

## Chapter 3: Quantum Entanglement

They don't think it be like it is,  
but it do.

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### I. QUANTUM STATES OF MULTI-PARTICLE SYSTEMS

So far, we have studied quantum mechanical systems consisting of single particles. The next important step is to look at systems of more than one particle. We shall see that the postulates of quantum mechanics, when applied to multi-particle systems, give rise to interesting and counterintuitive phenomena such as **quantum entanglement**.

Suppose we have two particles labeled  $A$  and  $B$ . If each individual particle is treated as a quantum system, the postulates of quantum mechanics require that its state be described by a vector in a Hilbert space. Let  $\mathcal{H}_A$  and  $\mathcal{H}_B$  denote the respective single-particle Hilbert spaces. Then the Hilbert space for the combined system of two particles is

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \quad (3.1)$$

The symbol  $\otimes$  refers to a **tensor product**, a mathematical operation that combines two Hilbert spaces to form another Hilbert space. It is most easily understood in terms of explicit basis vectors: let  $\mathcal{H}_A$  be spanned by a basis  $\{|\mu_1\rangle, |\mu_2\rangle, |\mu_3\rangle, \dots\}$ , and  $\mathcal{H}_B$  be spanned by  $\{|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle, \dots\}$ . Then  $\mathcal{H}_A \otimes \mathcal{H}_B$  is a space spanned by basis vectors consisting of *pairwise combinations* of basis vectors drawn from the  $\mathcal{H}_A$  and  $\mathcal{H}_B$  bases:

$$\left\{ |\mu_i\rangle \otimes |\nu_j\rangle \text{ for all } |\mu_i\rangle, |\nu_j\rangle \right\}. \quad (3.2)$$

Thus, if  $\mathcal{H}_A$  has dimension  $d_A$  and  $\mathcal{H}_B$  has dimension  $d_B$ , then  $\mathcal{H}_A \otimes \mathcal{H}_B$  has dimension  $d_A d_B$ . Any two-particle state can be written as a superposition of these basis vectors:

$$|\psi\rangle = \sum_{ij} c_{ij} |\mu_i\rangle \otimes |\nu_j\rangle. \quad (3.3)$$

The inner product between the tensor product basis states is defined as follows:

$$\left( |\mu_i\rangle \otimes |\nu_j\rangle, |\mu_p\rangle \otimes |\nu_q\rangle \right) \equiv \left( \langle \mu_i | \otimes \langle \nu_j | \right) \left( |\mu_p\rangle \otimes |\nu_q\rangle \right) \equiv \langle \mu_i | \mu_p \rangle \langle \nu_j | \nu_q \rangle = \delta_{ip} \delta_{jq}. \quad (3.4)$$

In other words, the inner product is performed “slot-by-slot”. We calculate the inner product for  $A$ , calculate the inner product for  $B$ , and then multiply the two resulting numbers. You can check that this satisfies all the formal requirements for an inner product in linear algebra (see [Exercise 1](#)).

For example, suppose  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are both 2D Hilbert spaces describing spin-1/2 degrees of freedom. Each space can be spanned by an orthonormal basis  $\{|+z\rangle, |-z\rangle\}$ , representing “spin-up” and “spin-down”. Then the tensor product space  $\mathcal{H}$  is a 4D space spanned by

$$\left\{ |+z\rangle \otimes |+z\rangle, |+z\rangle \otimes |-z\rangle, |-z\rangle \otimes |+z\rangle, |-z\rangle \otimes |-z\rangle \right\}. \quad (3.5)$$

We now make an important observation. If  $A$  is in state  $|\mu\rangle$  and  $B$  is in state  $|\nu\rangle$ , then the state of the combined system is fully specified:  $|\mu\rangle \otimes |\nu\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ . But the reverse is not generally true! There exist states of the combined system that *cannot* be expressed in terms of definite states of the individual particles. For example, consider the following quantum state of two spin-1/2 particles:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle \right). \quad (3.6)$$

This state is constructed from two of the four basis states in (3.5), and you can check that the factor of  $1/\sqrt{2}$  ensures the normalization  $\langle\psi|\psi\rangle = 1$  with the inner product rule (3.4). It is evident from looking at Eq. (3.6) that neither  $A$  nor  $B$  possesses a definite  $|+\rangle$  or  $|-\rangle$  state. Moreover, we shall show (in Section VII) that there's *no* choice of basis that allows this state to be expressed in terms of definite individual-particle states; i.e.,

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \quad \text{for any } |\psi_A\rangle \in \mathcal{H}_A, |\psi_B\rangle \in \mathcal{H}_B. \quad (3.7)$$

In such a situation, the two particles are said to be **entangled**.

It is cumbersome to keep writing  $\otimes$  symbols, so we will henceforth omit the  $\otimes$  in cases where the tensor product is obvious. For instance,

$$\frac{1}{\sqrt{2}} \left( |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle \right) \equiv \frac{1}{\sqrt{2}} \left( |+\rangle |-\rangle - |-\rangle |+\rangle \right). \quad (3.8)$$

For systems of more than two particles, quantum states can be defined using multiple tensor products. Suppose a quantum system contains  $N$  particles described by the individual Hilbert spaces  $\{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_N\}$  having dimensionality  $\{d_1, \dots, d_N\}$ . Then the overall system is described by the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_N, \quad (3.9)$$

which has dimensionality  $d = d_1 d_2 \dots d_N$ . The dimensionality scales exponentially with the number of particles! For instance, if each particle has a 2D Hilbert space, a 20-particle system has a Hilbert space with  $2^{20} = 1\,048\,576$  dimensions. Thus, even in quantum systems with a modest number of particles, the quantum state can carry huge amounts of information. This is one of the motivations behind the active research field of quantum computing.

Finally, a proviso: although we refer to subsystems like  $A$  and  $B$  as “particles” for narrative convenience, they need not be actual particles. All this formalism applies to general subsystems—i.e., subsets of a large quantum system’s degrees of freedom. For instance, if a quantum system has a position eigenbasis for 3D space, the  $x$ ,  $y$ , and  $z$  coordinates are distinct degrees of freedom, so each position eigenstate is really a tensor product:

$$|\mathbf{r} = (x, y, z)\rangle \equiv |x\rangle |y\rangle |z\rangle.$$

Also, if the subsystems really *are* particles, we are going to assume for now that the particles are distinguishable. There are other complications that arise if the particles are “identical”, which will be the subject of the next chapter (if you’re unsure what this means, just read on).

## II. PARTIAL MEASUREMENTS

Let us recall how measurements work in single-particle quantum theory. Each observable  $Q$  is described by some Hermitian operator  $\hat{Q}$ , which has an eigenbasis  $\{|q_i\rangle\}$  such that

$$\hat{Q}|q_i\rangle = q_i|q_i\rangle. \quad (3.10)$$

For simplicity, let the eigenvalues  $\{q_i\}$  be non-degenerate. Suppose a particle initially has quantum state  $|\psi\rangle$ . This can always be expanded in terms of the eigenbasis of  $\hat{Q}$ :

$$|\psi\rangle = \sum_i \psi_i |q_i\rangle, \quad \text{where } \psi_i = \langle q_i|\psi\rangle. \quad (3.11)$$

The **measurement postulate of quantum mechanics** states that if we measure  $Q$ , then (i) the probability of obtaining the measurement outcome  $q_i$  is  $P_i = |\psi_i|^2$ , the absolute square of the coefficient of  $|q_i\rangle$  in the basis expansion; and (ii) upon obtaining this outcome, the system instantly “collapses” into state  $|q_i\rangle$ .

Mathematically, these two rules can be summarized using the projection operator

$$\hat{\Pi}(q_i) = |q_i\rangle\langle q_i|. \quad (3.12)$$

Applying this operator to  $|\psi\rangle$  gives the non-normalized state vector

$$|\psi'\rangle = |q_i\rangle\langle q_i|\psi\rangle. \quad (3.13)$$

From this, we glean two pieces of information:

- The probability of obtaining this outcome is  $\langle\psi'|\psi'\rangle = |\langle q_i|\psi\rangle|^2$ .
- The post-collapse state is obtained by the re-normalization  $|\psi'\rangle \rightarrow |q_i\rangle$ .

For multi-particle systems, there is a new complication: what if a measurement is performed on just one particle?

