AMM PBL

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Class: ME/MS803M/AM05
The common values that we use are:

\[ G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^2 \]

Radius of Earth \((r) = 6371 \text{ km} \) [From Universe Today]

Mass of Earth \((M) = 5.9736 \times 10^{24} \text{ kg} \) [From Universe Today]

**Exercise 1**

Consider a “bullet rocket” fired upwards with an initial speed \(s_0\). Is it possible for the rocket to escape from the Earth if \(s_0\) is large enough, or will it always fall back to Earth? Find the escape speed \(s_0\) if possible.

It is possible for the rocket to escape from the Earth if \(s_0\) is large enough. The rocket will never fall back once it escapes from the Earth as the travelling distance is infinity because the velocity never reaches to zero.

Let \(s\) be the initial velocity of the bullet

\[
- \frac{GM}{r^2} = s \frac{ds}{dr} = \frac{d}{dr} \left(0.5s^2\right)
\]

\[
\frac{GM}{r} + c = 0.5s^2
\]

When \(r \Rightarrow \infty\), \(s \Rightarrow 0\)

\[
\therefore c = 0
\]

\[
\frac{GM}{r} = 0.5s^2
\]

\[
s = \sqrt{2GM/r}
\]

\[
s = \sqrt{(6.674 \times 10^{-20})(2)(5.9736 \times 10^{24})/6371}
\]

\[
s = 11.1872 \text{ km}/\text{s}
\]

\[
= 11.187 \text{ km}/\text{s}(3dp)
\]
Exercise 2

Consider a satellite launched from a rocket in a direction tangent to the Earth’s surface at a height of 100km.

i) What should the initial speed of the satellite be for it to go into a circular orbit around the Earth?

Let \( r \) be the distance between the satellite and the center of Earth

\[ x = r \cos \theta, \quad y = r \sin \theta \]

\( \theta = \omega t \)

\[ v = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = (-r\omega \sin(\omega t), \, r\omega \cos(\omega t)) \]

\[ a = \left( \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right) = (-r\omega^2 \cos(\omega t), \, -r\omega^2 \sin(\omega t)) \]

Since, the satellite is in a circular orbit, the satellite is always a fixed distance from the center of earth. Therefore, the acceleration is constant at all points.

The radius\((r)\) of the circle is 6471km.

When \( t=0 \), \( a = (-r\omega^2,0) \)

\[ -\frac{GM}{r^2} = a \]

\[ -r\omega^2 = -\frac{GM}{r^2} \]

\[ \omega^2 = \frac{GM}{r^3} \]

\[ \omega = \sqrt{\frac{GM}{r^3}} \]

\[ v = r\omega \]

\[ v = \sqrt{\frac{GMr^2}{r^3}} \]

\[ = \sqrt{GM/r} \]

\[ = \sqrt{(6.674 \times 10^{-20})(5.9736 \times 10^{24}) / 6471} \]

\[ = 7.8492 \text{ km/s} \]

\[ = 7.849 \text{ km/s (3dp)} \]
Checking using numerical estimation

Eccentricity: 0.000608283
Initial velocity: 7849m/s
Max Height: 6.47461e+006m
Min Height: 6.46674e+006m

New Velocity: 7849m/s
FPS: 62

Keys:
z: zoom out x: zoom in
y: increase newV0 by 1  h: decrease
u: increase newV0 by 10  j: decrease
i: increase newV0 by 100  k: decrease
hold shift for 5 times more
Diagram of the Earth and the path of satellite

- Centre of the ellipse
- Ellipse (Path of satellite)
- Foci of the ellipse
- Foci of the ellipse
- Circle (Earth)
- Centre of the circle (earth) /foci of the ellipse
Exercise 2

ii) What is the least initial speed it can have to remain in an orbit around the Earth, i.e. not fall back to Earth?

The acceleration of the satellite is not constant hence a elliptic orbit will be formed

Let \( r \) be the distance between the satellite and the center of Earth

Let one of the foci of the ellipse be the center of the earth

Let point 1 be the starting position of the satellite, 100km above Earth

Let point 2 be opposite of point 1, just above the surface of the Earth

\[
PE + KE = E_T
\]

\[
PE_1 + KE_1 = PE_2 + KE_2
\]

\[
KE_1 - KE_2 = PE_2 - PE_1
\]

\[
PE = -\frac{GMm}{r}
\]

\[
KE = \frac{1}{2} mV^2
\]

\[
\frac{1}{2} m_1 V_1^2 - \frac{1}{2} m_2 V_2^2 = -\frac{GMm_2}{r_2} + \frac{GMm_1}{r_1}
\]

Since mass is a constant, \( m_1 = m_2 \)

\[
\therefore \frac{1}{2} V_1^2 - \frac{1}{2} V_2^2 = GM/r_1 - GM/r_2
\]

