10. Green's functions

A **Green's function** is a solution to an inhomogenous differential equation with a "driving term" given by a delta function. It is used as a convenient method for solving more complicated inhomogenous differential equations. In physics, Green's functions methods are used to describe a wide variety of phenomena, ranging from the motion of complex mechanical oscillators to the emission of sound waves from loudspeakers.

10.1 The driven harmonic oscillator

As an introduction to the Green's function technique, we study the **driven harmonic oscillator**, which is a damped harmonic oscillator subjected to an arbitrary driving force. Its equation of motion is

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] x(t) = \frac{f(t)}{m}. \tag{1}$$

The left-hand side is the same as in the damped harmonic oscillator equation (see Chapter 4), where m is the mass of the particle, γ is the damping coefficient, and ω_0 is the natural frequency of the oscillator. On the right-hand side, we introduce a time-dependent driving force $f(t)$ (which acts alongside the pre-existing spring and damping forces). Given an arbitrarily complicated $f(t)$, our goal is to determine $x(t)$.

10.1.1 Green's function for the driven harmonic oscillator

Prior to solving the driven harmonic oscillator problem for a general driving force $f(t)$, let us first consider the following equation:

$$\left[\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \omega_0^2\right] G(t,t') = \delta(t-t'). \tag{2}$$

This is called the **Green's function equation**. The function $G(t,t')$, which depends on the two variables t and t', is called the **Green's function**. Note that the differential operator on the left-hand side involves only derivatives in t.

Comparing Eq. (2) to Eq. (1), we see that $G(t,t')$ describes the motion of a damped harmonic oscillator that is subjected to a particular choice of driving force—an infinitesimally sharp pulse centered at $t = t'$. This pulse is described by a delta function (see Section 9.4):

$$f(t) = m\,\delta(t-t'). \tag{3}$$

Why do we care about the Green's function? The reason is that as soon as $G(t,t')$ is known, we can easily produce a specific solution to the driven harmonic oscillator equation for *any* given driving force $f(t)$. That solution has the following form:

$$x(t) = \int_{-\infty}^{\infty} dt'\; G(t,t')\,\frac{f(t')}{m}. \tag{4}$$

To show mathematically that this is indeed a solution, plug Eq. (4) into Eq. (1):

$$\begin{aligned}\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] x(t) &= \int_{-\infty}^{\infty} dt' \,\left[\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \omega_0^2\right] G(t,t')\frac{f(t')}{m} \\ &= \int_{-\infty}^{\infty} dt'\; \delta(t-t')\,\frac{f(t')}{m} \\ &= \frac{f(t)}{m}.\end{aligned} \tag{5}$$

(Note that we can move the differential operator inside the integral over t' because t and t' are independent variables.) We have thus shown that the above expression for $x(t)$ satisfies the driven oscillator equation for driving force $f(t)$.

The Green's function concept is based on the principle of superposition of waves and oscillations. The motion of the oscillator is induced by the driving force, but the value of $x(t)$ at time t does not just depend on the instantaneous value of $f(t)$ at time t, but rather on the values of $f(t')$ over all times $t' < t$. (Indeed, the oscillations will continue even after the driving force is turned off.) The key idea is that any function $f(t)$ can be decomposed into a superposition of delta functions. Since the response of the oscillator to a delta function force is given by the Green's function, the solution $x(t)$ is given by a superposition of Green's functions.

10.1.2 Finding the Green's function

To solve the Green's function equation, we use the Fourier transform. Let us suppose for now that the Fourier transform of $G(t,t')$ with respect to t is convergent (we'll examine this assumption in detail later, in Section 10.1.4). We will also assume that the oscillator is not critically damped, i.e. $\omega_0 \neq \gamma$.

The Fourier-transformed Green's function is called the **frequency-domain Green's function**:

$$G(\omega, t') = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, G(t,t'). \tag{6}$$

Note that we have used the usual sign convention for time-domain Fourier transforms (see Section 9.2.3).

