# AMM PBL Year 2013/2014

**Singapore Polytechnic**  
Class: DASE/FT/2A/01

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## Constants Used in This PBL

<table>
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<tr>
<th>Description</th>
<th>Value</th>
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<tbody>
<tr>
<td>Radius of the Earth, $R$</td>
<td>$6.371 \times 10^6$ m</td>
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<tr>
<td>Gravitational Constant, $G$</td>
<td>$6.674 \times 10^{-11}$ m$^3$kg$^{-1}$s$^{-2}$</td>
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<tr>
<td>Mass of the Earth, $M$</td>
<td>$5.972 \times 10^{24}$ kg</td>
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PBL Exercise 1: Consider a “bullet rocket” fired upwards with an initial speed \( s_0 \). Is it possible for the rocket to escape from the Earth if \( s_0 \) is large enough, or will it always fall back to Earth? Find the escape speed \( s_0 \) if possible.

Solution:

We define the direction in which the bullet rocket is moving as the positive direction. So the force acting is a negative quantity, speed is a positive quantity, and acceleration is a positive quantity.

Due to Newton's laws of universal gravitation:

\[
F = -\frac{GMm}{r^2} \tag{1}
\]

\( F = \text{the force between masses.} \)
\( G = \text{Gravitational constant} \)
\( M = \text{mass of the earth} \)
\( m = \text{mass of the bullet rocket} \)
\( r = \text{distance from the center of the earth and the bullet rocket} \)

(Negative sign here is to show direction of \( F \) opposes the positive direction defined)

According to Newton's 2\(^{nd}\) Law,

\[
F = ma \tag{2}
\]

\( \text{(1)} = \text{(2)} \)

\[
-\frac{GMm}{r^2} = ma
\]

Since \( m \) is also a constant as the mass of the satellite does not change, divide both sides by \( m \)

\[
a = \frac{ds}{dt} = -\frac{GM}{r^2} \tag{3}
\]

\( s = \text{the speed of bullet rocket} \)
Using chain rule

\[
\frac{ds}{dt} = \frac{ds}{dr} \times \frac{dr}{dt}
\]

\[
\frac{ds}{dt} = \frac{ds}{dr} \tag{4}
\]

\[
3 = 4
\]

\[
\frac{ds}{dr} = -\frac{GM}{r^2}
\]

\[
\int s \, ds = \int -\frac{GM}{r^2} \, dr
\]

\[
\frac{1}{2} s^2 = \frac{GM}{r} + C
\]

At the moment of launch:

\[
r = R \quad S = S_0
\]

\[
R = \text{the radius of the Earth}
\]

\[
S_0 = \text{the initial speed of the bullet rocket}
\]

\[
\frac{1}{2} s_0^2 = \frac{GM}{R} + C
\]

\[
C = \frac{1}{2} s_0^2 - \frac{GM}{R}
\]

Therefore

\[
\frac{1}{2} s^2 = \frac{GM}{r} + \left(\frac{1}{2} s_0^2 - \frac{GM}{R}\right)
\]

\[
s = \pm \sqrt{\frac{2GM}{r} + \frac{s_0^2 - 2GM}{R}}
\]

For the escape speed (the minimum speed which allows the rocket to escape), the rocket will be able to travel infinite distance but with a speed indefinitely close to 0 at the final stage (from the positive side).

Mathematically, the equation becomes:
\[ 0 = \sqrt{\lim_{\infty} \frac{2GM}{s_0^2} + \frac{2GM}{R}} \]
\[ s_0^2 - \frac{2GM}{R} = 0 \]
\[ s_0 = \sqrt{\frac{2GM}{R}} \]
\[ s_0 \approx \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{6.371 \times 10^6}} \]
\[ s_0 = 11.2 \text{ km/s} \quad (3 \text{s.f.}) \]

The escape speed \( s_0 = 11.2 \text{ m/s} \).
PBL Exercise 2: Consider a satellite launched from a rocket in a direction tangent to the Earth’s surface at a height of 100km.

i. What should the initial speed of the satellite be for it to go into a circular orbit around the Earth?

ii. What is the least initial speed it can have to remain in an orbit around the Earth, i.e. not fall back to Earth?

iii. What is the greatest initial speed it can have to remain in an orbit around the Earth, i.e. not fly off into the solar system?

