.1 and Lemma 4.1, and the constructions below handle all remaining values of g. The required ITDs all exist [2]. Apply Lemma 4.5 to a $\mathrm{TD}(6, m) - \mathrm{TD}(6, h)$ with a $\{3,4\}$-GDD of type w^6 for $(m, h, w) \in \{(10,2,7), (15,2,4), (19,2,4), (23,4,10), (43,2,4)\}$ to obtain a $\{3,4\}$-GDD of type g^6 for $g \in \{31,43,55,67,127\}$. □

4.3. The Case $n = 5$

We show that conditions (i) and (vi) of the Main Theorem suffice to ensure the existence of $\{3,5\}$-GDDs of type g^6.

4.3.1. $t \equiv 1 \pmod 2$
All values of g are admissible here.

Lemma 4.7. *Let* $t \equiv 1 \pmod 2$. *Then there exists a* $\{3,5\}$*-GDD of type* g^t *for all* $g \notin \{2,10\}$.

Proof. Since $t \in B(\{3,5\})$ [15], there exists a $\{3,5\}$-GDD of type 1^t. For any $g \notin \{2,3,6,10\}$, assign weight g to each point of this GDD and apply WFC to obtain a $\{3,5\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4. $\{3\}$-GDDs of types 3^t and 6^t exist for all $t \equiv 1 \pmod 2$ by Theorem 1.1. $\square$

Lemma 4.8. *There exist* $\{3,5\}$*-GDDs of types* 2^t *and* 10^t *for all* $t \geq 3$.

Proof. When $t \equiv 0$ or $1 \pmod 3$, Theorem 1.1 gives $\{3\}$-GDDs of type 2^t. When $t \equiv 2 \pmod 3$, $t \in B(\{3,5^*\})$ [6]. In this case we take a $\{3,5^*\}$-PBD$(2t+1)$ and delete a point not on the unique block of size five to obtain a $\{3,5\}$-GDD of type 2^t. To obtain a $\{3,5\}$-GDD of type 10^t, assign weight five to each point of a $\{3,5\}$-GDD of type 2^t and apply Wilson's Fundamental Construction. The required ingredients exist by Theorem 2.4. $\square$

4.3.2. $t \equiv 0 \pmod 2$
We must have $g \equiv 0 \pmod 2$.

Lemma 4.9. *Let* $g \equiv 0 \pmod 2$. *Then there exists a* $\{3,5\}$*-GDD of type* g^t *for all* $t \geq 3$.

Proof. Lemma 4.8 shows the existence of $\{3,5\}$-GDDs of type 2^t for all $t \geq 3$. For $g \notin \{4,6,12,20\}$, assign weight $g/2$ to each point of this GDD and apply Wilson's Fundamental Construction to obtain a $\{3,5\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4. For $g \in \{6,12\}$, there exists a $\{3\}$-GDD of type g^t for all $t \geq 3$ by Theorem 1.1. For $g = 4$, we assign weight two to each point of a $\{3,5\}$-GDD of type 2^t and apply Wilson's Fundamental Construction to obtain a$\{3,5\}$-GDD of type 4^t. The required ingredients exist by Lemma 4.8. To obtain a $\{3,5\}$-GDD of type 20^t, assign weight five to each point of a $\{3,5\}$-GDD of type 4^t and apply WFC. The required ingredients exist by Theorem 2.4. $\square$

4.4. The Case $n = 6$

We show that conditions (i), (ii) and (v) of the Main Theorem suffice for the existence of $\{3,6\}$-GDDs of type g^t.

4.4.1. $t \equiv 0$ *or* $1 \pmod 3$
All values of g are admissible here.

Lemma 4.10. *Let* $t \equiv 0$ *or* $1 \pmod 3$, $t \notin \{4,10,12,22\}$. *Then there exists a* $\{3,6\}$*-GDD of type* g^t *for all* $g \geq 2$.

Proof. Since $t \in B(\{3,6\})$, there exists a $\{3,6\}$-GDD of type 1^t. For $g \notin \{2,3,4,6,10,14,18,22,42\}$, assign weight g to each point of this GDD and apply WFC to obtain a $\{3,6\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4. Theorem 1.1 gives $\{3\}$-GDDs of type g^t for all $g \equiv 0 \pmod 2$ when $t \equiv 0$ or $1 \pmod 3$. This covers the cases $g \in \{2,4,6,10,14,18,22,42\}$. The case $g = 3$ and $t \geq 5$ is covered by Lemma 4.16. Theorem 1.1 gives the existence of a $\{3\}$-GDD of type 3^3. □

It remains to consider $\{3,6\}$-GDDs of type g^t for $t \in \{4,10,12,22\}$.

Lemma 4.11. *There exists a* $\{3,6\}$-*GDD of type* g^4 *if and only if* $g \equiv 0 \pmod 2$.

Proof. If $g \equiv 1 \pmod 2$, condition (ii) of the Main Theorem is violated. When $g \equiv 0 \pmod 2$, there exists a $\{3\}$-GDD of type g^4 by Theorem 1.1. □

Lemma 4.12. *There exists a* $\{3,6\}$-*GDD of type* g^{10} *for all* $g \geq 2$.

Proof. When $g \equiv 0 \pmod 2$, the existence of $\{3,6\}$-GDDs of type g^{10} is handled by Theorem 1.1. When $g \equiv 1 \pmod 2$, the existence of $\{3,6\}$-GDDs of type g^{10} for $g \in \{5,7,11,13\}$ is handled by Lemma 4.1. For $g = 3$, existence is handled by Lemma 4.15. A $\{3,6\}$-GDD of type 9^{10} can be constructed by taking a $\{3,6\}$-GDD of type 3^{10}, assign weight three to each point, and apply WFC. The required ingredients exist by Lemma 4.10. For the remaining values of g, we proceed by induction and assume that $\{3,6\}$-GDDs of type g^{10} exist for all odd $g \leq k$, where $k \geq 13$. Let $g = k+2$. Then g is 3-good. Take a $\{3\}$-GDD of type $[g_1, \ldots, g_s]$, where $\sum_{i=1}^{s} g_i = g$, and $g_i \geq 3$ for $1 \leq i \leq s$. Apply Lemma 2.1 to this GDD to obtain a $\{3,6\}$-GDD of type g^{10}. The required ingredients exist by the induction hypothesis. □

The proof for the following lemmas mimic that for Lemma 4.12.

Lemma 4.13. *There exists a* $\{3,6\}$-*GDD of type* g^{12} *for all* $g \geq 2$.

Lemma 4.14. *There exists a* $\{3,6\}$-*GDD of type* g^{22} *for all* $g \geq 2$.

4.4.2. $t \equiv 2 \pmod 3$

We must have $g \equiv 0 \pmod 3$. We begin with the construction of some small ingredients.

Lemma 4.15. *There exists a* $\{3,6\}$-*GDD of type* 3^t *for* $t \in \{6,8,10,12,14,20\}$.

Proof. For $t = 6$, it is known that there is a TD(3,6) with a parallel class [2]. Apply Lemma 2.2 to obtain a $\{3,6\}$-GDD of type 3^6. For $t \in \{8,10,12,14,20\}$, there exist resolvable $\{3\}$-GDDs of type $6^{t/2}$ by Theorem 2.3. Apply Lemma 2.2 to obtain $\{3,6\}$-GDDs of type 3^t. □

Lemma 4.16. *There exists a* $\{3,6\}$-*GDD of type* 3^t *for all* $t \geq 5$.