Consider a system of two particles A and B, with two-particle Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ . We perform a measurement on particle A, corresponding to a Hermitian operator  $\hat{Q}_A$  that acts upon  $\mathcal{H}_A$  and has eigenvectors  $\{|\mu\rangle \mid \mu = 1, 2, \dots\}$  (i.e., the eigenvectors are enumerated by some index  $\mu$ ). We can write any state  $|\psi\rangle$  using the eigenbasis of  $\hat{Q}_A$  for the  $\mathcal{H}_A$  part, and an arbitrary basis  $\{|\nu\rangle\}$  for the  $\mathcal{H}_B$  part:

$$\begin{aligned} |\psi\rangle &= \sum_{\mu\nu} \psi_{\mu\nu} |\mu\rangle|\nu\rangle \\ &= \sum_{\mu} |\mu\rangle|\varphi_{\mu}\rangle, \quad \text{where } |\varphi_{\mu}\rangle \equiv \sum_{\nu} \psi_{\mu\nu} |\nu\rangle \in \mathcal{H}_B. \end{aligned} \quad (3.14)$$

Unlike the single-particle case, the “coefficient” of  $|\mu_i\rangle$  in this basis expansion is not a complex number, but a vector in  $\mathcal{H}_B$ .

Proceeding by analogy, the probability of obtaining the outcome labelled by  $\mu$  should be the “absolute square” of this “coefficient”,  $\langle\varphi_{\mu}|\varphi_{\mu}\rangle$ . Let us define the partial projector

$$\hat{\Pi}(\mu) = |\mu\rangle\langle\mu| \otimes \hat{I}. \quad (3.15)$$

The  $A$  slot of this operator contains a projector,  $|\mu\rangle\langle\mu|$ , while the  $B$  slot leaves the  $\mathcal{H}_B$  part of the two-particle space unchanged. Applying the partial projector to the state given in Eq. (3.14) gives

$$|\psi'\rangle = \hat{\Pi}(\mu)|\psi\rangle = |\mu\rangle|\varphi_\mu\rangle. \quad (3.16)$$

Now we follow the same measurement rules as before. The outcome probability is

$$P_\mu = \langle\psi'|\psi'\rangle = \langle\mu|\mu\rangle\langle\varphi_\mu|\varphi_\mu\rangle = \sum_\nu |\psi_{\mu\nu}|^2. \quad (3.17)$$

The post-measurement collapsed state is obtained by the re-normalization

$$|\psi'\rangle \rightarrow \frac{1}{\sqrt{\sum_{\nu'} |\psi_{\mu\nu'}|^2}} \sum_\nu \psi_{\mu\nu} |\mu\rangle|\nu\rangle. \quad (3.18)$$

*Example*—A system of two spin-1/2 particles is in the “singlet state”

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle|-\rangle - |-\rangle|+\rangle \right). \quad (3.19)$$

For each particle,  $|+\rangle$  and  $|-\rangle$  denote eigenstates of the operator  $\hat{S}_z$ , with eigenvalues  $+\hbar/2$  and  $-\hbar/2$  respectively. Suppose we measure  $S_z$  on particle A. What are the probabilities of the possible outcomes, and the associated post-collapse states?

- **First outcome:**  $+\hbar/2$ .

- The partial projector is  $|+\rangle\langle+| \otimes \hat{I}$ .
- Applying the projection to  $|\psi\rangle$  yields  $|\psi'\rangle = (1/\sqrt{2})|+\rangle|-\rangle$ .
- The outcome probability is  $P_+ = \langle\psi'|\psi'\rangle = \frac{1}{2}$ .
- The post-collapse state is  $\frac{1}{\sqrt{P_+}}|\psi'\rangle = |+\rangle|-\rangle$

- **Second outcome:**  $-\hbar/2$ .

- The partial projector is  $|-\rangle\langle-| \otimes \hat{I}$ .
- Applying the projection to  $|\psi\rangle$  yields  $|\psi'\rangle = (1/\sqrt{2})|-\rangle|+\rangle$ .
- The outcome probability is  $P_- = \langle\psi'|\psi'\rangle = \frac{1}{2}$ .
- The post-collapse state is  $\frac{1}{\sqrt{P_-}}|\psi'\rangle = |-\rangle|+\rangle$ .

The two possible outcomes,  $+\hbar/2$  and  $-\hbar/2$ , occur with equal probability. In either case, the two-particle state collapses so that  $A$  is in the observed spin eigenstate, and  $B$  has the opposite spin. After the collapse, the two-particle state is no longer entangled.

### III. THE EINSTEIN-PODOLSKY-ROSEN “PARADOX”

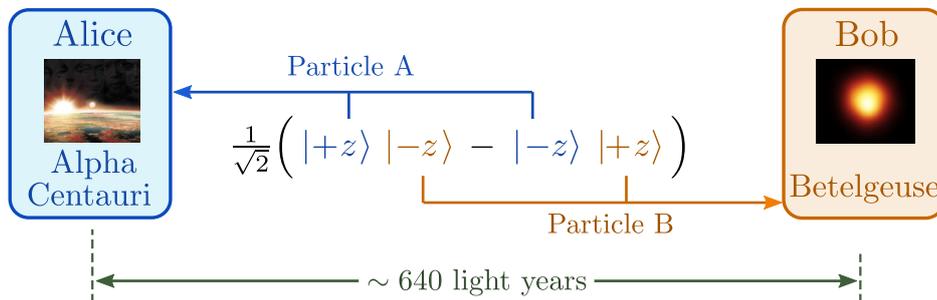
In 1935, Einstein, Podolsky, and Rosen (EPR) formulated a thought experiment, now known as the **EPR paradox**, that highlights the counter-intuitive features of quantum entanglement [3]. They tried to use this thought experiment to argue that quantum theory cannot serve as a fundamental description of reality. Subsequently, however, it was shown that the EPR paradox is not an actual paradox; physical systems *really do* have the strange behavior that the thought experiment highlighted.

Consider an entangled state, like the following “singlet state” of two spin-1/2 particles:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right). \quad (3.20)$$

As before, let the two particles be labeled  $A$  and  $B$ . Measuring  $S_z$  on  $A$  collapses the system into a two-particle state that is unentangled, where each particle has a definite spin. If the measurement outcome is  $+\hbar/2$ , the new state is  $|+z\rangle|-z\rangle$ , whereas if the outcome is  $-\hbar/2$ , the new state is  $|-z\rangle|+z\rangle$ .

The postulates of quantum theory seem to indicate that the state collapse happens instantaneously, regardless of the distance separating the particles. Imagine that we prepare the two-particle state in a laboratory on Earth. Particle  $A$  is then transported to the laboratory of Alice, in the Alpha Centauri star system, and particle  $B$  is transported to the laboratory of Bob, in the Betelgeuse system, separated by  $\sim 640$  light years. In principle, this can be done carefully enough to avoid disturbing the two-particle quantum state.



Once ready, Alice measures  $\hat{S}_z$  on particle  $A$ , which induces an instantaneous collapse of the two-particle state. Immediately afterwards, Bob measures  $\hat{S}_z$  on particle  $B$ , and obtains—with 100% certainty—the opposite spin. During the time interval between these two measurements, no classical signal could have traveled between the two star systems, not even at the speed of light. Yet the state collapse induced by Alice’s measurement has a definite effect on the result of Bob’s measurement.

There are three noteworthy aspects of this phenomenon:

First, it dispels some commonsensical but mistaken “explanations” for quantum state collapse in terms of perturbative effects. For instance, it is sometimes explained that if we want to measure a particle’s position, we need to shine a light beam on it, or disturb it in some way, and this disturbance generates an uncertainty in the particle’s momentum. The EPR paradox shows that such stories don’t capture the full weirdness of quantum state collapse, for we can collapse the state of a particle by doing a measurement on *another* particle far away!

Second, our experimentalists have a certain amount of control over the state collapse, due to the choice of what measurement to perform. So far, we have considered  $S_z$  measurements performed by Alice on particle  $A$ . But Alice can choose to measure the spin of  $A$  along another axis, say  $S_x$ . In the basis of spin-up and spin-down states, the operator  $\hat{S}_x$  has matrix representation

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.21)$$

The eigenvalues and eigenvectors are

$$\begin{aligned} s_x = \frac{\hbar}{2}, \quad |+\rangle &= \frac{1}{\sqrt{2}} (|+z\rangle + |-z\rangle) \\ s_x = -\frac{\hbar}{2}, \quad |-\rangle &= \frac{1}{\sqrt{2}} (|+z\rangle - |-z\rangle). \end{aligned} \quad (3.22)$$

Conversely, we can write the  $\hat{S}_z$  eigenstates in the  $\{|+\rangle, |-\rangle\}$  basis:

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle). \end{aligned} \quad (3.23)$$

This allows us to write the two-particle entangled state in the  $\hat{S}_x$  basis:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|-\rangle|+\rangle - |+\rangle|-\rangle). \quad (3.24)$$

Alice's measurement still collapses the particles into definite spin states with opposite spins—but now spin states of  $S_x$  rather than  $S_z$ .