Solutions:

Before we work out the solutions, let us analyze the orbital mechanics in general.

After the launch of the satellite, there will be no external forces (except gravitational force) acting on it.

According to the conservation of energy:

\[ E_i = E_k + E_p = \text{constant} \]

\[ E_i: \text{Total Energy} \]
\[ E_k: \text{Kinetic Energy} \]
\[ E_p: \text{Gravitational Potential Energy} \]

Since \( E_p \) = Work done by the gravitational field bringing a mass in from infinity to a certain point,
\[ E_p = W = \int_{\infty}^{r} Fdx = \int_{\infty}^{r} \frac{GmM}{r^2} dx = -\frac{GmM}{r} \]

\( G: \text{Gravitational constant} \)
\( M: \text{Mass of the earth} \)
\( m: \text{Mass of satellite} \)
\( r: \text{Distance between the centre of the earth and the satellite} \)

\[ \therefore E_t = E_k + E_p = \frac{1}{2}mv^2 - \frac{GmM}{r} \]

Since \( m \) is also a constant as the mass of the satellite does not change,

Let \( r = \text{constant} \),

\[ \frac{E_t}{m} = \frac{v^2}{2} - \frac{GM}{r} \quad (1) \]

By \textbf{Kepler's laws of planetary motion}:

The orbit is elliptical, and the center of the Earth is at one of the foci. (A circular orbit can be treated as a special elliptical orbit.)

At Periapsis (point of minimum distance) and Apoapsis (point of maximum distance):
\[ \alpha = \frac{v_a^2}{2} - \frac{GM}{r_a} = \frac{v_p^2}{2} - \frac{GM}{r_p} \]

\( v_a \): Speed at Apoapsis  
\( v_p \): Speed at Periapsis  
\( r_a \): Distance at Apoapsis  
\( r_p \): Distance at Periapsis

Rearranging:

\[ \frac{v_a^2}{2} - \frac{v_p^2}{2} = \frac{GM}{r_a} - \frac{GM}{r_p} \]  \( \text{(2)} \)

Since there are no external forces (except gravitational force), and hence no external torques, acting on the satellite,

\( \therefore \) By conservation of angular momentum,

\[ ||L|| = \text{Constant} = ||r \times p|| = ||r \times mv|| \]

\( r \): Position Vector  
\( p \): Momentum  
\( v \): Velocity  
\( m \): Mass of satellite

Since \( m \) is a constant as the mass of the satellite does not change,

Let \( \beta = \text{Constant} = \frac{||L||}{m} = ||r \times v|| = ||r|| ||v|| \sin \theta \)

\( \theta \): Angle between \( r \) and \( v \)

At Periapsis (point of minimum distance) and Apoapsis (point of maximum distance):

\( r \perp v \Rightarrow \theta = 90^\circ \Rightarrow \sin \theta = 1 \)

\( \therefore \beta = r_a v_a = r_p v_p \)

\[ v_p = \frac{r_a v_a}{r_p} \]  \( \text{(3)} \)

Sub \( \text{(3)} \) into \( \text{(2)} \),
\[
\frac{1}{2} \left( 1 - \frac{r_a^2}{r_p^2} \right) v_a^2 = \frac{GM}{r_a} - \frac{GM}{r_p}
\]
\[
\frac{1}{2} \left( \frac{r_p^2 - r_a^2}{r_p^2} \right) v_a^2 = \frac{GM}{r_a} - \frac{GM}{r_p}
\]
\[
\frac{1}{2} v_a^2 = GM \left[ \frac{\frac{r_p}{r_a}}{r_a(2a)} \right]
\]