Proof. Let $t \notin \{2, 4, 12, 14, 20\}$. Then $t \in B(\{3, 5, 6, 8, 10\})$. So there exists a $\{3, 5, 6, 8, 10\}$-GDD of type 1^t. Assign weight three to each point of this GDD and apply WFC to obtain a $\{3, 6\}$-GDD of type 3^t. The required ingredients exist by Theorem 1.1 and Lemma 4.15. When $t \in \{12, 14, 20\}$, existence of the GDDs is settled by Lemma 4.15. □

Lemma 4.17. *Let* $g \equiv 0 \pmod 3$. *Then there exists a* $\{3, 6\}$-*GDD of type* g^t *for all* $t \geq 5$.

Proof. When $g \equiv 0 \pmod 6$, Theorem 1.1 gives a $\{3\}$-GDD of type g^t for all $t \geq 3$. When $g \equiv 3 \pmod 6$, assign weight $g/3$ to each point of a $\{3, 6\}$-GDD of type 3^t (which exists by Lemma 4.16) and apply WFC to obtain a $\{3, 6\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4 and Lemma 4.10. □

4.5. The Case $n = 8$

This is the most difficult case. We begin with the construction of $\{3, 8\}$-GDDs of type g^t for $g \in \{2, 3, 4, 5\}$. The following results on PBD-closure are useful.

Lemma 4.18. $\{v \equiv 2 \pmod 3 \mid v \geq 44\} \subseteq B(\{3, 4, 6, 20, 23, 29\})$.

Proof. In [3], $\{3\}$-GDDs of types $20^1 4^6, 23^1 1^{24}, 20^1 10^1 4^5, 23^1 7^3 3^3, 20^1 10^3 6^1$, and $29^1 1^{30}$ were shown to exist. Since $7, 10 \in B(\{3, 4\})$, this gives $\{44, 47, 50, 53, 56, 59\} \subseteq B(\{3, 4, 6, 20, 23, 29\})$. Theorem 1.2 gives $\{3\}$-GDDs of types $6^t 20^1$ and $6^t 22^1$ for all $t \geq 6$. This shows $6t + 20 \in B(\{3, 6, 20\})$ and $6t + 23 \in B(\{3, 23\})$ for all $t \geq 6$. The result then follows. □

Lemma 4.19. $\{v \equiv 0 \pmod 2 \mid v \geq 34\} \setminus \{38, 44\} \subseteq B(\{3, 10, 12, 14, 16, 18, 20\})$.

Proof. Theorem 1.2 provides the existence of a $\{3\}$-GDD of type $12^t u^1$ for all $t \geq 3$ and $u \in \{10, 12, 14, 16, 18, 20\}$. This gives a $\{3, 10, 12, 14, 16, 18, 20\}$-PBD$(v)$ for all even $v \geq 46$. For $v = 36, 42$, view a TD$(3, 12)$ and TD$(3, 14)$ as PBDs. Adjoin a new point to the groups of a TD$(3, 11)$ and TD$(3, 13)$ to get the appropriate PBDs on 34 and 40 points, respectively. □

Lemma 4.20. *There exists a* $\{3, 8\}$-*GDD of type* 2^t *if and only if* $t \notin \{2, 5, 8, 11, 14, 17\}$.

Proof. Theorem 1.1 settles the case when $t \equiv 0$ or $1 \pmod 3$. Hence we deal only with the case $t \equiv 2 \pmod 3$. If $t < 44$, $t \notin \{5, 8, 11, 14, 17\}$, the existence of a $\{3, 8\}$-GDD of type 2^t is given by Lemma 4.1. When $t \geq 44$, there exists a $\{3, 4, 6, 20, 23, 29\}$-GDD of type 1^t by Lemma 4.18. Assign weight two to each point of this GDD and apply WFC to obtain a $\{3, 8\}$-GDD of type 2^t. The required ingredients exist by Lemma 4.1. Nonexistence of the remaining GDDs is handled by the necessary conditions of the Main Theorem. □

Lemma 4.21. *There exists a* $\{3, 8\}$-*GDD of type* 3^t *if and only if* $t \notin \{2, 4, 6, 10, 12\}$.

Proof. First note that the existence of $\{3,8\}$-GDDs of types 3^8 and 3^{16} follows from Lemma 2.2 and Theorem 2.3. For $t \in \{14,18,20,26,28,30,34\}$, the existence of a $\{3,8\}$-GDD of type 3^t is provided by Lemma 4.1. Let $t \notin \{4,6,10,12,14,16,18,20,26,28,30,34\}$. Then $t \in B(\{3,5,8\})$. Hence, there exists a $\{3,5,8\}$-GDD of type 1^t. Assign weight three to each point of this GDD and apply WFC to obtain a $\{3,8\}$-GDD of type 3^t. The required ingredients exist by Theorem 1.1 and the note above. Nonexistence of the remaining GDDs is handled by the necessary conditions of the Main Theorem. □

Lemma 4.22. *There exists a* $\{3,8\}$*-GDD of type* 4^t *if and only if* $t \notin \{2,5,8\}$.

Proof. The existence of a $\{3,8\}$-GDD of type 4^t exists for all $t \equiv 0$ or $1 \pmod 3$ by Theorem 1.1. For $t \equiv 5 \pmod 6$, first note that there exist $\{3,8\}$-GDDs of types 4^{11} and 4^{17} by Lemma 4.1. If $t \geq 23$, then $t \in B(\{3,11^*\})$ [10]. Hence, there exists a $\{3,11\}$-GDD of type 1^t. Assign weight four to each point of this GDD and apply WFC to obtain a $\{3,8\}$-GDD of type 4^t. The required ingredients exist by Theorem 1.1 and the note above. We now deal with the remaining case when $t \equiv 2 \pmod 6$. Lemma 4.1 gives the existence of $\{3,8\}$-GDDs of type 4^t for $t \in \{14,20,26,32,38\}$. We proceed by unduction on t, assuming that all $\{3,8\}$-GDDs of type 4^t exist for $t \leq k$, where $k \geq 38$, and $t \neq 2,5,8$. Let $t = k+6$. Then there exists a $\{3\}$-GDD of type $\left(\frac{t-14}{3}\right)^3 14^1$ by Theorem 1.2. View this GDD as a $\{3,14,\frac{t-14}{3}\}$-GDD of type 1^t. Assign weight four to each point of this GDD andapply WFC to obtain a $\{3,8\}$-GDD of type 4^t. The required ingredients exist by Theorem 1.1, the note above, and the induction hypothesis. Nonexistence of the remaining GDDs is handled by the necessary conditions of the Main Theorem. □

Lemma 4.23. *There exists a* $\{3,8\}$*-GDD of type* 5^t *if and only if* $t \notin \{2,4,5,6,8\}$.

