Chapter 3

Ramification Theory

This chapter introduces ramification theory, which roughly speaking asks the following question: if one takes a prime (ideal) $\mathfrak{p}$ in the ring of integers $\mathcal{O}_K$ of a number field K, what happens when $\mathfrak{p}$ is lifted to $\mathcal{O}_L$, that is $\mathfrak{p}\mathcal{O}_L$, where L is an extension of K. We know by the work done in the previous chapter that $\mathfrak{p}\mathcal{O}_L$ has a factorization as a product of primes, so the question is: will $\mathfrak{p}\mathcal{O}_L$ still be a prime? or will it factor somehow?

In order to study the behavior of primes in L/K, we first consider absolute extensions, that is when $K = \mathbb{Q}$, and define the notions of *discriminant*, *inertial degree* and *ramification index*. We show how the discriminant tells us about ramification. When we are lucky enough to get a "nice" ring of integers $\mathcal{O}_L$, that is $\mathcal{O}_L = \mathbb{Z}[\theta]$ for $\theta \in L$, we give a method to compute the factorization of primes in $\mathcal{O}_L$. We then generalize the concepts introduced to relative extensions, and study the particular case of Galois extensions.

3.1 Discriminant

Let K be a number field of degree n. Recall from Corollary 1.8 that there are n embeddings of K into $\mathbb{C}$.

Definition 3.1. Let K be a number field of degree n, and set

$$\begin{aligned} r_1 &= \text{number of real embeddings} \\ r_2 &= \text{number of pairs of complex embeddings} \end{aligned}$$

The couple (r_1, r_2) is called the signature of K. We have that

$$n = r_1 + 2r_2.$$

Examples 3.1.

1. The signature of $\mathbb{Q}$ is $(1, 0)$.
2. The signature of $\mathbb{Q}(\sqrt{d})$, $d > 0$, is $(2, 0)$.

3. The signature of $\mathbb{Q}(\sqrt{d})$, $d < 0$, is $(0,1)$.
4. The signature of $\mathbb{Q}(\sqrt[3]{2})$ is $(1,1)$.

Let K be a number field of degree n, and let $\mathcal{O}_K$ be its ring of integers. Let $\sigma_1, \dots, \sigma_n$ be its n embeddings into $\mathbb{C}$. We define the map

$$\begin{aligned} \sigma \quad : K \to & \quad \mathbb{C}^n \\ x \mapsto & \quad (\sigma_1(x), \dots, \sigma_n(x)). \end{aligned}$$

Since $\mathcal{O}_K$ is a free abelian group of rank n, we have a $\mathbb{Z}$-basis $\{\alpha_1, \dots, \alpha_n\}$ of $\mathcal{O}_K$. Let us consider the $n \times n$ matrix M given by

$$M = (\sigma_i(\alpha_j))_{1 \le i,j \le n}.$$

The determinant of M is a measure of the density of $\mathcal{O}_K$ in K (actually of $K/\mathcal{O}_K$). It tells us how sparse the integers of K are. However, $\det(M)$ is only defined up to sign, and is not necessarily in either $\mathbb{R}$ or K. So instead we consider

$$\begin{aligned} \det(M^2) &= \det(M^t M) \\ &= \det\left(\sum_{k=1}^{n} \sigma_k(\alpha_i)\sigma_k(\alpha_j)\right)_{i,j} \\ &= \det(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i\alpha_j))_{i,j} \in \mathbb{Z}, \end{aligned}$$

and this does not depend on the choice of a basis.

Definition 3.2. Let $\alpha_1, \dots, \alpha_n \in K$. We define

$$disc(\alpha_1, \dots, \alpha_n) = \det(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i\alpha_j))_{i,j}.$$

In particular, if $\alpha_1, \dots, \alpha_n$ is any $\mathbb{Z}$-basis of $\mathcal{O}_K$, we write Δ_K, and we call discriminant the integer

$$\Delta_K = \det(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i\alpha_j))_{1 \le i,j \le n}.$$

We have that $\Delta_K \neq 0$. This is a consequence of the following lemma.

Lemma 3.1. *The symmetric bilinear form*

$$\begin{aligned} K \times K &\to \mathbb{Q} \\ (x,y) &\mapsto \mathrm{Tr}_{K/\mathbb{Q}}(xy) \end{aligned}$$

is non-degenerate.

Proof. Let us assume by contradiction that there exists $0 \neq \alpha \in K$ such that $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha\beta) = 0$ for all $\beta \in K$. By taking $\beta = \alpha^{-1}$, we get

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha\beta) = \mathrm{Tr}_{K/\mathbb{Q}}(1) = n \neq 0.$$

□

Now if we had that $\Delta_K = 0$, there would be a non-zero column vector $(x_1, \ldots, x_n)^t$, $x_i \in \mathbb{Q}$, killed by the matrix $(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i\alpha_j))_{1\le i,j\le n}$. Set $\gamma = \sum_{i=1}^n \alpha_i x_i$, then $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_j\gamma) = 0$ for each j, which is a contradiction by the above lemma.

