1 Derivatives

The derivative of a function \( f \) is another function, \( f' \), defined as

\[
\frac{df}{dx} \equiv \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.
\] (1)

This kind of expression is called a limit expression because it involves a limit (in this case, the limit where \( \delta x \) goes to zero).

If the derivative exists within some domain of \( x \) (i.e., the above limit expression is mathematically well-defined), then we say \( f \) is differentiable in that domain. It can be shown that a differentiable function is automatically continuous.

Graphically, the derivative represents the slope of the graph of \( f(x) \), as shown below:

If \( f \) is differentiable, we can define its second-order derivative \( f'' \) as the derivative of \( f' \). Third-order and higher-order derivatives are defined similarly.

1.1 Properties of derivatives

1.1.1 Rules for limit expressions

Since derivatives are defined using limit expressions, let us review the rules governing limits.

First, the limit of a linear superposition is equal to the linear superposition of limits. Given two constants \( a_1 \) and \( a_2 \) and two functions \( f_1 \) and \( f_2 \),

\[
\lim_{x \to c} [a_1 f_1(x) + a_2 f_2(x)] = a_1 \lim_{x \to c} f_1(x) + a_2 \lim_{x \to c} f_2(x). \] (2)

Second, limits obey a product rule and a quotient rule:

\[
\lim_{x \to c} \left[ f_1(x) f_2(x) \right] = \left[ \lim_{x \to c} f_1(x) \right] \left[ \lim_{x \to c} f_2(x) \right],
\]

\[
\lim_{x \to c} \left[ \frac{f_1(x)}{f_2(x)} \right] = \frac{\lim_{x \to c} f_1(x)}{\lim_{x \to c} f_2(x)}. \] (3)

As a special exception, the product rule and quotient rule are inapplicable if they result in \( 0 \times \infty \), \( \infty/\infty \), or \( 0/0 \), which are undefined. As an example of why such combinations are problematic, consider this:

\[
\lim_{x \to 0} x = \lim_{x \to 0} \left[ x^2 \frac{1}{x} \right] = \lim_{x \to 0} \left[ x^2 \right] \lim_{x \to 0} \left[ \frac{1}{x} \right] = 0 \times \infty \text{ (??)}
\] (4)

In fact, the limit expression has the value of 0; it was not correct to apply the product rule in the second step.
1.1.2 Composition rules for derivatives

Using the rules for limit expressions, we can derive the elementary composition rules for derivatives:

\[
\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x) \quad \text{(linearity)} \tag{5}
\]

\[
\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x) \quad \text{(product rule)} \tag{6}
\]

\[
\frac{d}{dx}[f(g(x))] = f'(g(x)) g'(x) \quad \text{(chain rule)} \tag{7}
\]

These can all be proven by direct substitution into the definition of the derivative, and taking appropriate orders of limits. With the aid of these rules, we can prove various standard results, such as the “power rule” for derivatives:

\[
\frac{d}{dx}[x^n] = nx^{n-1}, \quad n \in \mathbb{N}. \tag{8}
\]

The linearity of the derivative operation implies that derivatives “commute” with sums, i.e. you can move them to the left or right of summation signs. This is a very useful feature. For example, we can use it to prove that the exponential is its own derivative, as follows:

\[
\frac{d}{dx}[\exp(x)] = \frac{d}{dx} \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \tag{9}
\]

\[
= \lim_{n \to \infty} \frac{d}{dx} \left(1 + \frac{x}{n}\right)^n \tag{10}
\]

\[
= \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{n-1} \tag{11}
\]

\[
= \exp(x). \tag{12}
\]

Derivatives also commute with limits. For example, we can use this on the alternative definition of the exponential function from Exercise 1 of Chapter 0:

\[
\frac{d}{dx}[\exp(x)] = \frac{d}{dx} \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \tag{13}
\]

\[
= \lim_{n \to \infty} \frac{d}{dx} \left(1 + \frac{x}{n}\right)^n \tag{14}
\]

\[
= \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{n-1} \tag{15}
\]

\[
= \exp(x). \tag{16}
\]

1.2 Taylor series

A function is \textbf{infinitely differentiable} at a point \(x_0\) if all orders of derivatives (i.e., the first derivative, the second derivative, etc.) are well-defined at \(x_0\). If a function is infinitely differentiable at \(x_0\), then near that point it can be expanded in a \textbf{Taylor series}:

\[
f(x) \leftrightarrow \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left[ \frac{d^n f}{dx^n} \right](x_0) \tag{17}
\]

\[
= f(x_0) + (x-x_0)f'(x_0) + \frac{1}{2}(x-x_0)^2f''(x_0) + \cdots \tag{18}
\]

Here, the “zeroth derivative” refers to the function itself. The Taylor series can be derived by assuming that \(f(x)\) can be written as a general polynomial involving terms of the form \((x-x_0)^n\), and then using the definition of the derivative to find the series coefficients.
Many common encountered functions have Taylor series that are exact (i.e., the series is convergent and equal to the value of the function itself) over some portion of their domain. But beware: it is possible for a function to have a divergent Taylor series, or a Taylor series that converges to a different value than the function itself! The conditions under which this happens is a complicated topic that we will not delve into.