Proof. The case $t \equiv 1$ or $3 \pmod 6$ is handled by Theorem 1.1. Existence of $\{3,8\}$-GDDs of types 5^{11} and 5^{17} is provided by Lemma 4.1. When $t \equiv 5 \pmod 6$, $t \geq 23$, we have $t \in B(\{3,11^*\})$ [10]. Hence, there exists a $\{3,11\}$-GDD of type 1^t. Assign weight five to each point of this GDD and apply WFC to obtain a $\{3,8\}$-GDD of type 5^t. The required ingredients exist by Theorem 2.4 and Lemma 4.1. This settles the case when t is odd. Lemma 4.1 gives the existence of $\{3,8\}$-GDDs of type 5^t for $t \in \{10,12,14,16,18,20,22,24,26,32,38,44\}$. When t is even and $t \geq 34$, $t \notin \{38,44\}$, there exists a $\{3,10,12,14,16,18,20\}$-GDD of type 1^t by Lemma 4.19. Assign weight five to each point of this GDD and apply WFC to obtain a $\{3,8\}$-GDD of type 5^t. The required ingredients exist by Theorem 2.4 and Lemma 4.1. Nonexistence of the remaining GDDs is handled by the necessary conditions of the Main Theorem. □

At this point, we need only consider the existence of $\{3,8\}$-GDDs of type g^t for $g \geq 6$.

Lemma 4.24. *Let* $a \geq 0$ *and* $0 \leq w \leq a$. *If there exist a* $TD(8,m+a)$–$TD(8,a)$ *and a* $\{3,8\}$*-GDD of type* w^8, *then there exists a* $\{3,8\}$*-GDD of type* $(m+w)^8$.

Proof. Take a $\mathrm{TD}(8, m+a) - \mathrm{TD}(8, a)$ and remove $a - w$ points from each hole to obtain a $\{7, 8\}$-IGDD of type $(m, w)^8$. Fill in the holes of this IGDD with a $\{3, 8\}$-GDD of type w^8 and replace each block of size seven by the blocks of a Steiner system $\mathrm{S}(2, 3, 7)$. $\square$

Lemma 4.25. *There exists a* $\{3, 8\}$-*GDD of type* g^8 *for all* $g \geq 6$.

Proof. Lemma 4.24 implies the existence of $\{3, 8\}$-GDDs of types $(m-1)^8$ and m^8, whenever a $\mathrm{TD}(8, m)$ exists. It follows that if $g \notin \{14, 20, 21, 33, 34, 35, 38, 51, 54, 57\}$, then there exists a $\{3, 8\}$-GDD of type g^8. The existence of $\{3, 8\}$-GDDs of types 14^8 and 20^8 follows from Lemma 4.1. For $g \in \{21, 33, 34, 35, 38, 51, 54, 74\}$, apply Lemma 2.1 with $\{3\}$-GDDs of types 1^{21}, 1^{33}, $10^1 8^3$, $17^1 1^{18}$, $8^1 6^5$, 1^{51}, $24^1 6^5$, and $14^1 10^6$, all of which exist by Theorem 1.2. $\square$

Lemma 4.26. *Let* $t \in B(\{3, 8\})$. *Then there exists a* $\{3, 8\}$-*GDD of type* g^t *for all* $g \geq 6$.

Proof. Take a $\{3, 8\}$-GDD of type 1^t. Assign weight g to each point of this GDD and apply WFC to obtain a $\{3, 8\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4 and Lemma 4.25. $\square$

It remains to deal with those cases when $t \notin B(\{3, 8\})$.

Lemma 4.27. *Let* $t \in \{4, 6\}$. *Then there exists a* $\{3, 8\}$-*GDD of type* g^t *if and only if* $g \equiv 0 \pmod 2$.

Proof. If $g \equiv 1 \pmod 2$, condition (ii) of the Main Theorem is violated. When $g \equiv 0 \pmod 2$, there exists a $\{3\}$-GDD of type g^t by Theorem 1.1. $\square$

Lemma 4.28. *There exists a* $\{3, 8\}$-*GDD of type* g^5 *if and only if* $g \equiv 0 \pmod 3$.

Proof. If $g \not\equiv 0 \pmod 3$, condition (ii) of the Main Theorem is violated. When $g \equiv 0 \pmod 3$, there exists a $\{3\}$-GDD of type g^5 by Theorem 1.1. $\square$

Lemma 4.29. *Let* $t \in \{10, 12\}$. *Then there exists a* $\{3, 8\}$-*GDD of type* g^t *for all* $g \geq 6$.

Proof. If $g \equiv 0 \pmod 2$, there exists a $\{3\}$-GDD of type g^t by Theorem 1.1. If $g \equiv 1 \pmod 2$ and $g \leq 25$, $g \notin \{15, 21\}$, existence is handled by Lemma 4.1. To obtain a $\{3, 8\}$-GDD of type g^t for $g \in \{15, 21\}$, we assign weight three to each point of a $\{3, 8\}$-GDD of type $(g/3)^{10}$ and $(g/3)^{12}$ and apply WFC. The required ingredients exist by Lemma 4.21. For $g \geq 27$, we proceed by induction, assuming that there exist $\{3, 8\}$-GDDs of type g^t for all $g \leq k$, where $k \geq 25$. Let $g = k + 2$. Then g is 5-good. Take a $\{3\}$-GDD of type $[g_1, \ldots, g_s]$, where $\sum_{i=1}^{s} g_i = g$, and $g_i \geq 5$ for $1 \leq i \leq s$. Apply Lemma 2.1 to this GDD to obtain a $\{3, 8\}$-GDD of type g^t. The required ingredients exist by the induction hypothesis. $\square$

Lemma 4.30. *Let* $t \in \{11, 14, 17\}$. *Then there exists a* $\{3, 8\}$*-GDD of type* g^t *for all* $g \geq 6$.

Proof. For $g \in \{7, 8, 10, 11, 12, 13, 14\}$, the result follows from Lemma 4.1. For $g = 6$, the result follows from Theorem 1.1. For $g = 9$, the result follows by assigning weight three to each point of a $\{3, 8\}$-GDD of type 3^t and applying WFC. The required master and ingredients exist by Lemma 4.21. For the remaining odd (even, respectively) values of g, note that g is 3-good (4-good, respectively) and mimic the proof of Lemma 4.29. □

Lemma 4.31. *Let* $t \in \{16, 18, 20, 23, 26, 28, 29, 30, 34, 35, 36, 38\}$. *Then there exists a* $\{3, 8\}$-GDD of type g^t *for all* $g \geq 6$.

Proof. Lemma 4.1 gives the existence of $\{3, 8\}$-GDDs of type g^t for $g \in \{7, 11, 13\}$. For the remaining odd (even, respectively) values of g, note that g is 3-good (2-good, respectively) and mimic the proof of Lemma 4.29. □

4.6. The Case $n = 10$

The case for even g is easily handled.

Lemma 4.32. *Let* $g \equiv 0 \pmod 6$. *Then there exists a* $\{3, 10\}$*-GDD of type* g^t *for all* $t \geq 3$.

Proof. Follows from Theorem 1.1. □

Lemma 4.33. *Let* $g \equiv 2$ *or* $4 \pmod 6$. *Then there exists a* $\{3, 10\}$*-GDD of type* g^t *if and only if* $t \equiv 0$ *or* $1 \pmod 3$.