Third, this ability to choose the measurement axis *does not* allow for superluminal communication. Alice can choose whether to (i) measure  $S_z$  or (ii) measure  $S_x$ , and this choice instantaneously affects the quantum state of particle  $B$ . If Bob can find a way to distinguish between the cases (i) and (ii), even statistically, this would serve as a method for instantaneous communication, violating the theory of relativity! Yet this turns out to be impossible. The key problem is that quantum states themselves cannot be measured; only observables can be measured. Suppose Alice's measurement is  $\hat{S}_z$ , which collapses  $B$  to either  $|+\rangle$  or  $|-\rangle$ , each with probability  $1/2$ . Bob must now choose which measurement to perform. If he measures  $S_z$ , the outcome is  $+\hbar/2$  or  $-\hbar/2$  with equal probabilities. If he measures  $S_x$ , the probabilities are:

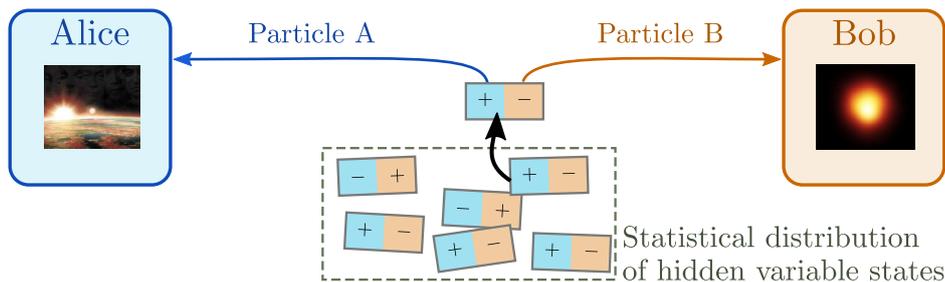
$$\begin{aligned} P(S_x = +\hbar/2) &= \frac{1}{2} |\langle +x|+\rangle|^2 + \frac{1}{2} |\langle +x|-\rangle|^2 = \frac{1}{2} \\ P(S_x = -\hbar/2) &= \frac{1}{2} |\langle -x|+\rangle|^2 + \frac{1}{2} |\langle -x|-\rangle|^2 = \frac{1}{2}. \end{aligned} \quad (3.25)$$

The probabilities are still equal! Repeating this analysis for any other choice of spin axis, we find that the two possible outcomes always have equal probability. Thus, Bob's measurement does not yield any information about Alice's choice of measurement axis.

Since quantum state collapse does not allow for superluminal communication, it is consistent *in practice* with the theory of relativity. However, state collapse is still **nonlocal**,

in the sense that unobservable ingredients of the theory (quantum states) can change faster than light can travel between two points. For this reason, EPR argued that quantum theory is *philosophically* inconsistent with relativity.

EPR suggested an alternative: maybe quantum mechanics is an approximation of some deeper theory, whose details are currently unknown, but which is deterministic and local. Such a “**hidden variable theory**” may give the appearance of quantum state collapse in the following way. Suppose each particle has a definite but “hidden” value of  $S_z$ , either  $S_z = +\hbar/2$  or  $S_z = -\hbar/2$ ; let us denote these as  $[+]$  or  $[-]$ . We can hypothesize that the two-particle quantum state  $|\psi\rangle$  is not an actual description of reality; rather, it corresponds to a *statistical* distribution of “hidden variable” states, denoted by  $[+; -]$  (i.e.,  $S_z = +\hbar/2$  for particle  $A$  and  $S_z = -\hbar/2$  for particle  $B$ ), and  $[-; +]$  (the other way around).



When Alice measures  $S_z$ , the value of the hidden variable is revealed. A result of  $+z$  implies  $[+; -]$ , whereas  $-z$  implies  $[-; +]$ . When Bob subsequently measures  $S_z$ , the result obtained is the opposite of Alice’s result. But those were simply the values all along—there is no instantaneous physical influence traveling between their two laboratories.

Clearly, there are many missing details in this hypothetical description. Any actual hidden variable theory would also need to replicate the huge list of successful predictions made by quantum theory. Trying to come up with a suitable theory of this sort seems difficult, but with enough hard work, one might imagine that it is doable.

#### IV. BELL’S THEOREM

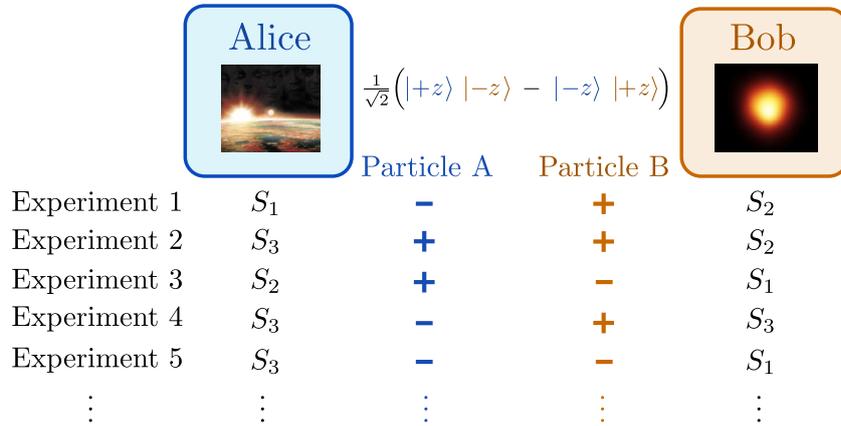
In 1964, John S. Bell published a bombshell paper showing that the predictions of quantum theory are *inherently inconsistent* with hidden variable theories [4]. The amazing thing about this result, known as **Bell’s theorem**, is that it requires no knowledge about the details of the hidden variable theory, just that it is deterministic and local. Here, we present a simplified version of Bell’s theorem due to Mermin [5].

We again consider spin-1/2 particle pairs, with particle  $A$  sent to Alice at Alpha Centauri, and particle  $B$  to Bob at Betelgeuse. Each experimentalist can measure the particle’s spin along three distinct choices of spin axis. These spin observables are denoted by  $S_1$ ,  $S_2$ , and  $S_3$ . We will not specify the actual directions of these spin axes until later in the proof. For now, just note that the axes need not correspond to orthogonal spatial directions.

We repeatedly prepare the particle pairs in the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle|-\rangle - |-\rangle|+\rangle \right), \quad (3.26)$$

and send the respective particles to Alice and Bob. During each round of the experiment, each experimentalist randomly chooses one of the three spin axes  $S_1$ ,  $S_2$ , or  $S_3$ , and performs that spin measurement. It doesn't matter which experimentalist performs the measurement first; the experimentalists can't influence each other, as there is not enough time for a light-speed signal to travel between the two locations. Many rounds of the experiment are conducted; for each round, both experimentalists' choices of spin axis are recorded, along with their measurement results.



At the end, the experimental records are brought together and examined. We assume that the results are consistent with the predictions of quantum theory. Among other things, this means that whenever the experimentalists happen to choose the same measurement axis, they always find opposite spins. (For example, this is the case during “Experiment 4” in the above figure, where both experimentalists happened to measure  $S_3$ .)

Can a hidden variable theory reproduce the results predicted by quantum theory? In a hidden variable theory, each particle must have a definite value for each spin observable. For example, particle  $A$  might have  $S_1 = +\hbar/2$ ,  $S_2 = +\hbar/2$ ,  $S_3 = -\hbar/2$ . Let us denote this by  $[+ + -]$ . To be consistent with the predictions of quantum theory, the hidden spin variables for the two particles must have opposite values along each direction. This means that there are 8 distinct possibilities, which we can denote as

$$\begin{aligned}
 &[+ + +; - - -], \quad [ + + -; - - +], \quad [ + - +; - + -], \quad [ + - -; - + +], \\
 &[- + +; + - -], \quad [- + -; + - +], \quad [- - +; + + -], \quad [- - -; + + +].
 \end{aligned}$$

For instance,  $[+ + -; - - +]$  indicates that for particle  $A$ ,  $S_1 = S_2 = +\hbar/2$  and  $S_3 = -\hbar/2$ , while particle  $B$  has the opposite spin values,  $S_1 = S_2 = -\hbar/2$  and  $S_3 = +\hbar/2$ . So far, however, we don't know anything about the relative probabilities of these 8 cases.