Since

\[
r_p + r_a = 2a \ (a: \text{Semi major axis})
\]

\[\therefore \frac{1}{2} v_a^2 = GM \left[ \frac{2a - r_a}{r_a(2a)} \right]\]

From what we had derived earlier:

\[
\alpha = \frac{v_a^2}{2} - \frac{GM}{r_a}
\]
\[
\alpha = GM \left[ \frac{2a - r_a}{r_a(2a)} - \frac{1}{r_a} \right]
\]
\[
\alpha = -\frac{GM}{2a} \quad (4)
\]

\[\begin{align*}
1 &= (4) \\
\frac{v^2}{2} - \frac{GM}{r} &= -\frac{GM}{2a} \\
v^2 &= GM \left( \frac{2}{r} - \frac{1}{a} \right) \quad (5)
\end{align*}\]

Therefore we can directly calculate the speed of the satellite with a given distance, when the satellite is travelling in a fixed elliptical orbit. This formula is extremely useful and simplifies the calculation of required initial speeds.
For all 3 cases below, distance at launching point, \( r = constant = R + 100 \times 10^3 \)

i. For a circular orbit, 

\[ a = radius \ of \ the \ orbit = R + 100 \times 10^3 \]

From (5)

\[ v_{\text{circular}}^2 = GM \left( \frac{2}{R + 100 \times 10^3} - \frac{1}{R + 100 \times 10^3} \right) \]

\[ v_{\text{circular}} = \sqrt{\frac{GM}{R + 100 \times 10^3}} \]

\[ v_{\text{circular}} = \frac{6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{\sqrt{6.371 \times 10^6 + 100 \times 10^3}} \]

\[ v_{\text{circular}} = 7.85 \text{ km/s} \]

**Speed required for the circular orbit,** \( v_{\text{circular}} = 7.85 \text{ km/s} \)

For the next 2 cases, the orbit will be elliptical. The launching point will be either the Apoapsis or the Periapsis of the elliptical orbit as \( \mathbf{r} \perp \mathbf{v} \) at the moment of the launch.

ii. To achieve minimum initial speed, the Semi major axis \( a \) should be as small as possible. This conclusion can be drawn from observing the formula (5).
Intuitively, such orbit would have its Apoapsis at the launching point, and its Periapsis indefinitely close to the surface of the Earth.

Hence:

\[ a = \frac{2R + 100}{2} = R + 50 \times 10^3 \]

\[ \therefore v_{\text{min}}^2 = GM \left( \frac{2}{R + 100 \times 10^3} - \frac{1}{R + 50 \times 10^3} \right) \]

\[ v_{\text{min}} = \sqrt{GM \left( \frac{2}{R + 100 \times 10^3} - \frac{1}{R + 50 \times 10^3} \right)} \]

\[ = \sqrt{6.674 \times 10^{-11} \times 5.972 \times 10^{24} \left( \frac{2}{6.371 \times 10^6 + 100 \times 10^3} - \frac{1}{6.371 \times 10^6 + 50 \times 10^3} \right)} \]

\[ v_{\text{min}} = 7.82 \text{km/s} \]

The minimum initial speed maintaining an orbit around the Earth is 7.82 km/s

iii. To achieve maximum initial speed, maintaining an orbit around the Earth, the elliptical orbit will be indefinitely large. The launching point is the Periapsis.
\[ \therefore a \to \infty \]

\[ v_{\text{max}}^2 = GM \left( \frac{2}{R + 100 \times 10^3} - \frac{1}{\infty} \right) \]

\[ v_{\text{max}}^2 = GM \left( \frac{2}{R + 100 \times 10^2} \right) \]

\[ v_{\text{max}} = \sqrt{\frac{2GM}{R + 100 \times 10^3}} \]

\[ v_{\text{max}} = \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{6.371 \times 10^6 + 100 \times 10^3}} \]

\[ v_{\text{max}} = 11.1 \text{km/s} \]

The greatest initial speed approaches but not equal to 11.1 km/s.