Proof. Condition (v) of the Main Theorem is violated if $t \equiv 2 \pmod 3$. If $t \equiv 0$ or $1 \pmod 3$, there exists a $\{3\}$-GDD of type g^t by Theorem 1.1. □

We now focus on the case $g \equiv 1 \pmod 2$.

Lemma 4.34. *There exists a* $\{3, 10\}$*-GDD of type* g^{10} *for all* $g \equiv 1 \pmod 2$.

Proof. The case $g = 3$ follows from Lemma 2.2 and Theorem 2.3. Lemma 4.1 settles the cases $g \in \{5, 7, 11, 13\}$. The remaining values of g are 3-good. The result then follows by induction using Lemma 2.1. □

Lemma 4.35. *Let* $t \in B(\{3, 10\})$. *Then there exists a* $\{3, 10\}$*-GDD of type* g^t *for all* $g \equiv 1 \pmod 2$.

Proof. Take a $\{3, 10\}$-GDD of type 1^t and assign weight g to each of the points. Apply WFC to obtain a $\{3, 10\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4 and Lemma 4.34. □

Lemma 4.36. *Let* $t \in \{4, 6\}$. *Then there exists a* $\{3, 10\}$-*GDD of type* g^t *if and only if* $g \equiv 0 \pmod 2$.

Proof. If $g \equiv 1 \pmod 2$, condition (ii) of the Main Theorem is violated. Existence follows from Lemmas 4.32 and 4.33. □

Lemma 4.37. *There exists a* $\{3, 10\}$-*GDD of type* g^{12} *for all* $g \equiv 1 \pmod 2$, *except when* $g = 3$.

Proof. First note that condition (iii) of the Main Theorem excludes the existence of a $\{3, 10\}$-GDD of type 3^{12}, since $36 \notin B(\{3, 10\})$ by Theorem 2.5.

Lemma 4.1 gives the existence of $\{3, 10\}$-GDDs of type g^{12} when $g \in \{5, 7, 9, 11, 13, 17, 19, 23, 25\}$. For $g \in \{15, 21\}$, take a $\{3, 10\}$-GDD of type $(g/3)^{12}$, which exists by Lemma 4.1, and assign weight three to each of its points. Apply WFC to obtain a $\{3, 10\}$-GDD of type g^{12}. The required ingredients exist by Theorem 1.1 and Lemma 4.35. If $g \geq 27$, then g is 5-good. Straightforward induction using Lemma 2.1 establishes the required result. □

Lemma 4.38. *Let* $t \in \{16, 18, 22, 24, 34, 36, 42\}$. *Then there exists a* $\{3, 10\}$-*GDD of type* g^t *for all* $g \equiv 1 \pmod 2$.

Proof. The case $g \in \{3, 5, 7, 11, 13\}$ is handled by Lemma 4.1. For the remaining odd values of g, note that g is 3-good and proceed by induction using Lemma 2.1. □

It remains to consider those values of $t \equiv 2 \pmod 3$. Condition (v) of the Main Theorem requires that $g \equiv 0 \pmod 3$ in this case.

Lemma 4.39. $\{v \equiv 2 \pmod 6 | v \geq 50\} \backslash \{56, 62\} \subseteq B(\{3, 10, 16, 18, 20, 26, 32\})$.

Proof. Theorem 1.2 provides the existence of a $\{3\}$-GDD of type $18^t u^1$ for all $t \geq 3$ and $u \in \{20, 26, 32\}$. This gives a $\{3, 10, 16, 18, 20, 26, 32\}$-PBD$(v)$ for $v \equiv 2 \pmod 6$, $v \geq 74$. For $v \in \{50, 68\}$, note that there exist $\{3\}$-GDDs of types $20^1 10^3$ and $20^1 16^3$ by Theorem 1.2. □

Lemma 4.40. *There exists a* $\{3, 10\}$-*GDD of type* 3^t *for all* $t \equiv 2 \pmod 3$, $t \notin \{2, 8, 14\}$.

Proof. The case $t \equiv 5 \pmod 6$ is handled by Theorem 1.1. If $t \equiv 2 \pmod 6$, $t \geq 50$ and $t \notin \{56, 62\}$, there exists a $\{3, 10, 16, 18, 20, 26, 32\}$-GDD of type 1^t by Lemma 4.39. Assign weight three to each point of this GDD and apply WFC to obtain a $\{3, 10\}$-GDD of type 3^t. The required ingredients exist by Lemma 4.1 and Lemma 4.34. The remaining values of t are handled by Lemma 4.1. □

Lemma 4.41. *Let* $t \equiv 2 \pmod 3$, $t \notin \{2, 8, 14\}$. *Then there exists a* $\{3, 10\}$-*GDD of type* g^t *for all* $g \equiv 0 \pmod 3$.

Proof. Take a $\{3,10\}$-GDD of type 3^t (which exists by Lemma 4.40) and assign weight $g/3$ to each of its points. Apply WFC to obtain a $\{3,10\}$-GDD of type g^t. The required ingredients exist by Theorem 2.4 and Lemma 4.34. $\square$

Lemma 4.42. *There exists a* $\{3,10\}$*-GDD of type* g^8 *if and only if* $g \equiv 0 \pmod 6$.

Proof. If $g \not\equiv 0 \pmod 6$, then conditions (ii) of the Main Theorem is violated. If $g \equiv 0 \pmod 6$, then the existence of a $\{3\}$-GDD of type g^8 is provided by Theorem 1.1. $\square$

Lemma 4.43. *There exists a* $\{3,10\}$*-GDD of type* g^{14} *if and only if* $g \equiv 0 \pmod 3$, $g \neq 3$.

Proof. If $g \equiv 0 \pmod 6$, there exists a $\{3\}$-GDD of type g^{14} by Theorem 1.1. When $g \equiv 3 \pmod 6$, there exists a $\{3,10\}$-GDD of type g^{14} for $g \in \{9,15,21,33,39\}$ by Lemma 4.1. Theorem 1.2 gives the existence of a $\{3\}$-GDD of type $9^{2r}u^1$ for all $r \geq 2$ and $u \in \{9,15,21\}$. Apply Lemma 2.1 to this GDD to obtain a $\{3,10\}$-GDD of type g^{14} for all $g \geq 45$. The result for the case $g = 27$ follows from the existence of a $\{3\}$-GDD of type 9^3. The nonexistence of a $\{3,10\}$-GDD of type 3^{14} is handled by the conditions of the Main Theorem. $\square$

5. Conclusion

The Main Theorem given in this paper is an extension of the results of Hanani [8] concerning uniform GDDs with block size three, as well as the results of Gronau, Mullin and Pietsch [7] on the existence of PBDs with block sizes three and n, $4 \leq n \leq 10$. It also dictates when the complete multipartite graph with equal-sized partitions has an edge decomposition into K_3's and K_n's, $4 \leq n \leq 10$, thus extending also some results in graph decompositions [9].

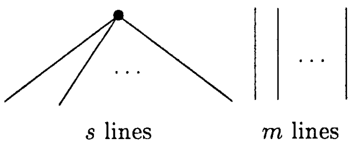

Fig. 1. Prestructure $P(s,m)$

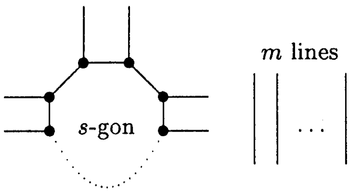

Fig. 2. Prestructure $Q(s,m)$

Acknowledgments. The authors would like to thank the referee for his careful reading and for pointing out some errors in an earlier draft of the paper.

