Low-lying Zeros of Families of L-functions

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The studies of L-functions are known to be very useful in number theory since the days of Riemann and Dirichlet. This is largely due to the fact that non-trivial zeroes of an L-function associated with an arithmetic object encode a lot of information of the object concerned. The philosophy of random matrix theory is that statistics associated to zeros of a family of L-functions can be modelled by that of eigenvalues of large random matrices in a suitable linear group. This has been confirmed for many families of L-functions by various authors.

In a joint work with S. Baier, we proved an *unconditional* density theorem for the low-lying zeros of Hasse-Weil L-functions for the family of all elliptic curves over the rationals. Let $L(s, E)$ denote the Hasse-Weil L-function associated with an elliptic curve E over $\mathbb{Q}$ and set

$$D(E, \phi) = \sum_{\rho_E} \phi\left(\frac{\rho_E - 1}{2\pi i} \log X\right),$$

where the ρ_E's run over the non-trivial zeros of $L(s, E)$, the parameter X be chosen arbitrarily and ϕ is an even function which, when restricted to $\mathbb{R}$, becomes a non-negative function with compactly support Fourier transform. Note that this $D(E, \phi)$ represents the density of zeros of $L(s, E)$ near the central point. In the work under discussion, we proved an unconditional bound for the average of $D(E, \phi)$ over all elliptic curves E over $\mathbb{Q}$.

This result follows a long line of works by A. Brumer, D. R. Heath-Brown and M. P. Young. But all these earlier results are conditional to various kinds of hitherto unresolved generalized Riemann Hypotheses(GRH). These assumptions were all removed in our result.

If we assume GRH for these Hasse-Weil L-functions (and only for these L-functions), the unconditional density theorem implies that the average analytic rank of all elliptic curves over the rationals is at most 27/14 which is strictly less than 2. The previous result that gives an average rank strictly less than 2 required additionally GRH for Dirichlet and symmetric square L-functions. As noted by D. R. Heath-Brown, knowing the average rank is strictly less than 2 is of "paramount importance" since from this we can infer that a positive proportion of elliptic curves have algebraic ranks equal to their analytic ranks and finite Tate-Shafarevich groups, due to the celebrated works of Gross-Zagier and Kolyvagin. We remark here that the hitherto unresolved conjecture due to Birch and Swinnerton-Dyer gives that all ellpitic curves have the same algebraic and analytic ranks.

In another collaboration with P. Gao, we proved some density results for the low-lying zeros of families of L-functions of holomorphic Hecke eigenforms of level one and weight k twisted with quadratic Dirichlet characters, as well as those of cubic and quartic Dirichlet L-functions. These results are very much of the same flavor as the elliptic curve result.

These mean-value density results all boil down to the study of average values of exponential and characters sums. These, in turn, can be handled using various types of large sieve inequalities, a very useful tool when studying mean-values in number theory.

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Selected Publications

1. P. Gao and L. Zhao, One Level Density of Low-lying Zeros of Families of L-Functions, *Compos. Math.*, to appear.
2. S. Baier and L. Zhao, On Hecke eigenvalues at Piatetski-Shapiro primes, J. London Math. Soc. (2), **81** (2010), no. 1, 175-201.
3. S. Baier and L. Zhao, The Sato-Tate Conjecture on Average for Small Angles, Trans. Amer. Math. Soc., **361** (2009), no. 4, 1811-1832.
4. S. Baier and L. Zhao, On the Low-lying zeros of Hasse-Weil L-functions for Elliptic Curves, Adv. Math., **219** (2008), no. 3, 952-985.
5. S. Baier and L. Zhao, Primes in Quadratic Progressions on Average, *Math. Ann.*, **338** (2007), no. 4, 2007, 963-982.
6. L. Zhao, Oscillations of Hecke Eigenvalues at Primes, *Rev. Mat. Iberoam.*, **22** (2006), no. 1, 323-337.
7. L. Zhao, Large Sieve Inequalities for Characters to Square Moduli, *Acta Arith.*, **112** (2004), no. 3, 2004, 297-308.