Next, we Fourier transform both sides of the Green's function equation (2), and make use of how derivatives behave under Fourier transformation. The result is

$$\left[-\omega^2 - 2i\gamma\omega + \omega_0^2\right] G(\omega, t') = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \delta(t-t') = e^{i\omega t'}. \tag{7}$$

The differential equation for $G(t,t')$ has thus been converted into an *algebraic* equation for $G(\omega, t')$. The latter is easily solved:

$$G(\omega, t') = -\frac{e^{i\omega t'}}{\omega^2 + 2i\gamma\omega - \omega_0^2}. \tag{8}$$

Finally, we retrieve the time-domain solution by using the inverse Fourier transform:

$$G(t,t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} G(\omega, t') \tag{9}$$

$$= -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\gamma\omega - \omega_0^2}. \tag{10}$$

The denominator of the integral is a quadratic expression, so this can be re-written as:

$$G(t,t') = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{e^{-i\omega(t-t')}}{(\omega-\omega_+)(\omega-\omega_-)} \quad \text{where} \;\; \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{11}$$

This can be evaluated by contour integration. The integrand has two poles, both lying in the negative complex plane (note: these are precisely the complex frequencies of the damped harmonic oscillator discussed in Chapter 4). For $t < t'$, Jordan's lemma requires us to close the contour in the upper half-plane; this encloses neither pole, so the integral is zero. For

$t > t'$, we must close the contour in the lower half-plane, enclosing both poles. The result is

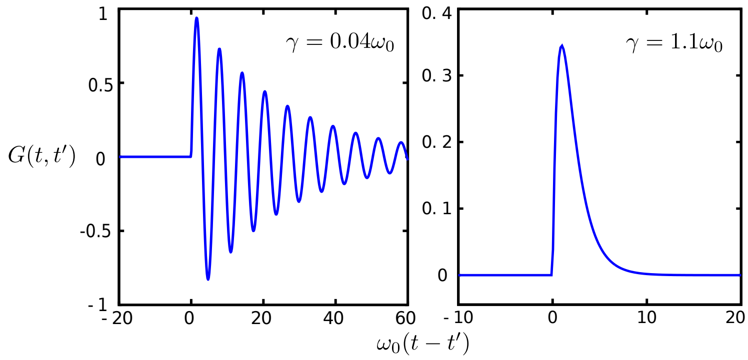
$$G(t,t') = i\Theta(t-t') \left[\frac{e^{-i\omega_+(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_-(t-t')}}{\omega_- - \omega_+}\right] \tag{12}$$

$$= \Theta(t-t')\, e^{-\gamma(t-t')} \times \begin{cases} \frac{1}{\sqrt{\omega_0^2-\gamma^2}} \sin\left[\sqrt{\omega_0^2-\gamma^2}(t-t')\right], & \gamma < \omega_0, \\ \frac{1}{\sqrt{\gamma^2-\omega_0^2}} \sinh\left[\sqrt{\gamma^2-\omega_0^2}(t-t')\right], & \gamma > \omega_0. \end{cases} \tag{13}$$

Here, $\Theta(t-t')$ refers to the step function

$$\Theta(\tau) = \begin{cases} 1, & \text{for } \tau \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

This result is plotted in the figure below, in both the under-damped and over-damped regimes. The solution for the critically-damped case, $\gamma = \omega_0$, is left as an exercise.



10.1.3 Features of the Green's function

As previously noted, the time-domain Green's function has a physical meaning: it represents the motion of the oscillator in response to a force "pulse", $f(t) = m\,\delta(t-t')$. The basic features of the results obtained in the previous section match our intuition of what this motion should look like.

The first thing to notice is that the Green's function depends on t and t' only in the combination $t-t'$. In physical terms, it is easy to see why: the response of the oscillator to an infintesimally sharp force pulse should only depend on the time elapsed since the pulse. To take advantage of this property, we could re-define the frequency-domain Green's function as

$$G(\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega(t-t')}\, G(t-t'), \tag{15}$$

which then obeys

$$\left[-\omega^2 - 2i\gamma\omega + \omega_0^2\right] G(\omega) = 1. \tag{16}$$

This is a little nicer to work with, because there isn't an extraneous t' variable in the definition.