Appendix

A Some Small Uniform $\{3, n\}$-GDDs

In this appendix, we give details on the computational procedures we employed to construct those GDDs given in Lemma 4.1.

```
P(int s, int m, int n, int g, int t)
{
    int a[100][50], b, block[100][10], i, j, k, ok, r;

    for (i = 0; i < t; i++)
        for (j = 0; j < g; j++)
            a[i][j] = j*t+i+1;

    a[0][0] = 0;
    for (b = 0; b < a; b++) {
        k = 0;
        block[b][k++] = 1;
        for (j = 0; j < g; j++) {
            for (i = 0; i < t; i++) {
                if ((a[i][j]%t != 1) && a[i][j]) {
                    block[b][k++] = a[i][j];
                    a[i][j] = 0;
                }
                if (k == n)
                    break;
            }
            if (k == n)
                break;
        }
    }

    for (b = s; b < s+m; b++) {
        k = 0;
        for (j = 0; j < g; j++) {
            for (i = 0; i < t; i++) {
                ok = 1;
                for (r = 0; r < k; r++)
                    if ((a[i][j]-block[b][r])%t == 0) {
                        ok = 0;
                        break;
                    }
                if (a[i][j] && ok) {
                    block[b][k++] = a[i][j];
                    a[i][j] = 0;
                }
                if (k == n)
                    break;
            }
            if (k == n)
                break;
        }
    }

    for (b = 0; b < s+m; b++) {
        for (k = 0; k < n; k++)
            printf(''%d'', block[b][k]);
        printf(''\n'');
    }
}
```

Fig. 3. C-function for generating $P(s, m)$

Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{3, n\}$-GDD of type g^t. We call the set $\{A \in \mathcal{A} \mid |A| = n\}$ the *prestructure* of the GDD. Given a prestructure, a set of blocks of size three (*triples*) can be added to complete it to a GDD. The heuristic we used to complete the prestructure is similar to the hill-climbing heuristic used in [7]. In what follows, we give a prestructure for each GDD listed in Table 1. The point set X is assumed to be $\{1, \ldots, gt\}$. Unless otherwise specified, the groups are $\{jt + i \,|\, 0 \leq j \leq g\}$, for $1 \leq i \leq t$. So we list only the blocks of size n. We also omit listing the triples here since they exhibit no particular structure and are space consuming. Moreover, the triples can be easily found using a simple hill-climbing heuristic. For space efficiency, we also represent all the integers $a, a+1, \ldots, b$ by $a - b$.

The intersection pattern of the blocks in many of the prestructures have one of the forms shown in Figures 1 and 2. The blocks in the prestructures of the form $P(s, m)$ are generated by calling the C-function presented in Figure 3. We are not able to find such succinct description for the blocks in $Q(s, m)$, and we content ourselves by exhibiting explicitly the blocks in the prestructure in this case. We use the notation $rQ(s, m)$ for r disjoint copies of $Q(s, m)$.

A.1. $n = 4$

We give prestructures of $\{3, 4\}$-GDDs of types listed in Table 1. For $g \in \{5, 7, 11, 13, 19\}$, a prestructure for a $\{3, 4\}$-GDD of type g^6 is $P(3, \frac{3g-5}{2})$.

A.2. $n = 6$

We give prestructures of $\{3, 6\}$-GDDs of types listed in Table 1.

Prestructures of $\{3, 6\}$-GDDs

Type	Prestructure	Type	Prestructure	Type	Prestructure
5^{10}	$P(5, 4)$	7^{12}	$P(0, 14)$	11^{22}	$P(5, 36)$
5^{12}	$P(0, 10)$	7^{22}	$P(3, 23)$	13^{10}	$P(3, 19)$
5^{22}	$P(5, 14)$	11^{10}	$P(5, 14)$	13^{12}	$P(0, 26)$
7^{10}	$P(3, 9)$	11^{12}	$P(0, 22)$	13^{22}	$P(3, 45)$

A.3. $n = 8$

We give prestructures of $\{3, 8\}$-GDDs of types listed in Table 1. First, we deal with the case when g is even.

For $t \in \{20, 23, 26, 29, 32, 35, 38, 41\}$, there exists a $\{3, 8\}$-PBD$(2t + 1)$ with precisely ten blocks of size eight. These ten blocks of size eight may be taken to be the following.

{1–7, 40}	{8–15}	$\{7, 15, 21, 27, 31, 35, 37, 39\}$
{8, 16–21, 40}	{1, 9, 22–27}	$\{6, 14, 20, 26, 30, 34, 36, 38\}$
{2, 10, 16, 22, 28–31}	{3, 11, 17, 23, 32–35}	$\{5, 13, 19, 25, 29, 33, 38, 39\}$
$\{4, 12, 18, 24, 28, 32, 36, 37\}$		

The existence of such a PBD can be verified by employing the hill-climbing heuristic on this set of blocks. From this PBD delete a point not on the block of size eight to obtain a $\{3, 8\}$-GDD of type 2^t.

Let $\mathscr{T}$ be the set consisting of the following ten blocks of size eight.

$\{1, 5, 9, 13, 17, 21, 25, 29\}$	$\{2, 6, 10, 14, 28, 33, 37, 41\}$	$\{1, 7, 22, 26, 30, 33, 38, 42\}$
$\{3, 5, 11, 15, 19, 23, 34, 37\}$	$\{4, 9, 22, 27, 31, 34, 39, 41\}$	$\{2, 8, 11, 13, 26, 35, 40, 43\}$
$\{3, 6, 12, 16, 17, 27, 30, 35\}$	$\{10, 15, 20, 21, 31, 36, 38, 43\}$	$\{8, 12, 14, 19, 25, 36, 39, 42\}$
$\{4, 7, 16, 20, 23, 28, 29, 40\}$		

For $g \in \{4, 8, 10, 14\}$ and $t \in \{11, 14, 17, 20, 26, 32, 38\}$, let $\{4j + i \mid 1 \le i \le 4\} \cup \{4(t + j) + i \mid 1 \le i \le g - 4\}$, for $0 \le j \le t$, be the groups of a $\{3, 8\}$-GDD of type g^t. A prestructure for this GDD is the set $\mathscr{T}$.

For $g \in \{14, 20\}$, let $\{jg + i \mid 1 \le i \le g\}$, for $0 \le j \le 8$, be the groups of a $\{3, 8\}$-GDD of type g^8. Define the 8×8 matrix $A = (a_{ij})$, with $0 \le i < 8$ and $1 \le j \le 8$, such that $a_{ij} = ig + j$. A prestructure for this GDD consists of the 16 blocks defined by each of the columns of A, as well as each of the generalized main diagonals of A.

For $t \in \{11, 14, 17\}$, let $\{12j + i \mid 1 \le i \le 12\}$, for $0 \le j < t$, be the groups of a $\{3, 8\}$-GDD of type 12^t. A prestructure for this GDD consists of the following nine blocks, which is isomorphic to the triangular scheme.