By conservation of angular momentum \((L)\)

\[
L = r_1 m_1 V_1 = r_2 m_2 V_2
\]

\[
r_1 V_1 = r_2 V_2
\]

\[
V_2 = \frac{r_1 V_1}{r_2}
\]

\[
\frac{1}{2} V_1^2 - \frac{1}{2} \left( \frac{r_1 V_1}{r_2} \right)^2 = GM/r_1 - GM/r_2
\]

\[
\frac{1}{2} V_1^2 \left( 1 - \left( \frac{r_1}{r_2} \right)^2 \right) = GM \left( \frac{1}{r_1} - \frac{1}{r_2} \right)
\]

\[
V_1^2 = 2GM \left( \frac{1}{r_1} - \frac{1}{r_2} \right) + \left( \frac{r_2^2 - r_1^2}{r_2^2} \right)
\]

\[
V_1^2 = 2GM \left( \frac{r_2 - r_1}{r_2 r_1} \right) \left( \frac{r_2^2}{r_2^2 - r_1^2} \right)
\]
\[ V_1^2 = 2GM \left( \frac{r_2 - r_1}{r_2 r_1} \right) \left( \frac{r_2^2}{r_2 + r_1} \right) \]

\[ V_1^2 = 2GM \left( \frac{r_2}{r_1 (r_2 + r_1)} \right) \]

\[ V_1 = \sqrt{2GM \left( \frac{r_2}{r_1 (r_2 + r_1)} \right)} \]

\[ V = \sqrt{2 \times 6.674 \times 10^{-20} \times 5.9736 \times 10^{24} \left( \frac{6371}{6471(6371 + 6471)} \right)} \]

\[ = 7.8185 \text{km/s} \]
\[ = 7.819 \text{km/s (3d.p.)} \]

**Checking with numerical estimation:**

Eccentricity: 0.00770499
Initial velocity: 7819 m/s
Max Height: 6.47116e+006 m
Min Height: 6.3722e+006 m
New Velocity: 7819 m/s
FPS: 58

Keys:
- enter: update path
- z: zoom out
- x: zoom in
- y: increase new V0 by 1
- h: decrease
- u: increase new V0 by 10
- j: decrease
- i: increase new V0 by 100
- k: decrease
- hold shift + key: achieve 5 time more effect
Exercise 2

iii.) What is the greatest initial speed it can have to remain in an orbit around the Earth, i.e. not fly off into the solar system?

From the law of conservation of energy. Energy can neither be created nor destroyed, however it can convert from one form to another form without any energy loss.

By default, the satellite only possesses kinetic and gravitational potential energy, neglecting other forms of energy.

Let \( PE \) be the gravitational potential energy of the satellite.

Let \( KE \) be the kinetic energy of the satellite.

Let \( E_T \) be the total energy of the satellite.

\[
PE + KE = E_T
\]

When the satellite at infinite distance away from the Earth, the kinetic energy will become minimum which is closing to zero. It is due to the speed of the satellite decreasing to the minimum as its escapes infinitely away from the Earth.

To prove that:

\[
-\frac{1}{2}mv_0^2 = PE_\infty - PE_0
\]

Where \( V_0 = \sqrt{2GM/r} \)

From RHS:

\[
= PE_\infty - PE_0 \quad \text{(From: Splung.com physics)}
\]

\[
= -m \sum (GM/r)
\]

Since the \( \sum \) only apply when there is more than one object.

\[
= -m(GM/r)
\]

From LHS:

\[
KE = -\frac{1}{2}mv_0^2
\]

\[
= -\frac{1}{2}m\left(\sqrt{2GM/r}\right)^2
\]

\[
= -\frac{1}{2}m(2GM/r)
\]

\[
= -m(GM/r)
\]

\[
= RHS
\]

When \( KE \Rightarrow 0 \)

\[
PE = E_T
\]

\[
KE_0 + PE_0 = KE_\infty + PE_\infty
\]

\[
KE_0 + PE_0 = 0 + PE_\infty
\]

\[
KE_0 = PE_\infty - PE_0
\]

\[
\therefore -\frac{1}{2}mv_0^2 = PE_\infty - PE_0
\]

Where \( V_0 = \sqrt{2GM/r} \)

Since \( KE \propto V_0^2 \)

\[
V_0 = \sqrt{(6.674 \times 10^{-20})(2)(5.9736 \times 10^{24})/6471}
\]

\[
= 11.1004 km/s
\]

\[
= 11.100 km/s (3dp)
\]

Satellite must never reach 11.1 km/s to stay in orbit.
Approximating with numerical estimation

(Plotting 50,000,000 points at 1 sec interval of t)
Bibliography