The next thing to notice is how the Green's function behaves just before and after the pulse. Its value is zero for $t - t' < 0$ (i.e., prior to the application of the pulse). This feature is referred to as "causality", and we will discuss it in greater detail in Section 10.1.4. Moreover, there is no discontinuity in $x(t)$ at $t - t' = 0$, which means that the force pulse does not cause the oscillator to "teleport" instantaneously to a different position. Rather, it produces a discontinuity in the oscillator's *velocity*.

To see this explicitly, let us integrate the Green's function equation (2) over an infinitesimal interval of time surrounding t':

$$\begin{aligned} \lim_{\epsilon\rightarrow 0}\int_{t'-\epsilon}^{t'+\epsilon} dt\left[\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\omega_0^2\right]G(t,t') &= \lim_{\epsilon\rightarrow 0}\int_{t'-\epsilon}^{t'+\epsilon} dt\;\delta(t-t') \\ = \lim_{\epsilon\rightarrow 0}\left\{\left.\frac{\partial G(t,t')}{\partial t}\right|_{t=t'+\epsilon}-\left.\frac{\partial G(t,t')}{\partial t}\right|_{t=t'-\epsilon}\right\} &= 1 \end{aligned} \tag{17}$$

On the last line, the expression on the left-hand side represents the difference between the oscillator velocity just after the pulse, and the velocity just before the pulse. Evidently, the force pulse imparts one unit of velocity at $t = t'$. If you look at the solutions obtained in the previous section, you can verify that indeed $\partial G/\partial t = 0$ immediately before the pulse, and $\partial G/\partial t = 1$ immediately after the pulse.

For $t-t' > 0$, the applied force goes back to zero, and the system behaves like the undriven harmonic oscillator. If the oscillator is under-damped ($\gamma < \omega_0$), it undergoes oscillation around the origin, which decays exponentially back to the origin as the energy imparted by the pulse is damped away. If the oscillator is over-damped ($\gamma > \omega_0$), the oscillator moves for a distance, then decays exponentially back to the origin without oscillating.

10.1.4 Causality

We have noted that the motion $x(t)$ ought to depend on the driving force $f(t')$ at all past times $t' < t$, but should *not* depend on the force at future times. Because of the relation

$$x(t) = \int_{-\infty}^{\infty} dt'\; G(t,t')\,\frac{f(t')}{m}, \tag{18}$$

this means that the Green's function ought to satisfy

$$G(t,t') = 0 \;\;\text{for}\;\; t-t' < 0. \tag{19}$$

This condition is referred to as **causality**, because it is equivalent to saying that *cause* must precede *effect*. A Green's function that satisfies it is called a **causal Green's function**. The specific solution for $G(t,t')$ which we derived in Section 10.1.2 is a causal Green's function.

The time-domain Green's function satisfies a second-order differential equation, so it has a family of solutions; the general solution contains two free parameters. Not all solutions for the time-domain Green's function are causal. We need to ask ourselves whether the specific solution we found in the previous section is the *only* possible causal solution. It turns out that the answer is yes, and there are a couple of ways to see why.

The first approach is to observe that for $t > t'$, the Green's function satisfies the differential equation for the *undriven* harmonic oscillator. But as we noted in Section 10.1.3, the Green's function must also obey two conditions at $t = t'+0^+$: (i) $G = 0$, and (ii) $\partial G/\partial t = 1$. These act as two boundary conditions for the undriven harmonic oscillator equation over the range $t > t'$. Hence, the solution for $G(t,t')$ in this range is completely specified.

The other way to see that the causal Green's function is unique is to imagine adding to our specific solution any solution $x_1(t)$ for the undriven harmonic oscillator. It is easy to verify that the new form of $G(t,t')$ is still a solution to the differential equation for the Green's function. As we saw during our study of the damped harmonic oscillator (Chapter 4), the general solution for $x_1(t)$ contains two free parameters. The general solution is always infinite for $t \to -\infty$, with one exception: the "trivial" solution $x_1(t) = 0$. The trivial solution is a *specific* solution. Thus, the causality requirement is equivalent to setting *two* boundary conditions, so the causal Green's function that we have found is the only possible one.