$\{1, 13, 25, 37, 49, 61, 73, 85\}$	$\{1, 26, 38, 50, 62, 74, 86, 98\}$	$\{13, 26, 51, 63, 75, 87, 99, 111\}$
$\{25, 38, 51, 76, 88, 100, 112, 124\}$	$\{5, 37, 50, 63, 76, 101, 113, 125\}$	$\{6, 18, 49, 62, 75, 88, 101, 126\}$
$\{19, 31, 61, 74, 87, 100, 113, 126\}$	$\{6, 19, 43, 73, 86, 99, 112, 125\}$	$\{5, 18, 31, 43, 85, 98, 111, 124\}$

We now settle the case when g is odd.

For $t \in \{11, 17\}$, let $\{4j + i \mid 1 \le i \le 4\} \cup \{4(t + j) + 1\}$, for $0 \le j < t$, be thegroups of a $\{3, 8\}$-GDD of type 5^t. A prestructure for this GDD is the set $\mathscr{T}$.

For $g \in \{7, 9, 11, 13, 17, 19, 23, 25\}$, a prestructure for a $\{3, 8\}$-GDD of type g^{12} has the following blocks, which is isomorphic to $Q(6, \frac{3(g-7)}{2})$.

$\{7, 36–42\}$	$\{7, 71–76, 80\}$	$\{8, 77–79, 81–84\}$	$\{15, 43–49\}$	$\{22, 50–56\}$	$\{29, 57–63\}$
$\{36, 64–70\}$	$\{7j + i \mid 1 \le i \le 8\}, 0 \le j \le 4$		$\{8j + i \mid 5 \le i \le 12\}, 10 \le j \le \frac{3(g-1)}{2}$		

For $g \in \{7, 11, 13\}$ and $t \in \{11, 17, 23, 29, 35\}$, let $\{4j + i \mid 1 \le i \le 4\} \cup \{4(t + j) + i \mid 1 \le i \le g - 4\}$, for $0 \le j \le t$, be the groups of a $\{3, 8\}$-GDD of type g^t. A prestructure for this GDD is the set $\mathscr{T}$.

Prestructures of several other GDDs of the form $Q(s,m)$ are given in the next table.

Type	Prestructure	Blocks in prestructure
3^{14}	$Q(3,0)$	$\{1–8\}$, $\{8–14,16\}$, $\{1,16–22\}$, $\{1,23–28,30\}$, $\{8,15,37–42\}$, $\{16,29,31–36\}$
3^{28}	$2Q(3,0)$	$\{1–8\}+i$, $\{8–15\}+i$, $\{1,15–21\}+i$, $\{1,22–28\}+i$, $\{8,29–35\}+i$, $\{15,36–42\}+i$, $i\in\{0,42\}$
5^{14}	$Q(5,0)$	$\{1,28,30–35\}$, $\{1,36–42\}$, $\{8,43–49\}$, $\{15,50–56\}$, $\{22,57–62,70\}$, $\{28,63–69\}$, $\{7j+i \mid 1\le i\le 8\}$, $0\le j\le 3$
7^{18}	$3Q(3,0)$	$\{1–8\}+i$, $\{8–15\}+i$, $\{1,15–18,20–22\}+i$, $\{1,23–29\}+i$, $\{8,19,30–35\}+i$, $\{15,36–42\}+i$, $i\in\{0,42,84\}$
9^{10}	$Q(3,6)$	$\{1–8\}$, $\{8–15\}$, $\{2,15–21\}$, $\{2,23–29\}$, $\{8,22,30,31,33–35,46\}$, $\{15,36–42\}$, $\{32,43–45,47–50\}$, $\{8j+i \mid 3\le i\le 10\}$, $6\le j\le 10$
11^{10}	$Q(5,5)$	$\{6,29–35\}$, $\{6,57–63\}$, $\{8,53,64–67,69,70\}$, $\{15,36–42\}$, $\{22,43–49\}$, $\{29,50–52,54–56,68\}$, $\{7j+i \mid 1\le i\le 8\}$, $0\le j\le 3$, $\{8j+i \mid 7\le i\le 14\}$, $8\le j\le 12$
11^{14}	$Q(11,0)$	$\{1,70,72–77\}$, $\{1,78–84\}$, $\{8,85–91\}$, $\{15,92–98\}$, $\{22,99–105\}$, $\{29,106–112\}$, $\{36,113–119\}$, $\{43,120–126\}$, $\{50,127–133\}$, $\{57,134–140\}$, $\{64,141–146,154\}$, $\{70,147–153\}$, $\{7j+i \mid 1\le i\le 8\}$, $0\le j\le 9$
11^{16}	$2Q(4,4)$	$\{1–8\}+i$, $\{8–15\}+i$, $\{15–22\}+i$, $\{1,22–28\}+i$, $\{1,50–56\}+i$, $\{8,29–35\}+i$, $\{15,36–42\}+i$, $\{22,43–49\}+i, i\in\{0,56\}$
11^{18}	$3Q(3,3)$	$\{1–8\}+i$, $\{8–15\}+i, \{1,15–18,20–22\}+i$, $\{1,23–29\}+i$, $\{8,19,30–35\}+i$, $\{15,36–42\}+i$, $i\in\{0,42,84\}$
13^{10}	$Q(7,4)$	$\{1,43–49\}$, $\{1,92–98\}$, $\{8,50–56\}, \{15,57–63\}$, $\{22,64–70\}$, $\{29,71–77\}$, $\{36,78–84\}$, $\{43,85–91\}$, $\{7j+i \mid 1\le i\le 8\}$, $0\le j\le 5$, $\{8j+i \mid 3\le i\le 10\}$, $12\le j\le 15$
17^{10}	$Q(11,2)$	$\{1,70,72–77\}$, $\{1,134–140\}$, $\{8,141–147\}$, $\{15,78–84\}$, $\{22,85–91\}$, $\{29,92,94–98,150\}$, $\{36,99–105\}$, $\{43,106–112\}$, $\{50,113–119\}$, $\{57,120–126\}$, $\{64,127–133\}$, $\{70,148,149,151–155\}$, $\{93,156–162\}$, $\{163–170\}$, $\{7j+i \mid 1\le i\le 8\}$, $0\le j\le 9$
23^{10}	$Q(5,20)$	$\{6,29–35\}$, $\{6,57–63\}$, $\{8,51,64–67,69,70\}$, $\{15,36–42\}$, $\{22,43–49\}$, $\{29,50,52–56,68\}$, $\{7j+i \mid 1\le i\le 8\}$, $0\le j\le 3$, $\{8j+i \mid 7\le i\le 14\}$, $8\le j\le 27$
25^{10}	$Q(7,19)$	$\{1,43–49\}$, $\{1,64,85–90\}$, $\{8,71,92–97\}$, $\{15,78–84\}$, $\{22,73–77,91,98\}$, $\{29,50–56\}$, $\{36,57–63\}$, $\{43,65–70,72\}$, $\{7j+i \mid 1\le i\le 8\}$, $0\le j\le 5$ $\{8j+i \mid 3\le i\le 10\}$, $12\le j\le 30$

The following are prestructures of the form $P(s, m)$.