Here are some useful Taylor series:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1 \tag{19}
\]

\[
\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad \text{for } |x| < 1 \tag{20}
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{21}
\]

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{22}
\]

\[
\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \tag{23}
\]

\[
\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \tag{24}
\]

The last four Taylor series, (21)–(24), converge to the value of the function for all \(x \in \mathbb{R}\).

### 1.3 Ordinary differential equations

A differential equation is an equation that involves derivatives of a function. For example, here is a differential equation involving \(f\) and its first derivative:

\[
\frac{df}{dx} = f(x) \tag{25}
\]

This is called an ordinary differential equation because it involves a derivative with respect to a single variable \(x\), rather than multiple variables.

Finding a solution for the differential equation means finding a function that satisfies the equation. There is no single method for solving differential equations. In some cases, we can guess the solution; for example, by trying different elementary functions, we can discover that the above differential equation can be solved by

\[
f(x) = A \exp(x). \tag{26}\]

Certain classes of differential equation can be solved using techniques like Fourier transforms, Green’s functions, etc., some of which will be taught in this course. On the other hand, many differential equations simply have no known exact analytic solution.

**Example**—The following differential equation describes a damped harmonic oscillator:

\[
\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x(t) = 0. \tag{27}
\]

In this case, note that \(x(t)\) is the function, and \(t\) is the input variable. This is unlike our previous notation where \(x\) was the input variable, so don’t get confused! This equation is obtained by applying Newton’s second law to an object moving in one dimension subject to both a damping force and a restoring force, with \(x(t)\) representing the position as a function of time.
1.3.1 Specific solutions and general solutions

When confronted with an ordinary differential equation, the first thing you should check for is the highest derivative appearing in the equation. This is called the order of the differential equation. If the equation has order \( N \), then its general solution contains \( N \) free parameters that can be assigned any value (this is similar to the concept of integration constants, which we’ll discuss in the next chapter). Therefore, if you happen to guess a solution, but that solution does not contain \( N \) free parameters, then you know the solution isn’t the most general one.

For example, the ordinary differential equation
\[
\frac{df}{dx} = f(x) 
\]
has order one. We have previously guessed the solution \( f(x) = A\exp(x) \), which has one free parameter, \( A \). So we know our work is done: there is no solution more general than the one we found.

A specific solution to a differential equation is a solution containing no free parameters. One way to get a specific solution is to start from a general solution, and assign actual values to each of the free parameters. In physics problems, the assigned values are commonly determined by boundary conditions. For example, you may be asked to solve a second-order differential equation given the boundary conditions \( f(0) = a \) and \( f(1) = b \); alternatively, you might be given the boundary conditions \( f(0) = c \) and \( f'(0) = d \), or any other combination of two conditions. For an ordinary differential equation of order \( N \), we need \( N \) conditions to define a specific solution.

1.4 Partial derivatives

So far, we have focused on functions which take a single input. Functions can also take multiple inputs; for instance, a function \( f(x, y) \) maps two input numbers, \( x \) and \( y \), and outputs a number. In general, the inputs are allowed to vary independently of one another. The partial derivative of such a function is its derivative with respect to one of its inputs, keeping the others fixed. For example,
\[
f(x, y) = \sin(2x - 3y^2) 
\]
has partial derivatives
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2 \cos(2x - 3y^2), \\
\frac{\partial f}{\partial y} &= -6 \cos(2x - 3y^2).
\end{align*}
\]

1.4.1 Change of variables

We saw in Section 1.1.2 that single-variable functions obey a derivative composition rule,
\[
\frac{d}{dx} f(g(x)) = g'(x) f'(g(x)).
\]
This composition rule has a important generalization for partial derivatives, which is related to the physical concept of a change of coordinates. Suppose a function \( f(x, y) \) takes two inputs \( x \) and \( y \), and we wish to express them using a different coordinate system denoted by \( u \) and \( v \). In general, each coordinate in the old system depends on both coordinates in the new system:
\[
x = x(u, v), \quad y = y(u, v).
\]
Expressed in the new coordinates, the function is

\[ F(u, v) \equiv f(x(u, v), y(u, v)). \]  (34)

It can be shown that the transformed function’s partial derivatives obey the composition rule

\[ \frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \]  (35)

\[ \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}. \]  (36)

On the right-hand side of these equations, the partial derivatives are to be expressed in terms of the new coordinates \((u, v)\). For example,

\[ \frac{\partial f}{\partial x} = \left. \frac{\partial f}{\partial x} \right|_{x=x(u, v), \ y=y(u, v)} \]  (37)

