Sforza’s formula for the volume of a hyperbolic tetrahedron

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Various formulas for the volume of a non-Euclidean tetrahedron have been a subject of much recent research. Surprisingly, Italian mathematician Gaetano Sforza came up with a fairly simple formula in 1906 but his result was only published in Italian and has been completely forgotten by the mathematical community. It was only in August 2006 after a discussion between A. D. Mednykh and J. M. Montesinos at the conference in El Burgo de Osma (Spain) when Sforza’s theorem became widely known again.

Consider a hyperbolic (or spherical tetrahedron) $T$ with dihedral angles $A, B, C, D, E, F$. Assume that $A, B, C$ are dihedral angles at the edges adjacent to one vertex and $D, E, F$ are opposite to them correspondingly. Then the Gram matrix $G(T)$ is defined as

$$G(T) = \begin{pmatrix}
1 & -\cos A & -\cos B & -\cos F \\
-\cos A & 1 & -\cos C & -\cos E \\
-\cos B & -\cos C & 1 & -\cos D \\
-\cos F & -\cos E & -\cos D & 1 \\
\end{pmatrix}$$

Further, let $c_{ij} = (-1)^{i+j}G_{ij}$ be the $(i,j)$-cofactor, where $G_{ij}$ is the $(i,j)$-minor of the matrix $G$.

**Theorem (Sforza)** Let $T$ be a compact hyperbolic tetrahedron with Gram matrix $G$. Consider $G = G(A)$ as a function of the dihedral angle $A$. Then the volume of $T$ is given by the formula

$$V(T) = \frac{1}{4} \int_{A_0}^A \log \left( \frac{c_{34}(A) - \sqrt{-\det G(A)} \sin A}{c_{34}(A) + \sqrt{-\det G(A)} \sin A} \right) dA,$$

where $A_0$ is a suitable root of the equation $\det G(A) = 0$.

In this talk we’ll provide an outline of an easy proof of Sforza’s formula.

**Speaker Biography**
Dr. Nikolay Abrosimov is a young active mathematician from Novosibirsk, Russia. He holds a lecturer position at Novosibirsk State University and a research fellow position at Sobolev Institute of Mathematics. His notable achievements are including his work in the list of “the most important results of Russian Academy of Sciences” in 2009 and Alexandrov Prize for young researchers of Russian Academy 2011. Dr Abrosimov’s hobbies are piano (he got a Youth Prize in piano in his hometown of Zheleznogorsk in 1997) and shooting (he was a member of the Novosibirsk State University team).

**Host: Dr. Fedor Duzhin, Division of Mathematical Sciences, School of Physical and Mathematical Sciences**