Let's now focus on the subset of experiments in which the two experimentalists happened to choose *different* spin axes (e.g., Alice chose  $S_1$  and Bob chose  $S_2$ ). Within this subset, what is the probability *for the two measurement results to have opposite signs* (i.e., one + and one -)? To answer this question, we first look at the following 6 cases:

$$\begin{aligned}
 &[ + + -; - - +], \quad [ + - +; - + -], \quad [ + - -; - + +], \\
 &[- + +; + - -], \quad [- + -; + - +], \quad [- - +; + + -].
 \end{aligned}$$

These are the cases which do not have all + or all - for each particle. Consider one of these, say  $[+ + -; - - +]$ . The two experimentalists picked their measurement axes at random

each time, and amongst the experiments where they picked different axes, there are two ways for the measurement results to have opposite signs:  $(S_1, S_2)$  or  $(S_2, S_1)$ . There are four ways to get the same sign:  $(S_1, S_3)$ ,  $(S_2, S_3)$ ,  $(S_3, S_1)$  and  $(S_3, S_2)$ . Thus, for this particular set of hidden variables, the probability for measurement results with opposite signs is  $1/3$ . If we go through all 6 of the cases listed above, we find that in all cases, the probability for opposite signs is  $1/3$ .

Now look at the remaining 2 cases:

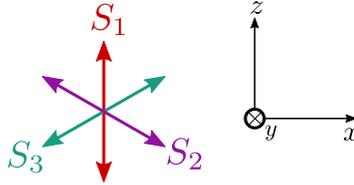
$$[+++; ---], \quad [---; +++].$$

For these, Alice and Bob always obtain results with opposite signs. Combining this with the findings from the previous paragraph, we obtain the following statement:

*Given that the two experimentalists choose different spin axes, the probability that their results have opposite signs is  $P \geq 1/3$ .*

This is called **Bell's inequality**. If we can arrange a situation where quantum theory predicts a probability  $P < 1/3$  (i.e., a violation of Bell's inequality), that would mean that quantum theory is inherently inconsistent with local deterministic hidden variables. This conclusion would hold regardless of the "inner workings" of the hidden variable theory. In particular, note that the above derivation made no assumptions about the relative probabilities of the hidden variable states.

To complete the proof, we must find a set  $\{S_1, S_2, S_3\}$  such that the predictions of quantum mechanics violate Bell's inequality. One simple choice is to align  $S_1$  with the  $z$  axis, and align  $S_2$  and  $S_3$  along the  $x$ - $z$  plane at  $120^\circ$  ( $2\pi/3$  radians) from  $S_1$ , as shown below:



The corresponding spin operators can be written in the eigenbasis of  $\hat{S}_z$ :

$$\begin{aligned}\hat{S}_1 &= \frac{\hbar}{2} \sigma_3 \\ \hat{S}_2 &= \frac{\hbar}{2} [\cos(2\pi/3)\sigma_3 + \sin(2\pi/3)\sigma_1] \\ \hat{S}_3 &= \frac{\hbar}{2} [\cos(2\pi/3)\sigma_3 - \sin(2\pi/3)\sigma_1].\end{aligned}\tag{3.27}$$

Suppose Alice chooses  $S_1$ , and obtains  $+\hbar/2$ . Particle  $A$  collapses to state  $|+z\rangle$ , and particle  $B$  collapses to state  $|-z\rangle$ . Bob is assumed to choose a different spin axis. If the choice is  $S_2$ , the expectation value is

$$\begin{aligned}\langle -z | S_2 | -z \rangle &= \frac{\hbar}{2} \left[ \cos(2\pi/3) \langle -z | \sigma_3 | -z \rangle + \sin(2\pi/3) \langle -z | \sigma_1 | -z \rangle \right] \\ &= \frac{\hbar}{2} \cdot \frac{1}{2}\end{aligned}\tag{3.28}$$

If  $P_+$  and  $P_-$  respectively denote the probability of measuring  $+\hbar/2$  and  $-\hbar/2$  in this measurement, the above equation implies that  $P_+ - P_- = +1/2$ . Moreover,  $P_+ + P_- = 1$  by probability conservation. Hence, the probability of obtaining a negative value (the opposite sign from Alice's measurement) is  $P_- = 1/4$ . All the other possible scenarios are worked out similarly. The result is that the overall probability of the two experimentalists obtaining opposite results (in the cases where they choose different measurement axis) is  $1/4$ . Bell's inequality is violated!

Last of all, we must consult Nature itself. Is it possible to observe, in an actual experiment, probabilities that violate Bell's inequality? In the decades following Bell's 1964 paper, many experiments were performed to answer this question. These experiments are all substantially more complicated than the simple two-particle spin-1/2 model that we've studied, and they are subject to various uncertainties and "loopholes" that are beyond the scope of our discussion. But in the end, the experimental consensus appears to be a clear *yes*: Nature really does behave according to quantum mechanics, and in a manner that cannot be replicated by deterministic local hidden variables! A summary of the experimental evidence is given in a review paper by Aspect [6].

## V. QUANTUM CRYPTOGRAPHY

One of the most remarkable consequences of Bell's thought experiment is that it provides a way to perform cryptography that is more secure, in certain respects, than conventional cryptography. This possibility was first raised by Ekert [7], and it has led to a huge amount of research into **quantum cryptography**, which is poised to be one of the most important technological applications of quantum mechanics.

Ekert's quantum cryptography scheme allows two participants, Alice and Bob, to share with each other a string of random binary digits (0 or 1), called a "key", in such a manner that no one else can learn the key by eavesdropping on their communications. Once Alice and Bob have established a secret shared key, it can be used to encrypt subsequent messages between them, which nobody else can decipher (e.g., by using [one-time pads](#)).

The scheme follows almost immediately from the Bell thought experiment of Section IV. In each round, a pair of spin-1/2 particles is prepared in the singlet state, with particle  $A$  sent to Alice and  $B$  sent to Bob. Alice and Bob each randomly choose a measurement axis ( $S_1$ ,  $S_2$ , or  $S_3$ ), and measure the spin of their particle along that axis.

After an appropriate number of rounds, Alice and Bob publicly announce their choices of measurement axes. These announcements are assumed to take place over a classical communication channel that cannot be jammed or manipulated by any hostile party (though it can be eavesdropped upon). From the announcements, Alice and Bob determine the rounds in which they happened to pick the same axes. Their measurement results during these rounds are guaranteed to be the opposites of each other. Hence, they have established a random binary string known to each other but to no one else.

How might an eavesdropper, Eve, attempt to foil this scheme? Suppose Eve can intercept some or all of the particles  $B$  destined for Bob. She might try to substitute her own measurements, in a manner that could let her work out the secret key. However, Eve is hampered by the fact that she is unable to predict or influence Bob's choices of measurement

axes (i.e., Bob's choices are truly random), nor is she able to impersonate Bob during the announcements of the axis choices (i.e., the classical communication channel is unjammable). Under these assumptions, it can be shown that any attempt by Eve to substitute her own measurements can be detected by Alice and Bob, by performing a statistical analysis of their measurement results in the rounds with different different axis choices. The detection of the eavesdropper turns out to be essentially the same as checking for Bell's inequality. For details, refer to Ref. [7].

Alternatively, Eve might try to "clone" the quantum state of particle  $B$  before passing it along to Bob. If this can be done, Eve can retain the cloned quantum state, wait for Bob to announce his choice of measurement axis for that round, and then perform the corresponding measurement to reproduce Bob's result. Though plausible at first glance, this turns out to be fundamentally unworkable, as it is incompatible with the laws of quantum mechanics.

The so-called **no-cloning theorem** can be proven as follows. Eve desires to clone an arbitrary state of a spin-half particle  $B$  onto another spin-half particle  $C$ . The two-particle Hilbert space is  $\mathcal{H} \otimes \mathcal{H}$ . With particle  $C$  initially prepared in some state  $|0\rangle$ , Eve must devise a unitary operation  $\hat{U}$ , representing the cloning process, such that

$$\hat{U}|\psi\rangle|0\rangle = e^{i\phi}|\psi\rangle|\psi\rangle \quad (3.29)$$

for all  $|\psi\rangle \in \mathcal{H}$ , and for some phase factor  $\phi$  that could depend on  $|\psi\rangle$ . Note that the value of  $\phi$  does not affect the outcomes of measurements.