The generalization of this rule to more than two inputs is straightforward. For a function \(f(x_1, \ldots, x_N)\), a change of coordinates \(x_i = x_i(u_1, \ldots, u_N)\) involves the composition

\[ F(u_1, \ldots, u_N) = f(x_1(u_1, \ldots, u_N), \ldots), \quad \frac{\partial F}{\partial u_i} = \sum_{j=1}^{N} \frac{\partial x_j}{\partial u_i} \frac{\partial f}{\partial x_j}. \]  (38)

**Example**—In two dimensions, Cartesian and polar coordinates are related by

\[ x = r \cos \theta, \quad y = r \sin \theta. \]  (39)

Given a function \(f(x, y)\), we can re-write it in polar coordinates as \(F(r, \theta)\). The partial derivatives are related by

\[ \frac{\partial F}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta, \]  (40)

\[ \frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = -\frac{\partial f}{\partial x} r \sin \theta + \frac{\partial f}{\partial y} r \cos \theta. \]  (41)

### 1.4.2 Partial differential equations

A **partial differential equation** is a differential equation involving multiple partial derivatives (as opposed to an ordinary differential equation, which involves derivatives with respect to a single variable). An example of a partial differential equation encountered in physics is Laplace’s equation,

\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \]  (42)

which describes the electrostatic potential \(\Phi(x, y, z)\) at position \((x, y, z)\), in the absence of any electric charges.

Partial differential equations are considerably harder to solve than ordinary differential equations. In particular, their boundary conditions are more complicated to specify: whereas each boundary condition for an ordinary differential equation consists of a single number (e.g., the value of \(f(x)\) at some point \(x = x_0\)), each boundary condition for a partial differential equation consists of a **function** (e.g., the values of \(\Phi(x, y, z)\) along some curve \(g(x, y, z) = 0\)).
1.5 Exercises

1. Show that if a function is differentiable, then it is also continuous.

2. Prove that the derivative of \( \ln(x) \) is \( 1/x \). [solution available]

3. Using the definition of non-natural powers, prove that

\[
\frac{d}{dx}[x^y] = yx^{y-1}, \text{ for } x \in \mathbb{R}^+, \ y \notin \mathbb{N}.
\] (43)

4. Consider \( f(x) = \tanh(\alpha x) \).

(a) Sketch \( f(x) \) versus \( x \), for two cases: (i) \( \alpha = 1 \) and (ii) \( \alpha \gg 1 \).

(b) Sketch the derivative function \( f'(x) \) for the two cases, based on your sketches in part (A) (i.e., without evaluating the derivative directly).

(c) Evaluate the derivative function, and verify that the result matches your sketches in part (B).

5. Prove geometrically that the derivatives of the sine and cosine functions are:

\[
\frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x).
\] (44)

Hence, derive their Taylor series, Eqs. (21) and (21).

6. For each of the following functions, derive the Taylor series around \( x = 0 \):

(a) \( f(x) = \ln[\alpha \cos(x)] \), to the first 3 non-vanishing terms.

(b) \( f(x) = \cos[\pi \exp(x)] \), to the first 4 non-vanishing terms.

(c) \( f(x) = \frac{1}{\sqrt{1 \pm x}} \), to the first 4 non-vanishing terms. Keep track of the signs (i.e., \( \pm \) versus \( \mp \)).

7. For each of the following functions, sketch the graph and state the domains over which the function is differentiable:

(a) \( f(x) = |\sin(x)| \)

(b) \( f(x) = [\tan(x)]^2 \)

(c) \( f(x) = \frac{1}{1 - x^2} \)

8. Let \( \vec{v}(x) \) be a vectorial function which takes an input \( x \) (a number), and gives an output value \( \vec{v} \) that is a 2-component vector. The derivative of this vectorial function is defined in terms of the derivatives of each vector component:

\[
\vec{v}(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} \Rightarrow \frac{d\vec{v}}{dx} = \begin{bmatrix} dv_1/dx \\ dv_2/dx \end{bmatrix}.
\] (45)

Now suppose \( \vec{v}(x) \) obeys the vectorial differential equation

\[
\frac{d\vec{v}}{dx} = A\vec{v},
\] (46)

where

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\] (47)

is a matrix that has two distinct real eigenvectors with real eigenvalues.
(a) How many independent numbers do we need to specify for the general solution?
(b) Let \( \vec{u} \) be one of the eigenvectors of \( A \), with eigenvalue \( \lambda \):

\[
A \vec{u} = \lambda \vec{u}.
\]

Show that \( \vec{v}(x) = \vec{u} e^{\lambda x} \) is a specific solution to the vectorial differential equation. Hence, find the general solution.

[solution available]