Moreover, if we add a non-trivial undriven oscillator solution $x_1(t)$ to the Green's function, the Green's function would no longer have a convergent Fourier transform—it wouldn't be square-integrable due to blowing up in the $t \to -\infty$ limit. Hence, requiring the time-domain Green's function to obey causality is equivalent to requiring the frequency-domain Green's function to be well-defined.

10.2 Space-time Green's functions

We can also use the Green's function method to study the wave equation, in order to describe the emission (and absorption) of waves. For simplicity, we restrict our discussion to waves propagating through a uniform medium (i.e., a medium that is featureless through all of space and time). Let us also assume that there is a single space dimension, given by coordinate x. (The generalization of the following discussion to the case of multiple spatial dimensions will be pretty straightforward.)

As we saw in Chapter 5, it is convenient to describe a wave using a complex wavefunction $\psi(x,t)$, which obeys the wave equation

$$\left[\frac{\partial^2}{\partial x^2} - \left(\frac{1}{c}\right)^2 \frac{\partial^2}{\partial t^2}\right]\psi(x,t) = 0. \tag{20}$$

Here, c is the speed of wave propagation. Henceforth, to simplify the equations, we will set $c = 1$. (You can go back to the previous case by replacing all instances of t with ct, and ω with ω/c, in the formulas that we will derive.)

The wave equation describes how waves propagate *after* they have already been created. To describe how the waves are generated in the first place, we must modify the wave equation by introducing a term on the right-hand side, called a **source**:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right]\psi(x,t) = f(x,t). \tag{21}$$

The source term turns the wave equation into an inhomogenous partial differential equation. It plays a role very similar to the driving force in the driven harmonic oscillator problem from Section 10.1. In acoustics, for example, such a source term can describe the motion of a loudspeaker's membrane, and $\psi(x,t)$ describes the waves that are emitted by the loudspeaker.

10.2.1 Time-domain Green's function

The wave equation's **time-domain Green's function** is defined by setting the source term to delta functions in both space and time:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right] G(x,x';t-t') = \delta(x-x')\,\delta(t-t'). \tag{22}$$

As can be seen, G is a function of two spatial variables, x and x', as well as two temporal variables t and t'. It corresponds to the wave generated by a pulse

$$f(x,t) = \delta(x-x')\,\delta(t-t'). \tag{23}$$

The differential operator in the Green's function equation only involves x and t, so we can regard x' and t' as parameters specifying where the pulse is localized in space and time. This Green's function ought to depend on the time variables only in the combination $t-t'$, similar to our previous discussion of the Green's function for the driven harmonic oscillator (Section 10.1); to emphasize this, we have written it as $G(x,x';t-t')$.

Just as for the harmonic oscillator, we can use the Green's function to construct a solution for the driven wave equation with an arbitrary source:

$$\psi(x,t) = \int dx' \int_{-\infty}^{\infty} dt'\; G(x,x';t-t')\,f(x',t'). \tag{24}$$

This difference from the harmonic oscillator problem is that the Green's function and the source now depend on space as well as time, so we also need to integrate over space in order to obtain $\psi(x,t)$. The Green's function thus contains information about how the influence of the source at one point in space and time *propagates* to other points in both space and time.