Now replace  $|\psi\rangle$  in the above equation with two arbitrary states denoted by  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , and take their inner product. According to Eq. (3.29),

$$\left(\langle\psi_1|\langle 0|\hat{U}^\dagger\right)\left(\hat{U}|\psi_2\rangle|0\rangle\right) = \left(\langle\psi_1|\langle\psi_1|e^{-i\phi_1}\right)\left(e^{i\phi_2}|\psi_2\rangle|\psi_2\rangle\right) \quad (3.30)$$

$$= e^{-i(\phi_1-\phi_2)}\left(\langle\psi_1|\psi_2\rangle\right)^2. \quad (3.31)$$

Here,  $\phi_1$  and  $\phi_2$  are the phase factors from Eq. (3.29) for the two chosen states. On the other hand, since  $\hat{U}$  is unitary,

$$\langle\psi_1|\langle 0|\hat{U}^\dagger\hat{U}|\psi_2\rangle|0\rangle = \left(\langle\psi_1|\langle 0|\right)\left(|\psi_2\rangle|0\rangle\right) \quad (3.32)$$

$$= \langle\psi_1|\psi_2\rangle. \quad (3.33)$$

Here we have used the fact that  $\langle 0|0\rangle = 1$ . Comparing the magnitudes of (3.31) and (3.33),

$$|\langle\psi_1|\psi_2\rangle|^2 = |\langle\psi_1|\psi_2\rangle| \Rightarrow |\langle\psi_1|\psi_2\rangle| = 0 \text{ or } 1. \quad (3.34)$$

But aside from the trivial case of a one-dimensional Hilbert space, this cannot be true for arbitrary  $|\psi_1\rangle$  and  $|\psi_2\rangle$ . For instance, for a two-dimensional space spanned by an orthonormal basis  $\{|0\rangle, |1\rangle\}$ , we can pick

$$|\psi_1\rangle = |0\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \Rightarrow |\langle\psi_1|\psi_2\rangle| = \frac{1}{\sqrt{2}}. \quad (3.35)$$

## VI. DENSITY OPERATORS

We now introduce the **density operator**, which helps to streamline many calculations in multi-particle quantum mechanics.

Consider a quantum system with a  $d$ -dimensional Hilbert space  $\mathcal{H}$ . Given an arbitrary state  $|\psi\rangle \in \mathcal{H}$ , define

$$\hat{\rho} = |\psi\rangle\langle\psi|. \quad (3.36)$$

This is just the projection operator for  $|\psi\rangle$ , but in this context we call it a “density operator”. Some other authors call it a **density matrix**, based on the fact that linear operators can be represented as matrices. It has the following noteworthy features:

1. It is Hermitian.
2. Suppose  $\hat{Q}$  is an observable with eigenvalues  $\{q_\mu\}$  and eigenstates  $\{|\mu\rangle\}$  (where  $\mu$  is some label that enumerates the eigenstates). If we do a  $\hat{Q}$  measurement on  $|\psi\rangle$ , the probability of obtaining  $q_\mu$  is

$$P_\mu = |\langle\mu|\psi\rangle|^2 = \langle\mu|\hat{\rho}|\mu\rangle. \quad (3.37)$$

3. Moreover, the expectation value of the observable is

$$\langle Q \rangle = \sum_\mu q_\mu P_\mu = \sum_\mu q_\mu \langle\mu|\hat{\rho}|\mu\rangle = \text{Tr}[\hat{Q}\hat{\rho}]. \quad (3.38)$$

In the last equality,  $\text{Tr}[\dots]$  denotes the trace, which is the sum of the diagonal elements of the matrix representation of the operator. The value of the trace is basis-independent.

Now consider, once again, a composite system consisting of two subsystems  $A$  and  $B$ , with Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . Let's say we are interested in the physical behavior of  $A$ , that is to say the outcome probabilities and expectation values of any measurements performed on  $A$ . These can be calculated from  $|\psi\rangle$ , the state of the combined system; however,  $|\psi\rangle$  also carries information about  $B$ , which is not relevant to us as we only care about  $A$ .

There is a more economical way to encode just the information about  $A$ . We can define the density operator for subsystem  $A$  (sometimes called the **reduced density operator**):

$$\hat{\rho}_A = \text{Tr}_B[\hat{\rho}]. \quad (3.39)$$

Here,  $\text{Tr}_B[\dots]$  refers to a **partial trace**. This means tracing over the  $\mathcal{H}_B$  part of the Hilbert space  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ , which yields an operator acting on  $\mathcal{H}_A$ .

To better understand Eq. (3.39), let us go to an explicit basis. Let  $\hat{Q}_A$  be an observable for  $\mathcal{H}_A$  with eigenbasis  $\{|\mu\rangle\}$ , and let  $\hat{Q}_B$  be an observable for  $\mathcal{H}_B$  with eigenbasis  $\{|\nu\rangle\}$ . If the density operator of the combined system is  $\hat{\rho} = |\psi\rangle\langle\psi|$ , then

$$\hat{\rho}_A = \sum_\nu \left( \hat{I} \otimes \langle\nu| \right) |\psi\rangle\langle\psi| \left( \hat{I} \otimes |\nu\rangle \right). \quad (3.40)$$

This is a Hermitian operator acting on the  $\mathcal{H}_A$  space. In the  $\{|\mu\rangle\}$  basis, its diagonal matrix elements are

$$\begin{aligned}\langle\mu|\hat{\rho}_A|\mu\rangle &= \sum_{\nu} \left( \langle\mu|\langle\nu| \right) |\psi\rangle\langle\psi| \left( |\mu\rangle|\nu\rangle \right) \\ &= \langle\psi| \left[ |\mu\rangle\langle\mu| \otimes \left( \sum_{\nu} |\nu\rangle\langle\nu| \right) \right] |\psi\rangle \\ &= \langle\psi| \left( |\mu\rangle\langle\mu| \otimes \hat{I}_B \right) |\psi\rangle.\end{aligned}\tag{3.41}$$

According to the rules of partial measurements discussed in Section II, this is precisely the probability of obtaining  $q_\mu$  when measuring  $\hat{Q}_A$  on subsystem  $A$ :

$$P_\mu = \langle\mu|\hat{\rho}_A|\mu\rangle.\tag{3.42}$$

It follows that the expectation value for observable  $\hat{M}$  is

$$\langle Q_A \rangle = \sum_{\mu} q_\mu \langle\mu|\hat{\rho}_A|\mu\rangle = \text{Tr} \left[ \hat{Q}_A \hat{\rho}_A \right].\tag{3.43}$$

These results hold for any choice of basis. Hence, knowing the density operator for  $A$ , we can determine the outcome probabilities of *any* partial measurement performed on  $A$ .

To better understand the properties of  $\hat{\rho}_A$ , let us write  $|\psi\rangle$  explicitly as

$$|\psi\rangle = \sum_{\mu\nu} \psi_{\mu\nu} |\mu\rangle|\nu\rangle,\tag{3.44}$$

where  $\sum_{\mu\nu} |\psi_{\mu\nu}|^2 = 1$ . Then

$$\begin{aligned}\hat{\rho} &= \sum_{\mu\mu'\nu\nu'} \psi_{\mu\nu} \psi_{\mu'\nu'}^* |\mu\rangle|\nu\rangle\langle\mu'|\langle\nu'| \\ \hat{\rho}_A &= \sum_{\mu\mu'\nu} \psi_{\mu\nu} \psi_{\mu'\nu}^* |\mu\rangle\langle\mu'| \\ &= \sum_{\nu} \left( \sum_{\mu} \psi_{\mu\nu} |\mu\rangle \right) \left( \sum_{\mu'} \psi_{\mu'\nu}^* \langle\mu'| \right) \\ &= \sum_{\nu} |\varphi_\nu\rangle\langle\varphi_\nu|, \quad \text{where } |\varphi_\nu\rangle = \sum_{\mu} \psi_{\mu\nu} |\mu\rangle.\end{aligned}\tag{3.45}$$

But  $|\varphi_\nu\rangle$  is not necessarily normalized to unity:  $\langle\varphi_\nu|\varphi_\nu\rangle = \sum_{\mu} |\psi_{\mu\nu}|^2 \leq 1$ . Let us define

$$|\tilde{\varphi}_\nu\rangle = \frac{1}{\sqrt{P_\nu}} |\varphi_\nu\rangle, \quad \text{where } P_\nu = \sum_{\mu} |\psi_{\mu\nu}|^2.\tag{3.46}$$

Note that each  $P_\nu$  is a non-negative real number in the range  $[0, 1]$ . Then

$$\hat{\rho}_A = \sum_{\nu} P_\nu |\tilde{\varphi}_\nu\rangle\langle\tilde{\varphi}_\nu|, \quad \text{where } \begin{cases} \text{each } P_\nu \text{ is a real number in } [0, 1], \text{ and} \\ \text{each } |\tilde{\varphi}_\nu\rangle \in \mathcal{H}_A, \text{ with } \langle\tilde{\varphi}_\nu|\tilde{\varphi}_\nu\rangle = 1. \end{cases}\tag{3.47}$$

In general, we can define a density operator as any operator that has the form of Eq. (3.47), regardless of whether or not it was formally derived via a partial trace. We can interpret it as describing an ensemble of quantum states weighted by a set of classical probabilities. Each term in the sum consists of (i) a weighting coefficient  $P_\nu$  which can be regarded as a probability (the coefficients are all real numbers in the range  $[0, 1]$ , and sum to 1), and (ii) a projection operator associated with some normalized state vector  $|\tilde{\varphi}_\nu\rangle$ . Note that the states in the ensemble do not have to be orthogonal to each other.