Example 3.2. Consider the quadratic field $K = \mathbb{Q}(\sqrt{5})$. Its two embeddings into $\mathbb{C}$ are given by

$$\sigma_1 : a + b\sqrt{5} \mapsto a + b\sqrt{5},\ \sigma_2 : a + b\sqrt{5} \mapsto a - b\sqrt{5}.$$

Its ring of integers is $\mathbb{Z}[(1+\sqrt{5})/2]$, so that the matrix M of embeddings is

$$M = \begin{pmatrix} \sigma_1(1) & \sigma_2(1) \\ \sigma_1\left(\frac{1+\sqrt{5}}{2}\right) & \sigma_2\left(\frac{1+\sqrt{5}}{2}\right) \end{pmatrix}$$

and its discriminant Δ_K can be computed by

$$\Delta_K = \det(M^2) = 5.$$

3.2 Prime decomposition

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$. Then $\mathfrak{p}\cap\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. Indeed, one easily verifies that this is an ideal of $\mathbb{Z}$. Now if a, b are integers with $ab \in \mathfrak{p}\cap\mathbb{Z}$, then we can use the fact that $\mathfrak{p}$ is prime to deduce that either a or b belongs to $\mathfrak{p}$ and thus to $\mathfrak{p}\cap\mathbb{Z}$ (note that $\mathfrak{p}\cap\mathbb{Z}$ is a proper ideal since $\mathfrak{p}\cap\mathbb{Z}$ does not contain 1, and $\mathfrak{p}\cap\mathbb{Z} \neq \emptyset$, as $\mathrm{N}(\mathfrak{p})$ belongs to $\mathfrak{p}$ and $\mathbb{Z}$ since $\mathrm{N}(\mathfrak{p}) = |\mathcal{O}/\mathfrak{p}| < \infty$).

Since $\mathfrak{p}\cap\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$, there must exist a prime number p such that $\mathfrak{p}\cap\mathbb{Z} = p\mathbb{Z}$. We say that $\mathfrak{p}$ is above p.

$$\begin{array}{c} \mathfrak{p} \subset \mathcal{O}_K \subset K \\ \Big| \\ p\mathbb{Z} \subset \mathbb{Z} \subset \mathbb{Q} \end{array}$$

We call residue field the quotient of a commutative ring by a maximal ideal. Thus the residue field of $p\mathbb{Z}$ is $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. We are now interested in the residue field $\mathcal{O}_K/\mathfrak{p}$. We show that $\mathcal{O}_K/\mathfrak{p}$ is a $\mathbb{F}_p$-vector space of finite dimension. Set

$$\phi : \mathbb{Z} \to \mathcal{O}_K \to \mathcal{O}_K/\mathfrak{p},$$

where the first arrow is the canonical inclusion ι of $\mathbb{Z}$ into $\mathcal{O}_K$, and the second arrow is the projection π, so that $\phi = \pi \circ \iota$. Now the kernel of ϕ is given by

$$ker(\phi) = \{a \in \mathbb{Z} \mid a \in \mathfrak{p}\} = \mathfrak{p}\cap\mathbb{Z} = p\mathbb{Z},$$

so that ϕ induces an injection of $\mathbb{Z}/p\mathbb{Z}$ into $\mathcal{O}_K/\mathfrak{p}$, since $\mathbb{Z}/p\mathbb{Z} \simeq Im(\phi) \subset \mathcal{O}_K/\mathfrak{p}$. By Lemma 2.1, $\mathcal{O}_K/\mathfrak{p}$ is a finite set, thus a finite field which contains $\mathbb{Z}/p\mathbb{Z}$ and we have indeed a finite extension of $\mathbb{F}_p$.