10.2.2 Frequency-domain Green's function for waves

The wave equation's **frequency-domain Green's function** is obtained by Fourier transforming the time-domain Green's function in the $t - t'$ coordinate:

$$G(x,x';\omega) = \int_{-\infty}^{\infty} d\tau\; e^{i\omega\tau}\, G(x,x';\tau). \tag{25}$$

It obeys the differential equation

$$\left[\frac{\partial^2}{\partial x^2} + \omega^2\right] G(x,x';\omega) = \delta(x-x'). \tag{26}$$

Just as we can write the time-domain solution to the wave equation in terms of the time-domain Green's function, we can do the same for the frequency-domain solution:

$$\Psi(x,\omega) = \int dx'\; G(x,x';\omega)\, F(x',\omega), \tag{27}$$

where

$$\Psi(x,\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\, \psi(x,t), \quad F(x,\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\, f(x,t). \tag{28}$$

Note that there is no integral over frequencies: the ω component of the solution involves only the ω components of the Green's function and the $-\omega$ components of the source contribute, without any contribution from other frequencies.

10.2.3 Outgoing boundary conditions

In order to find the time-domain Green's function, it is simplest to solve for the frequency-domain Green's function first (we pursued a similar strategy when dealing with the harmonic oscillator Green's function).

Before looking for a solution, we need to specify the boundary conditions. That, however, depends on the situation we are interested in. One common setup is a enclosed cavity, where the waves propagate inside a finite domain, say $x \in (x_a, x_b)$. For such situations, we usually impose **Dirichlet boundary conditions**, which state that $G(x,x';\omega) = 0$ at the edges of the domain. The frequency-domain Green's function can then be solved by decomposing it into a series of basis functions defined within the finite domain, analogous to the Fourier series.

We will instead focus on a more interesting situation, which occurs when the space coordinate is unbounded, and runs over the entire space $x \in (-\infty, \infty)$. This describes, for example, a loudspeaker that emits sound into empty space, or an antenna that radiates electromagnetic waves into empty space. The emitted waves should propagate outward from the source, and the boundary conditions describing this condition are called **outgoing boundary conditions**. In 1D, this means that the Green's function should correspond to a left-moving wave for all x to the left of the source, and to a right-moving wave for all x to the right of the source.

We can guess the form of the Green's function which obeys such boundary conditions:

$$G(x,x';\omega) = \left\{\begin{array}{ll} A\, e^{-i\omega(x-x')}, & x \le x', \\ B\, e^{i\omega(x-x')}, & x \ge x' \end{array}\right. \quad \text{for some}\;\; A,B \in \mathbb{C}. \tag{29}$$

It is straightforward to verify that this formula for $G(x,x',\omega)$ satisfies the wave equation in both the regions $x < x'$ and $x > x'$, and that it satisfies outgoing boundary conditions. We simply have to determine the A and B coefficients. To do this, first note that $G(x,x')$ should be continuous at $x = x'$, so $A = B$.

Secondly, integrating the Green's function equation across x' gives

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{x'-\epsilon}^{x'+\epsilon}\left[\frac{\partial^2}{\partial x^2}+\omega^2\right]G(x-x') &= \lim_{\epsilon\to 0}\int_{x'-\epsilon}^{x'+\epsilon}\delta(x-x') \\ = \lim_{\epsilon\to 0}\left\{\frac{\partial G}{dx}(x,x')\Big|_{x=x'+\epsilon} - \frac{\partial G}{\partial x}(x,x')\Big|_{x=x'-\epsilon}\right\} &= i\omega(B+A) = 1.\end{aligned} \tag{30}$$

Combining these two equations gives $A = B = 1/2i\omega$. Hence, the Green's function for outgoing boundary conditions is

$$G(x,x';\omega) = \frac{e^{i\omega|x-x'|}}{2i\omega}. \tag{31}$$

10.2.4 Causality and the time-domain Green's function

The outgoing frequency-domain Green's function derived in the previous section can be converted to a time-domain Green's function by the usual inverse Fourier transform:

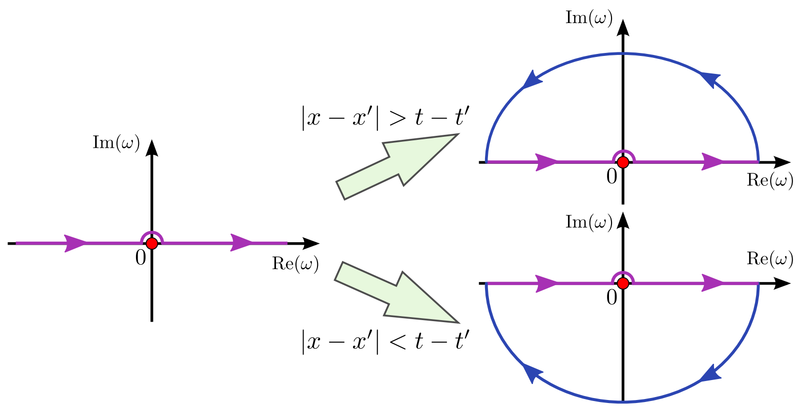
$$G(x,x';t-t') = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega(t-t')}\,G(x,x';\omega) \tag{32}$$

$$= \int_{-\infty}^{\infty}d\omega\,\frac{e^{i\omega\left[|x-x'|-(t-t')\right]}}{4\pi i\omega} \qquad (!) \tag{33}$$

There is a problem with this formula: the integral runs over the real-ω line, yet the integrand has a pole on the real axis (at the origin). The result is therefore ill-defined. In order to resolve this, we redefine $G(x,x';\omega)$ as a contour integral taking place over a deformed contour, which is infinitesimally close to the real line but avoids the pole:

$$G(x,x';t-t') \equiv \int_\Gamma d\omega\,\frac{e^{i\omega\left[|x-x'|-(t-t')\right]}}{4\pi i\omega}. \tag{34}$$

We will choose to deform the contour in a very specific way, which turns out to be the way that satisfies the principle of causality (Section 10.1.4). The deformed contour runs along the real axis, but skips above the pole at the origin:



The contour integral can be solved by either closing the contour in the upper half-plane, or in the lower half-plane. If we close the contour above, then the loop contour does not enclose the pole, and hence $G(x, x'; t - t') = 0$. According to Jordan's lemma, we must do this if the exponent in the integrand obeys

$$|x - x'| - (t - t') > 0 \quad \Rightarrow \quad |x - x'| > t - t'. \tag{35}$$

This inequality is satisfied in two cases: either (i) $t < t'$ (in which case the inequality is satisfied for all x, x' because $|x - x'|$ is strictly non-negative), or (ii) $t > t'$ but the value of $t - t'$ is smaller than $|x - x'|$. To understand the physical meaning of these two cases, recall that $G(x, x'; t - t')$ represents the field at position x and time t resulting from a pulse at the space-time point (x', t'). Thus, case (i) corresponds to times occurring before the pulse, and case (ii) corresponds to times occurring after the pulse but positions which are too far away from the location of the pulse for a wave to reach x from x'.

By contrast, for $|x-x'|-(t-t') < 0$ we can use the residue theorem to obtain $G(x, x'; t-t') = -1/2$. From this description, it is clear that the resulting time-domain Green's function is causal. Unlike the harmonic oscillator case, causality for the wave equation imposes the addition requirement that waves must be given enough time to propagate between two specified points.

The time-domain wavefunction can therefore be written as

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' \left[-\frac{1}{2}\,\Theta(t - t' - |x - x'|)\right] f(x', t'), \tag{36}$$

where Θ denotes the unit step function. In other words, the wavefunction at each space-time point (x, t) receives equal contribution from the sources $f(x', t')$ at space-time points (x', t') which lie within the "past light cone".

10.3 Looking ahead

Green's functions are widely used to describe the emission of acoustic and electromagnetic waves, and how they interact with various sorts of media. This is a vast topic whose details are covered in advanced courses in theoretical physics and electrical engineering. Here, we give a brief sketch of some future directions of study.

So far, we have focused our attentions on the simplest case of an infinite one-dimensional uniform medium. In most practical applications, one is interested in three spatial dimensions, and in non-uniform media. For such cases, the wave equation's frequency-domain Green's function can be generalized to

$$\left[\nabla^2 + n^2(\vec{r})\,\left(\frac{\omega}{c}\right)^2\right]\, G(\vec{r}, \vec{r}'; \omega) = \delta^3(\vec{r} - \vec{r}'), \tag{37}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the three-dimensional Laplacian operator, and $n(\vec{r})$ is the refractive index (see the discussion in Chapter 5). On the right-hand side of this equation is the three-dimensional delta function (see Section 9.5), which describes a point source located at position $\vec{r}'$ in the three-dimensional space.