Prestructures of $\{3, 8\}$-GDDs

Type	Prestructure	Type	Prestructure	Type	Prestructure
3^{30}	$P(7, 5)$	7^{26}	$P(19, 6)$	11^{38}	$P(23, 32)$
5^{18}	$P(7, 5)$	7^{28}	$P(21, 6)$	13^{16}	$P(9, 18)$
5^{20}	$P(5, 8)$	7^{30}	$P(7, 20)$	13^{18}	$P(7, 23)$
5^{22}	$P(11, 4)$	7^{34}	$P(3, 27)$	13^{20}	$P(13, 21)$
5^{24}	$P(0, 15)$	7^{36}	$P(13, 20)$	13^{26}	$P(7, 36)$
5^{32}	$P(17, 5)$	7^{38}	$P(7, 27)$	13^{28}	$P(21, 27)$
5^{38}	$P(11, 14)$	11^{20}	$P(5, 23)$	13^{30}	$P(19, 32)$
5^{44}	$P(5, 23)$	11^{26}	$P(11, 26)$	13^{34}	$P(15, 42)$
7^{10}	$P(3, 6)$	11^{28}	$P(5, 34)$	13^{36}	$P(13, 47)$
7^{14}	$P(7, 6)$	11^{30}	$P(7, 35)$	13^{38}	$P(19, 45)$
7^{16}	$P(9, 6)$	11^{34}	$P(11, 37)$	19^{10}	$P(3, 21)$
7^{20}	$P(13, 6)$	11^{36}	$P(13, 38)$		

Prestructures for the remaining GDDs do not have one of the forms $P(s, m)$ or $Q(s, m)$, and are presented in the next table. In each case, the groups of the GDD are taken to be $\{jg + i \mid 1 \le i \le g\}$, $0 \le j < t$.

Type	Blocks in prestructure
3^{18}	$\{1,4,7,10,13,16,19,22\}$, $\{2,5,8,25,28,31,34,37\}$, $\{3,6,9,40,43,46,49,52\}$, $\{1,5,9,11,14,17,20,23\}$, $\{2,6,7,26,29,32,35,38\}$, $\{3,4,8,41,44,47,50,53\}$, $\{3,5,7,12,15,18,21,24\}$, $\{2,4,9,27,30,33,36,39\}$, $\{1,6,8,42,45,48,51,54\}$
3^{20}	$\{1,4,7,10,13,16,19,22\}$, $\{22,25,28,31,34,37,40,43\}$, $\{2,5,43,46,49,52,55,58\}$, $\{5,8,11,14,17,20,23,26\}$, $\{26,29,32,35,38,41,44,47\}$, $\{3,6,7,47,50,53,56,59\}$, $\{5,7,27,30,33,36,39,42\}$, $\{22,26,45,48,51,54,57,60\}$, $\{9,12,15,18,21,24,43,47\}$
3^{26}	$\{1,4,7,10,13,16,19,22\}$, $\{22,25,28,31,34,37,40,43\}$, $\{2,5,43,46,49,52,55,58\}$ $\{5,8,11,14,17,20,23,26\}$, $\{26,29,32,35,38,41,44,47\}$, $\{3,6,7,47,50,53,56,59\}$, $\{5,7,27,30,33,36,39,42\}$, $\{22,26,45,48,51,54,57,60\}$, $\{9,12,15,18,21,24,43,47\}$, $\{7,26,61,64,67,70,73,76\}$, $\{26,43,62,65,68,71,74,77\}$, $\{7,43,63,66,69,72,75,78\}$
3^{34}	$\{1,4,7,10,13,16,19,22\}$, $\{22,25,28,31,34,37,40,43\}$, $\{2,5,43,46,49,52,55,58\}$ $\{5,8,11,14,17,20,23,26\}$, $\{26,29,32,35,38,41,44,47\}$, $\{3,6,7,47,50,53,56,59\}$, $\{5,7,27,30,33,36,39,42\}$, $\{22,26,45,48,51,54,57,60\}$, $\{9,12,15,18,21,24,43,47\}$, $\{7,26,61,64,67,70,73,76\}$, $\{26,43,62,65,68,71,74,77\}$, $\{7,43,63,66,69,72,75,78\}$, $\{j + 3i \mid 0 \le i \le 7\}$, $79 \le j \le 81$
5^{10}	$\{1,6,11,16,21,26,31,36\}$, $\{1,7,12,17,22,27,41,46\}$, $\{1,8,13,18,32,37,42,47\}$, $\{2,6,23,28,32,38,41,48\}$, $\{2,9,11,19,33,37,43,46\}$, $\{2,14,18,24,27,34,39,44\}$, $\{5,6,15,25,27,35,37,50\}$, $\{3,20,25,30,32,40,45,46\}$, $\{4,10,11,18,25,29,41,49\}$
5^{12}	$\{1,6,11,16,21,26,31,36\}$, $\{1,7,12,17,41,46,51,56\}$, $\{1,22,27,32,37,42,47,52\}$, $\{3,6,13,18,23,28,33,57\}$, $\{2,13,17,38,43,48,53,58\}$, $\{4,8,13,19,22,29,34,39\}$, $\{6,14,24,40,44,49,54,59\}$, $\{9,15,17,25,30,35,40,45\}$, $\{5,10,20,22,40,50,55,60\}$

Type	Blocks in prestructure
5^{16}	$\{1,6,11,16,21,26,31,36\}$, $\{1,41,46,51,56,61,66,71\}$, $\{1,7,12,17,22,27,32,76\}$, $\{2,6,37,42,47,52,57,62\}$, $\{3,8,11,18,37,67,72,77\}$, $\{13,23,28,33,37,43,48,80\}$, $\{4,6,53,58,63,68,73,78\}$, $\{9,14,19,24,29,34,53,67\}$, $\{11,23,39,44,49,53,59,64\}$, $\{5,10,15,20,54,69,74,79\}$, $\{20,25,30,35,40,45,50,67\}$, $\{20,23,38,55,60,65,70,75\}$
5^{26}	$\{1,6,11,16,21,26,31,36\}$, $\{1,7,12,17,27,41,46\}$, $\{1,8,13,18,32,37,42,47\}$, $\{2,6,23,28,32,38,41,48\}$, $\{2,9,11,19,33,37,43,46\}$, $\{2,14,18,24,27,34,39,44\}$, $\{5,6,15,25,27,35,37,50\}$, $\{3,20,25,30,32,40,45,46\}$, $\{4,10,11,18,25,29,41,49\}$, $\{j+5i \mid 0 \le i \le 7\}$, $51 \le j \le 55$ and $91 \le j \le 95$

A.4. $n = 10$

We give prestructures of $\{3,10\}$-GDDs of types listed in Table 1. A prestructure for a $\{3,10\}$-GDD of type 5^t for $t \in \{10,12,16,18,22,24,34,36,42\}$ is $P(0,t/2)$.

The following blocks constitute a prestructure for a $\{3,10\}$-GDD of type 13^{12}.

$\{1–10\}$	$\{1,11,12,14–20\}$	$\{1,21–24,26–30\}$	$\{10,31–33,35–40\}$	$\{10,41–45,47–50\}$
$\{13,51–59\}$	$\{25,99–107\}$	$\{34,60–68\}$	$\{46,108,116\}$	
$\{10j+i \mid 9 \le i \le 18\}, 6 \le j \le 8$			$\{10j+i \mid 7 \le i \le 16\}, 11 \le j \le 14$	

Prestructures of the form $Q(s,m)$ for several other GDDs are given in the next table.