From this point of view, a density operator of the form  $|\psi\rangle\langle\psi|$  corresponds to the special case of an ensemble containing only one quantum state  $|\psi\rangle$ . Such an ensemble is called a **pure state**, and describes a quantum system that is not entangled with any other system. If an ensemble is not a pure state, we call it a **mixed state**; it describes a system that is entangled with some other system.

We can show that any linear operator  $\hat{\rho}_A$  obeying Eq. (3.47) has the following properties:

1.  $\hat{\rho}_A$  is Hermitian.
2.  $\langle\varphi|\hat{\rho}_A|\varphi\rangle \geq 0$  for any  $|\varphi\rangle \in \mathcal{H}_A$  (i.e., the operator is positive semidefinite).
3. For any observable  $\hat{Q}_A$  acting on  $\mathcal{H}_A$ ,

$$\begin{aligned}
\langle Q_A \rangle &\equiv \sum_{\nu} P_{\nu} \langle \tilde{\varphi}_{\nu} | \hat{Q}_A | \tilde{\varphi}_{\nu} \rangle \\
&= \sum_{\mu\nu} P_{\nu} \langle \tilde{\varphi}_{\nu} | \mu \rangle \langle \mu | \hat{Q}_A | \tilde{\varphi}_{\nu} \rangle \quad (\text{using some basis } \{|\mu\rangle\}) \\
&= \sum_{\mu} \langle \mu | \hat{Q}_A \left( \sum_{\nu} |\tilde{\varphi}_{\nu}\rangle\langle\tilde{\varphi}_{\nu}| \right) | \mu \rangle \\
&= \text{Tr} \left[ \hat{Q}_A \hat{\rho}_A \right].
\end{aligned} \tag{3.48}$$

This property can be used to deduce the probability of obtaining any measurement outcome: if  $|\mu\rangle$  is the eigenstate associated with the outcome, the outcome probability is  $\langle\mu|\hat{\rho}_A|\mu\rangle$ , consistent with Eq. (3.42). To see this, take  $\hat{Q} = |\mu\rangle\langle\mu|$  in Eq. (3.48).

4. The eigenvalues of  $\hat{\rho}_A$ , denoted by  $\{p_1, p_2, \dots, p_{d_A}\}$ , satisfy

$$p_j \in \mathbb{R} \quad \text{and} \quad 0 \leq p_j \leq 1 \quad \text{for } j = 1, \dots, d_A, \quad \text{with} \quad \sum_{j=1}^{d_A} p_j = 1. \tag{3.49}$$

In other words, the eigenvalues can be interpreted as probabilities. This also implies that  $\text{Tr}[\hat{\rho}_A] = 1$ .

This property follows from Property 3 by taking  $\hat{Q} = |\varphi\rangle\langle\varphi|$ , where  $|\varphi\rangle$  is any eigenvector of  $\hat{\rho}_A$ , and then taking  $\hat{Q} = \hat{I}_A$ .

## VII. ENTANGLEMENT ENTROPY

Previously, we said that a multi-particle system is entangled if the individual particles lack definite quantum states. It would be nice to make this statement more precise, and in

fact physicists have come up with several different quantitative measures of entanglement. In this section, we will describe the most common measure, **entanglement entropy**, which is closely related to the entropy concept from thermodynamics, statistical mechanics, and information theory.

We have seen from the previous section that if a subsystem  $A$  is (possibly) entangled with some other subsystem  $B$ , the information required to calculate all partial measurement outcomes on  $A$  is stored within a reduced density operator  $\hat{\rho}_A$ . We can use this to define a quantity called the **entanglement entropy** of  $A$ :

$$S_A = -k_b \text{Tr}_A \left\{ \hat{\rho}_A \ln[\hat{\rho}_A] \right\}. \quad (3.50)$$

In this formula,  $\ln[\dots]$  denotes the logarithm of an operator, which is the inverse of the exponential:  $\ln(\hat{P}) = \hat{Q} \Rightarrow \exp(\hat{Q}) = \hat{P}$ . The prefactor  $k_b$  is Boltzmann's constant, and ensures that  $S_A$  has the same units as thermodynamic entropy.

The definition of the entanglement entropy is based on an analogy with the entropy concept from classical thermodynamics, statistical mechanics and information theory. In those classical contexts, entropy is a quantitative measure of uncertainty (i.e, lack of information) about a system's underlying microscopic state, or "microstate". Suppose a system has  $W$  possible microstates that occur with probabilities  $\{p_1, p_2, \dots, p_W\}$ , satisfying  $\sum_i p_i = 1$ . Then we define the classical entropy

$$S_{\text{cl.}} = -k_b \sum_{i=1}^W p_i \ln(p_i). \quad (3.51)$$

In a situation of complete certainty where the system is known to be in a specific microstate  $k$  ( $p_i = \delta_{ik}$ ), the formula gives  $S_{\text{cl.}} = 0$ . (Note that  $x \ln(x) \rightarrow 0$  as  $x \rightarrow 0$ ). In a situation of complete uncertainty where all microstates are equally probable ( $p_i = 1/W$ ), we get  $S_{\text{cl.}} = k_b \ln W$ , the entropy of a microcanonical ensemble in statistical mechanics. For any other distribution of probabilities, it can be shown that the entropy lies between these two extremes:  $0 \leq S_{\text{cl.}} \leq k_b \ln W$ . For a review of the properties of entropy, see Appendix C.

The concept of entanglement entropy aims to quantify the uncertainty arising from a quantum (sub)system's lack of a definite quantum state, due to it being possibly entangled with another (sub)system. When formulating it, the key issue we need to be careful about is how to extend classical notions of probability to quantum systems. We have seen that when performing a measurement on  $A$  whose possible outcomes are  $\{q_\mu\}$ , the probability of getting  $q_\mu$  is  $P_\mu = \langle \mu | \hat{\rho}_A | \mu \rangle$ . However, it is problematic to directly substitute these probabilities  $\{P_\mu\}$  into the classical entropy formula, since they are basis-dependent (i.e., the set of probabilities is dependent on the choice of measurement). Eq. (3.50) bypasses this problem by using the trace, which is basis-independent.

In the special case where  $\{|\mu\rangle\}$  is the eigenbasis for  $\hat{\rho}_A$ , the connection is easier to see. From (3.49), the eigenvalues  $\{p_\mu\}$  are all real numbers between 0 and 1, and summing to unity, so they can be regarded as probabilities. Then the entanglement entropy is

$$\begin{aligned} S_A &= -k_b \sum_{\mu} \langle \mu | \hat{\rho}_A \ln(\hat{\rho}_A) | \mu \rangle \\ &= -k_b \sum_{\mu} p_{\mu} \ln(p_{\mu}). \end{aligned} \quad (3.52)$$

Therefore, in this particular basis the expression for the entanglement entropy is consistent with the classical definition of entropy, with the eigenvalues of  $\hat{\rho}_A$  serving as the relevant probabilities.

By analogy with the classical entropy formula (see Appendix C), the entanglement entropy has the following bounds:

$$0 \leq S_A \leq k_b \ln(d_A), \quad (3.53)$$

where  $d_A$  is the dimension of  $\mathcal{H}_A$ .

The lower bound  $S_A = 0$  holds if and only if system  $A$  is in a pure state (i.e., it is not entangled with any other system). This is because the bound corresponds to a situation where  $\hat{\rho}_A$  has one eigenvalue that is 1, and all the other eigenvalues are 0 (see Appendix C). If we denote the eigenvector associated with the non-vanishing eigenvalue by  $|\psi\rangle$ , then the density matrix can be written as  $\hat{\rho}_A = |\psi\rangle\langle\psi|$ , which has the form of a pure state.