When $n = 1$, the above equation is similar to the frequency-domain Green's function equation that we studied in Section 10.2.3, except that the problem is three-dimensional rather than one-dimensional. Again assuming outgoing boundary conditions (Section 10.2.3), the solution for the Green's function in three dimensions can be determined analytically. We will not cover the solution method, but the result turns out to be

$$G(\vec{r}, \vec{r}'; \omega) = -\frac{e^{i(\omega/c)|\vec{r}-\vec{r}'|}}{4\pi|\vec{r} - \vec{r}'|}. \tag{38}$$

Similar to the solution in one dimension, this depends on the distance from the source, $|\vec{r} - \vec{r}'|$, and thus describes waves that are emitted isotropically from the source at $\vec{r}'$. One difference is that the value of G now decreases to zero with distance, due to the $|\vec{r} - \vec{r}'|$ in the denominator. This matches our everyday experience that the sound emitted from a point source grows fainter with distance, and it happens because the energy carried by the outgoing wave is spread out over a larger area with increasing distance from the source. By contrast, in one-dimensional space, the emitted waves do not spread as they travel, and hence G does not decrease with distance.

When $n(\vec{r})$ is not a constant but varies with position $\vec{r}$, then the waves emitted by the source do not radiate outwards in a simple way. The variations in the refractive index cause the waves to scatter in complicated ways. In most situations, the exact solution for the Green's function cannot be obtained analytically, but must be computed using specialized numerical methods. If the variations in $n(\vec{r})$ are sufficiently weak, one can sometimes derive approximate expressions for the Green's function, using a family of techniques known as **perturbation theory**.

For electromagnetic waves, there is another important complication: electromagnetic fields are described by vectors (i.e., the electric field vector and the magnetic field vector), not scalars. The propagation of electromagnetic waves is therefore described by a vectorial wave equation, not the scalar wave equation that we have assumed so far. Moreover, electromagnetic waves are not generated by scalar sources, but by vector sources (electrical currents). The corresponding Green's function is not a scalar quantity, but something called a **dyadic Green's function**. This is a tensorial object that describes the *vector* waves emitted by a *vector* source.

Finally, even though we have dealt so far with classical (non-quantum) waves, the Green's function concept is also crucial to the theory of *quantum* waves. **Quantum field theory** describes how fields like the electromagnetic field behave according to the laws of quantum mechanics. In this theory, the Green's functions no longer have simple numerical values, but are quantum mechanical *operators*. Almost all problems in quantum field theory essentially involve calculating a quantum mechanical Green's function. Quantum field theory is widely used in the subjects of condensed-matter physics and high-energy particle physics.

10.4 Exercises

1. Find the time-domain Green's function of the critically-damped harmonic oscillator, for which $\gamma = \omega_0$.

2. Consider an overdamped harmonic oscillator ($\gamma > \omega_0$) subjected to a *random* driving force $f(t)$, which fluctuates between random values, which can be either positive or negative, at each time t. The random force satisfies

$$\langle f(t) \rangle = 0 \quad \text{and} \quad \langle f(t) f(t') \rangle = A\delta(t - t'), \tag{39}$$

where $\langle \cdots \rangle$ denotes an average taken over many realizations of the random force and A is some constant which characterizes the overall magnitude of the random force. Using the causal Green's function, compute the "correlation function" $\langle x(t_1)\, x(t_2) \rangle$. Hence, show that for small time displacements Δt, the motion obeys

$$\left\langle [x(t + \Delta t) - x(t)]^2 \right\rangle \propto D\Delta t. \tag{40}$$

Compute the "diffusion constant" D. This result is called the **fluctuation-dissipation theorem**, which relates the strength of the fluctuations in a mechanical motion to the "diffusiveness" of its motion.