Type	Prestructure	Blocks in prestructure
3^{18}	$Q(3,0)$	$\{1–10\}$, $\{1,11–18,20\}$, $\{2,11,19,21–27\}$, $\{1,28–36\}$, $\{2,37,39–45,47\}$, $\{11,38,46,48–54\}$
3^{24}	$Q(4,0)$	$\{1–10\}$, $\{1,11–19\}$, $\{11,20–28\}$, $\{2,20,29–36\}$, $\{1,37–45\}$, $\{2,46–49,51–55\}$, $\{11,50,56–58,60–63,68\}$, $\{20,59,64–67,69–72\}$
7^{12}	$Q(3,3)$	$\{1–10\}$, $\{10–19\}$, $\{1,19–24,26–28\}$, $\{1,29–33,35,36,38,46\}$, $\{10,25,39–45,47\}$, $\{19,34,37,48,50–54,56\}$, $\{49,55,57–60,62–65\}$, $\{61,66–72,74,75\}$, $\{73,76–84\}$
7^{16}	$Q(4,4)$	$\{1–10\}$, $\{1,11–16,18–20\}$, $\{11,17,21–26,28,29\}$, $\{2,21,27,30–33,35,36,38\}$, $\{1,34,39–46\}$, $\{2,37,47–49,51,52,54–56\}$, $\{11,50,53,57,58,60–64\}$, $\{21,59,65–68,70–73\}$, $\{69,74–82\}$, $\{10j+i \mid 3 \le i \le 12\}$, $8 \le j \le 10$
11^{12}	$Q(4,6)$	$\{1–10\}$, $\{10–19\}$, $\{19–28\}$, $\{1,28–36\}$, $\{1,64–72\}$, $\{10,37–45\}$, $\{19,46–54\}$, $\{28,55–63\}$, $\{10j+i \mid 3 \le i \le 12\}$, $7 \le j \le 12$
17^{12}	$Q(3,15)$	$\{1–10\}$, $\{10–19\}$, $\{1,19–24,26–28\}$, $\{1,29–36,38\}$, $\{10,25,39–45,47\}$, $\{19,37,46,48,50–54,203\}$, $\{49,195–202,204\}$, $\{10j+i \mid 5 \le i \le 14\}$, $5 \le j \le 18$

Prestructures for the remaining cases are given below.

Type	Prestructure	Type	Prestructure	Type	Prestructure	Type	Prestructure
3^{16}	$P(3,2)$	3^{62}	$P(5,14)$	11^{16}	$P(5,13)$	13^{24}	$P(9,23)$
3^{20}	$P(0,6)$	7^{10}	$P(0,7)$	11^{18}	$P(3,17)$	13^{34}	$P(9,36)$
3^{22}	$P(5,2)$	7^{18}	$P(5,8)$	11^{22}	$P(9,16)$	13^{36}	$P(3,44)$
3^{26}	$P(3,5)$	7^{22}	$P(7,9)$	11^{24}	$P(7,20)$	13^{42}	$P(5,50)$
3^{32}	$P(5,5)$	7^{24}	$P(3,14)$	11^{34}	$P(7,31)$	15^{14}	$P(0,21)$
3^{34}	$P(9,2)$	7^{34}	$P(3,21)$	11^{36}	$P(5,35)$	19^{12}	$P(3,20)$
3^{36}	$P(3,8)$	7^{36}	$P(9,17)$	11^{42}	$P(9,38)$	21^{14}	$P(7,23)$
3^{38}	$P(7,5)$	7^{42}	$P(7,23)$	13^{10}	$P(0,13)$	23^{12}	$P(5,23)$
3^{42}	$P(5,8)$	9^{12}	$P(3,8)$	13^{16}	$P(3,18)$	25^{12}	$P(0,30)$
3^{44}	$P(9,5)$	9^{14}	$P(5,8)$	13^{18}	$P(7,17)$	33^{14}	$P(9,38)$
3^{56}	$P(3,14)$	11^{10}	$P(0,11)$	13^{22}	$P(5,24)$	39^{14}	$P(5,50)$

This completes the construction of all GDDs whose existence is claimed in Lemma 4.1.

References

1. Abel, R.J.R., Brouwer, A.E., Colbourn, C.J., Dinitz, J.H.: Mutually orthogonal Latin squares (MOLS). In: C.J. Colbourn and J.H. Dinitz: The CRC Handbook of Combinatorial Designs, Boca Raton: CRC Press (1996)
2. Abel, R.J.R., Colbourn, C.J., Dinitz, J.H.: Incomplete MOLS. In: C.J. Colbourn and J.H. Dinitz: The CRC Handbook of Combinatorial Designs, ch. II. 3, pp. 142–172, Boca Raton: CRC Press (1996)
3. Colbourn, C.J.: Small group divisible designs with block size three. J. Comb. Math. Comb. Comput. **14**, 153–171 (1993)
4. Colbourn, C.J., Cusack, C.A., Kreher, D.L.: Partial Steiner triple systems with equal-sized holes. J. Comb. Theory Ser. A **70**, 56–65 (1995)
5. Colbourn, C.J., Hoffman, D.G., Rees, R.: A new class of group divisible designs with block size three. J. Comb. Theory Ser. A **59**, 73–89 (1992)
6. Fort, J.M.K., Hedlund, G.A.: Minimal coverings of pairs by triples. Pac. J. Math. **8**, 709–719 (1958)
7. Gronau, H.-D.O.F., Mullin, R.C., Pietsch, C.: The closure of all subsets of $\{3, 4, \ldots, 10\}$ which include 3. Ars Comb. 129–161 (1995)
8. Hanani, H.: Balanced incomplete block designs and related designs. Discrete Math. **11**, 255–369 (1975)
9. Heinrich, K.: Graph decompositions and designs. In: C.J. Colbourn and J.H. Dinitz: The CRC Handbook of Combinatorial Designs, ch. IV. 22, pp. 361–366, Boca Raton: CRC Press (1996)
10. Huang, C., Mendelsohn, E., Rosa, A.: On partially resolvable *t*-partitions. Ann. Discrete Math. **12**, 169–183 (1982)
11. Mendelsohn, E., Rees, R.: Mandatory representation designs. J. Comb. Theory Ser. A **49**, 349–362 (1988)
12. Rees, R.: Two new direct product-type constructions for resolvable group-divisible designs. J. Comb. Designs **1**, 15–26 (1993)
13. Stinson, D.R.: Hill-climbing algorithms for the construction of combinatoral designs. Ann. Discrete Math. **26**, 321–334 (1985)

14. Wilson, R.M.: An existence theory for pairwise balanced designs I: Composition theorems and morphisms. J. Comb. Theory Ser. A **13**, 220–245 (1971)
15. Wilson, R.M.: Constructions and uses of pairwise balanced designs. In: Combinatorics Part I: Theory of Designs, Finite Geometry and Coding Theory, vol. 55 of Mathematical Centre Tracts, pp. 18–41, Amsterdam: Mathematisch Centrum (1975)

Received: July 21, 1999
Final version received: September 10, 2001