As a corollary, if we find that  $S_A \neq 0$ , then  $\hat{\rho}_A$  cannot be written as a pure state  $|\psi\rangle\langle\psi|$  for *any*  $|\psi\rangle$ , and hence it must describe a mixed state.

A system is said to be **maximally entangled** if it saturates the upper bound of (3.53),  $S_A = k_b \ln(d_A)$ . This occurs if and only if the eigenvalues of the density operator are all equal: i.e.,  $p_j = 1/d_A$  for all  $j = 1, \dots, d_A$ .

*Example*—Consider the following state of two spin-1/2 particles:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle|-\rangle - |-\rangle|+\rangle \right). \quad (3.54)$$

The density operator for the two-particle system is

$$\hat{\rho}(\psi) = \frac{1}{2} \left( |+\rangle|-\rangle - |-\rangle|+\rangle \right) \left( \langle+\langle-\langle-\langle+\right). \quad (3.55)$$

Tracing over system  $B$  (the second slot) yields the reduced density operator

$$\hat{\rho}_A(\psi) = \frac{1}{2} \left( |+\rangle\langle+\langle-\langle-\right). \quad (3.56)$$

This can be expressed as a matrix in the  $\{|+\rangle, |-\rangle\}$  basis:

$$\hat{\rho}_A(\psi) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \quad (3.57)$$

Now we can use  $\hat{\rho}_A$  to compute the entanglement entropy

$$S_A = -k_b \text{Tr} \{ \hat{\rho}_A \ln(\hat{\rho}_A) \} = -k_b \text{Tr} \begin{pmatrix} \frac{1}{2} \ln \left( \frac{1}{2} \right) & 0 \\ 0 & \frac{1}{2} \ln \left( \frac{1}{2} \right) \end{pmatrix} = k_b \ln(2). \quad (3.58)$$

Hence, the particles are maximally entangled.

## VIII. THE MANY WORLDS INTERPRETATION

We conclude this chapter by discussing a set of compelling but controversial ideas arising from the phenomenon of quantum entanglement: the **Many Worlds Interpretation** as formulated by Hugh Everett [8].

So far, when describing the phenomenon of state collapse, we have relied on the measurement postulate (see Section II), which is part of the **Copenhagen Interpretation** of quantum mechanics. This is how quantum mechanics is typically taught, and how physicists think about the theory when doing practical, everyday calculations.

However, the measurement postulate has two bad features:

1. It stands apart from the other postulates of quantum mechanics, for it is the only place where randomness (or “indeterminism”) creeps into quantum theory. The other postulates do not refer to probabilities. In particular, the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \quad (3.59)$$

is completely deterministic. If you know  $\hat{H}(t)$  and are given the state  $|\psi(t_0)\rangle$  at some time  $t_0$ , you can in principle determine  $|\psi(t)\rangle$  for all  $t$ . This time-evolution consists of a smooth, non-random rotation of the state vector within its Hilbert space. A measurement process, however, has a completely different character: it causes the state vector to jump discontinuously to a randomly-selected value. It is strange that quantum theory contains two completely different ways for a state to change.

2. The measurement postulate is silent on what constitutes a measurement. Does measurement require a conscious observer? Surely not: as Einstein once exasperatedly asked, are we really expected to believe that the Moon exists only when we look at it? But if a given device interacts with a particle, what determines whether it acts via the Schrödinger equation, or performs a measurement?

The Many Worlds Interpretation seeks to resolve these problems by positing that *the measurement postulate is not a fundamental postulate of quantum mechanics*. Rather, what we call “measurement”, including state collapse and the apparent randomness of measurement results, is an emergent phenomenon that can be derived from the behavior of complex many-particle quantum systems obeying the Schrödinger equation. The key idea is that a measurement process can be described by applying the Schrödinger equation to a quantum system containing both the thing being measured *and* the measurement apparatus itself.

We can study this using a toy model formulated by Albrecht [9]. Consider a spin-1/2 particle, and an apparatus designed to measure  $S_z$ . Let  $\mathcal{H}_S$  be the spin-1/2 Hilbert space (which is 2D), and  $\mathcal{H}_A$  be the Hilbert space of the apparatus (which has dimension  $d$ ). We will assume that  $d$  is very large, as actual experimental apparatuses are macroscopic objects containing  $10^{23}$  or more atoms! The Hilbert space of the combined system is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A, \quad (3.60)$$

and is  $2d$ -dimensional. Let us suppose the system is prepared in an initial state

$$|\psi(0)\rangle = \left( a_+ |+\rangle + a_- |-\rangle \right) \otimes |\Psi\rangle, \quad (3.61)$$

where  $a_{\pm} \in \mathbb{C}$  are the quantum amplitudes for the particle to be initially spin-up or spin-down, and  $|\Psi\rangle \in \mathcal{H}_A$  is the initial state of the apparatus.

The combined system now evolves via the Schrödinger equation. We aim to show that if the Hamiltonian has the form

$$\hat{H} = \hat{S}_z \otimes \hat{V}, \quad (3.62)$$

where  $\hat{S}_z$  is the operator corresponding to the observable  $S_z$ , then time evolution has an effect equivalent to the measurement of  $S_z$ .

It turns out that we can show this without making any special choices for  $\hat{V}$  or  $|\Psi\rangle$ . We only need  $d \gg 2$ , and for both  $\hat{V}$  and  $|\Psi\rangle$  to be “sufficiently complicated”. We choose  $|\Psi\rangle$  to be a random state vector, and choose random matrix components for the operator  $\hat{V}$ . The precise generation procedures will be elaborated on later. Once we decide on  $|\Psi\rangle$  and  $\hat{V}$ , we can evolve the system by solving the Schrödinger equation

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle, \quad \text{where } \hat{U}(t) = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]. \quad (3.63)$$

Because the part of  $\hat{H}$  acting on the  $\mathcal{H}_S$  subspace is  $\hat{S}_z$ , the result necessarily has the following form:

$$\begin{aligned} |\psi(t)\rangle &= \hat{U}(t)\left(a_+|+z\rangle + a_-|-z\rangle\right) \otimes |\Psi\rangle \\ &= a_+|+z\rangle \otimes |\Psi_+(t)\rangle + a_-|-z\rangle \otimes |\Psi_-(t)\rangle. \end{aligned} \quad (3.64)$$

Here,  $|\Psi_+(t)\rangle$  and  $|\Psi_-(t)\rangle$  are apparatus states that are “paired up” with the  $|+z\rangle$  and  $|-z\rangle$  states of the spin-1/2 subsystem. At  $t = 0$ , both  $|\Psi_+(t)\rangle$  and  $|\Psi_-(t)\rangle$  are equal to  $|\Psi\rangle$ ; for  $t > 0$ , they rotate into different parts of the state space  $\mathcal{H}_A$ . If the dimensionality of  $\mathcal{H}_A$  is sufficiently large, and both  $\hat{V}$  and  $|\Psi\rangle$  are sufficiently complicated, we can guess (and we will verify numerically) that the two state vectors rotate into completely different parts of the state space, so that

$$\langle\Psi_+(t)|\Psi_-(t)\rangle \approx 0 \quad \text{for sufficiently large } t. \quad (3.65)$$

Once this is the case, the two terms in the above expression for  $|\psi(t)\rangle$  can be interpreted as two decoupled “worlds”. In one world, the spin has a definite value  $+\hbar/2$ , and the apparatus is in a state  $|\Psi_+\rangle$  (which might describe, for instance, a macroscopically-sized physical pointer that is pointing to a “ $S_z = +\hbar/2$ ” reading). In the other world, the spin has a definite value  $-\hbar/2$ , and the apparatus has a different state  $|\Psi_-\rangle$  (which might describe a physical pointer pointing to a “ $S_z = -\hbar/2$ ” reading). Importantly, the  $|\Psi_+\rangle$  and  $|\Psi_-\rangle$  states are orthogonal, so they can be rigorously distinguished from each other. The two worlds are “weighted” by  $|a_+|^2$  and  $|a_-|^2$ , which correspond to the probabilities of the two possible measurement results.

The above description can be tested numerically. Let us use an arbitrary basis for the apparatus space  $\mathcal{H}_A$ ; in that basis, let the  $d$  components of the initial apparatus state vector  $|\Psi\rangle$  be random complex numbers:

$$|\Psi\rangle = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \vdots \\ \Psi_{d-1} \end{pmatrix}, \quad \text{where } \text{Re}(\Psi_j), \text{Im}(\Psi_j) \sim N(0, 1). \quad (3.66)$$

In other words, the real and imaginary parts of each complex number  $\Psi_j$  are independently drawn from the the standard normal (Gaussian) distribution, denoted by  $N(0, 1)$ . The normalization constant  $\mathcal{N}$  is defined so that  $\langle \Psi | \Psi \rangle = 1$ .