Definition 3.3. We call inertial degree, and we denote by $f_{\mathfrak{p}}$, the dimension of the $\mathbb{F}_p$-vector space $\mathcal{O}/\mathfrak{p}$, that is

$$f_{\mathfrak{p}} = \dim_{\mathbb{F}_p}(\mathcal{O}/\mathfrak{p}).$$

Note that we have

$$\mathrm{N}(\mathfrak{p}) = |\mathcal{O}/\mathfrak{p}| = |\mathbb{F}_p^{\dim_{\mathbb{F}_p}(\mathcal{O}/\mathfrak{p})}| = |\mathbb{F}_p|^{f_{\mathfrak{p}}} = p^{f_{\mathfrak{p}}}.$$

Example 3.3. Consider the quadratic field $K = \mathbb{Q}(i)$, with ring of integers $\mathbb{Z}[i]$, and let us look at the ideal $2\mathbb{Z}[i]$:

$$2\mathbb{Z}[i] = (1+i)(1-i)\mathbb{Z}[i] = \mathfrak{p}^2,\ \mathfrak{p} = (1+i)\mathbb{Z}[i]$$

since $(-i)(1+i) = 1-i$. Furthermore, $\mathfrak{p} \cap \mathbb{Z} = 2\mathbb{Z}$, so that $\mathfrak{p} = (1+i)$ is said to be above 2. We have that

$$\mathrm{N}(\mathfrak{p}) = N_{K/\mathbb{Q}}(1+i) = (1+i)(1-i) = 2$$

and thus $f_{\mathfrak{p}} = 1$. Indeed, the corresponding residue field is

$$\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_2.$$

Let us consider again a prime ideal $\mathfrak{p}$ of $\mathcal{O}$. We have seen that $\mathfrak{p}$ is above the ideal $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$. We can now look the other way round: we start with the prime $p \in \mathbb{Z}$, and look at the ideal $p\mathcal{O}$ of $\mathcal{O}$. We know that $p\mathcal{O}$ has a unique factorization into a product of prime ideals (by all the work done in Chapter 2). Furthermore, we have that $p \subset \mathfrak{p}$, thus $\mathfrak{p}$ has to be one of the factors of $p\mathcal{O}$.

Definition 3.4. Let $p \in \mathbb{Z}$ be a prime. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ above p. We call ramification index of $\mathfrak{p}$, and we write $e_{\mathfrak{p}}$, the exact power of $\mathfrak{p}$ which divides $p\mathcal{O}$.

We start from $p \in \mathbb{Z}$, whose factorization in $\mathcal{O}$ is given by

$$p\mathcal{O} = \mathfrak{p}_1^{e_{\mathfrak{p}_1}} \cdots \mathfrak{p}_g^{e_{\mathfrak{p}_g}}.$$

We say that p is ramified if $e_{\mathfrak{p}_i} > 1$ for some i. On the contrary, p is non-ramified if

$$p\mathcal{O} = \mathfrak{p}_1 \cdots \mathfrak{p}_g,\ \mathfrak{p}_i \neq \mathfrak{p}_j,\ i \neq j.$$

Both the inertial degree and the ramification index are connected via the degree of the number field as follows.

Proposition 3.2. *Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $p \in \mathbb{Z}$ and let*

$$p\mathcal{O} = \mathfrak{p}_1^{e_{\mathfrak{p}_1}} \cdots \mathfrak{p}_g^{e_{\mathfrak{p}_g}}$$

be its factorization in $\mathcal{O}$. We have that

$$n = [K:\mathbb{Q}] = \sum_{i=1}^{g} e_{\mathfrak{p}_i} f_{\mathfrak{p}_i}.$$

Proof. By Lemma 2.1, we have

$$\mathrm{N}(p\mathcal{O}) = |\mathrm{N}_{K/\mathbb{Q}}(p)| = p^n,$$

where $n = [K : \mathbb{Q}]$. Since the norm N is multiplicative (see Corollary 2.12), we deduce that

$$\mathrm{N}(\mathfrak{p}_1^{e_{\mathfrak{p}_1}} \cdots \mathfrak{p}_g^{e_{\mathfrak{p}_g}}) = \prod_{i=1}^{g} \mathrm{N}(\mathfrak{p}_i)^{e_{\mathfrak{p}_i}} = \prod_{i=1}^{g} p^{f_{\mathfrak{p}_i} e_{\mathfrak{p}_i}}.$$

□

There is, in general, no straightforward method to compute the factorization of $p\mathcal{O}$. However, in the case where the ring of integers $\mathcal{O}$ is of the form $\mathcal{O} = \mathbb{Z}[\theta]$, we can use the following result.

Proposition 3.3. *Let K be a number field, with ring of integers $\mathcal{O}_K$, and let p be a prime. Let us assume that there exists θ such that $\mathcal{O} = \mathbb{Z}[\theta]$, and let f be the minimal polynomial of θ, whose reduction modulo p is denoted by $\bar{f}$. Let*

$$\bar{f}(X) = \prod_{i=1}^{g} \phi_i(X)^{e_i}$$

be the factorization of $f(X)$ in $\mathbb{F}_p[X]$, with $\phi_i(X)$ coprime and irreducible. We set

$$\mathfrak{p}_i = (p, f_i(\theta)) = p\mathcal{O} + f_i(\theta)\mathcal{O}$$

where f_i is any lift of ϕ_i to $\mathbb{Z}[X]$, that is $\bar{f}_i = \phi_i \mod p$. Then

$$p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$$

is the factorization of $p\mathcal{O}$ in $\mathcal{O}$.