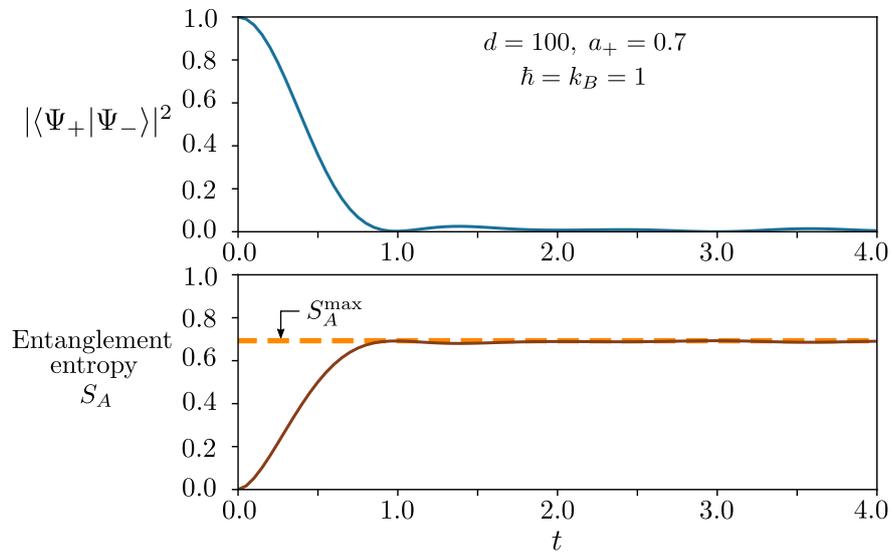
Likewise, we generate the matrix elements of  $\hat{V}$  according to the following random scheme:

$$A_{ij} \sim u_{ij} + iv_{ij}, \quad \text{where } u_{ij}, v_{ij} \sim N(0, 1)$$

$$\hat{V} = \frac{1}{2\sqrt{d}} (\hat{A} + \hat{A}^\dagger). \quad (3.67)$$

This scheme produces a  $d \times d$  matrix with random components, subject to the requirement that the overall matrix be Hermitian. The factor of  $1/2\sqrt{d}$  is relatively unimportant; it ensures that the eigenvalues of  $\hat{V}$  lie in the range  $[-2, 2]$ , instead of scaling with  $d$  [10].

The Schrödinger equation can now be solved numerically. The results are shown below:



In the initial state, we let  $a_+ = 0.7$ , so  $a_- = \sqrt{1 - 0.7^2} = 0.71414\dots$ . The upper panel plots the overlap between the two apparatus states,  $|\langle \Psi_+ | \Psi_- \rangle|^2$ , versus  $t$ . In accordance with the preceding discussion, the overlap is unity at  $t = 0$ , but subsequently decreases to nearly zero. For comparison, the lower panel plots the entanglement entropy between the two subsystems,  $S_A = -k_b \text{Tr}_A \{ \hat{\rho}_A \ln \hat{\rho}_A \}$ , where  $\hat{\rho}_A$  is the reduced density matrix obtained by tracing over the spin subspace. We find that  $S_A = 0$  at  $t = 0$ , due to the fact that the spin and apparatus subsystems start out with definite quantum states in  $|\psi(0)\rangle$ . As the system evolves, the subsystems become increasingly entangled, and  $S_A$  increases up to

$$S_A^{\max}/k_b = -\left( |a_+|^2 \ln |a_+|^2 + |a_-|^2 \ln |a_-|^2 \right) \approx 0.693 \quad (3.68)$$

This value is indicated in the figure by a horizontal dashed line, and corresponds to the result of the classical entropy formula for probabilities  $\{|a_+|^2, |a_-|^2\}$ . Moreover, we see that the entropy reaches  $S_A^{\max}$  at around the same time that  $|\langle \Psi_+ | \Psi_- \rangle|^2$  reaches zero. This demonstrates the close relationship between “measurement” and “entanglement”.

For details about the numerical linear algebra methods used to perform the above calculation, refer to Appendix D.

The “many worlds” concept can be generalized from the above toy model to the universe as a whole. In the viewpoint of the Many Worlds Interpretation of quantum mechanics, the entire universe can be described by a mind-bogglingly complicated quantum state, evolving deterministically according to the Schrödinger equation. This evolution involves repeated “branchings” of the universal quantum state, which continuously produces more and more worlds. The classical world that we appear to inhabit is just one of a vast multitude. It is up to you to decide whether this conception of reality seems reasonable. It is essentially a matter of preference, because the Copenhagen Interpretation and the Many Worlds Interpretation have identical physical consequences, which is why they are referred to as different “interpretations” of quantum mechanics, rather than different theories.

### Exercises

1. Let  $\mathcal{H}_A$  and  $\mathcal{H}_B$  denote single-particle Hilbert spaces with well-defined inner products. That is to say, for all vectors  $|\mu\rangle, |\mu'\rangle, |\mu''\rangle \in \mathcal{H}_A$ , that Hilbert space’s inner product satisfies the inner product axioms

- (a)  $\langle \mu | \mu' \rangle = \langle \mu' | \mu \rangle^*$
- (b)  $\langle \mu | \mu \rangle \in \mathbb{R}_0^+$ , and  $\langle \mu | \mu \rangle = 0$  if and only if  $|\mu\rangle = 0$ .
- (c)  $\langle \mu | (|\mu'\rangle + |\mu''\rangle) \rangle = \langle \mu | \mu' \rangle + \langle \mu | \mu'' \rangle$
- (d)  $\langle \mu | (c|\mu'\rangle) \rangle = c\langle \mu | \mu' \rangle$  for all  $c \in \mathbb{C}$ ,

and likewise for vectors from  $\mathcal{H}_B$  with that Hilbert space’s inner product.

In Section I, we defined a tensor product space  $\mathcal{H}_A \otimes \mathcal{H}_B$  as the space spanned by the basis vectors  $\{|\mu\rangle \otimes |\nu\rangle\}$ , where the  $|\mu\rangle$ ’s are basis vectors for  $\mathcal{H}_A$  and the  $|\nu\rangle$ ’s are basis vectors for  $\mathcal{H}_B$ . Prove that we can define an inner product using

$$\left(\langle \mu | \otimes \langle \nu | \right) \left( |\mu'\rangle \otimes |\nu'\rangle \right) \equiv \langle \mu | \mu' \rangle \langle \nu | \nu' \rangle = \delta_{\mu\mu'} \delta_{\nu\nu'} \quad (3.69)$$

which satisfies the inner product axioms.

2. Consider the density operator

$$\hat{\rho} = \frac{1}{2} | +z \rangle \langle +z | + \frac{1}{2} | +x \rangle \langle +x | \quad (3.70)$$

where  $| +x \rangle = \frac{1}{\sqrt{2}} (| +z \rangle + | -z \rangle)$ . This can be viewed as an equal-probability sum of two different pure states. However, the density matrix can also be written as

$$\hat{\rho} = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2| \quad (3.71)$$

where  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are the eigenvectors of  $\hat{\rho}$ . Show that  $p_1$  and  $p_2$  are *not* 1/2.

3. Consider two distinguishable particles,  $A$  and  $B$ . The 2D Hilbert space of  $A$  is spanned by  $\{|m\rangle, |n\rangle\}$ , and the 3D Hilbert space of  $B$  is spanned by  $\{|p\rangle, |q\rangle, |r\rangle\}$ . The two-particle state is

$$|\psi\rangle = \frac{1}{3} |m\rangle |p\rangle + \frac{1}{\sqrt{6}} |m\rangle |q\rangle + \frac{1}{\sqrt{18}} |m\rangle |r\rangle + \frac{\sqrt{2}}{3} |n\rangle |p\rangle + \frac{1}{\sqrt{3}} |n\rangle |q\rangle + \frac{1}{3} |n\rangle |r\rangle. \quad (3.72)$$

Find the entanglement entropy.

### Further Reading

- [1] Bransden & Joachain, §14.1—14.4, §17.1–17.5
- [2] Sakurai, §3.9
- [3] A. Einstein, B. Podolsky, and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Physical Review **47**, 777 (1935). [[link](#)]
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- [10] A. Edelman and N. R. Rao, *Random matrix theory*, Acta Numerica **14**, 233 (2005). [[link](#)]