Proof. Let us first notice that we have the following isomorphism

$$\mathcal{O}/p\mathcal{O} = \mathbb{Z}[\theta]/p\mathbb{Z}[\theta] \simeq \frac{\mathbb{Z}[X]/f(X)}{p(\mathbb{Z}[X]/f(X))} \simeq \mathbb{Z}[X]/(p, f(X)) \simeq \mathbb{F}_p[X]/\bar{f}(X),$$

where $\bar{f}$ denotes $f \mod p$. Let us call A the ring

$$A = \mathbb{F}_p[X]/\bar{f}(X).$$

The inverse of the above isomorphism is given by the evaluation in θ, namely, if $\psi(X) \in \mathbb{F}_p[X]$, with $\psi(X) \mod \bar{f}(X) \in A$, and $g \in \mathbb{Z}[X]$ such that $\bar{g} = \psi$, then its preimage is given by $g(\theta)$. By the Chinese Theorem, recall that we have

$$A = \mathbb{F}_p[X]/\bar{f}(X) \simeq \prod_{i=1}^{g} \mathbb{F}_p[X]/\phi_i(X)^{e_i},$$

since by assumption, the ideal $(\bar{f}(X))$ has a prime factorization given by $(\bar{f}(X)) = \prod_{i=1}^{g}(\phi_i(X))^{e_i}$.

We are now ready to understand the structure of prime ideals of both $\mathcal{O}/p\mathcal{O}$ and A, thanks to which we will prove that $\mathfrak{p}_i$ as defined in the assumption is prime, that any prime divisor of $p\mathcal{O}$ is actually one of the $\mathfrak{p}_i$, and that the power e_i appearing in the factorization of $\bar{f}$ are bigger or equal to the ramification index $e_{\mathfrak{p}_i}$ of $\mathfrak{p}_i$. We will then invoke the proposition that we have just proved to show that $e_i = e_{\mathfrak{p}_i}$, which will conclude the proof.

By the factorization of A given above by the Chinese theorem, the maximal ideals of A are given by $(\phi_i(X))A$, and the degree of the extension $A/(\phi_i(X))A$ over $\mathbb{F}_p$ is the degree of ϕ_i. By the isomorphism $A \simeq \mathcal{O}/p\mathcal{O}$, we get similarly that the maximal ideals of $\mathcal{O}/p\mathcal{O}$ are the ideals generated by $f_i(\theta) \mod p\mathcal{O}$.

We consider the projection $\pi : \mathcal{O} \to \mathcal{O}/p\mathcal{O}$. We have that

$$\pi(\mathfrak{p}_i) = \pi(p\mathcal{O} + f_i(\theta)\mathcal{O}) = f_i(\theta)\mathcal{O} \mod p\mathcal{O}.$$

Consequently, $\mathfrak{p}_i$ is a prime ideal of $\mathcal{O}$, since $f_i(\theta)\mathcal{O}$ is. Furthermore, since $\mathfrak{p}_i \supset p\mathcal{O}$, we have $\mathfrak{p}_i \mid p\mathcal{O}$, and the inertial degree $f_{\mathfrak{p}_i} = [\mathcal{O}/\mathfrak{p}_i : \mathbb{F}_p]$ is the degree of ϕ_i, while $e_{\mathfrak{p}_i}$ denotes the ramification index of $\mathfrak{p}_i$.

Now, every prime ideal $\mathfrak{p}$ in the factorization of $p\mathcal{O}$ is one of the $\mathfrak{p}_i$, since the image of $\mathfrak{p}$ by π is a maximal ideal of $\mathcal{O}/p\mathcal{O}$, that is

$$p\mathcal{O} = \mathfrak{p}_1^{e_{\mathfrak{p}_1}} \cdots \mathfrak{p}_g^{e_{\mathfrak{p}_g}}$$

and we are thus left to look at the ramification index.

The ideal $\phi_i^{e_i}A$ of A belongs to $\mathcal{O}/p\mathcal{O}$ via the isomorphism between $\mathcal{O}/p\mathcal{O} \simeq A$, and its preimage in $\mathcal{O}$ by π^{-1} contains $\mathfrak{p}_i^{e_i}$ (since if $\alpha \in \mathfrak{p}_i^{e_i}$, then α is a sum of products $\alpha_1 \cdots \alpha_{e_i}$, whose image by π will be a sum of product $\pi(\alpha_1) \cdots \pi(\alpha_{e_i})$ with $\pi(\alpha_i) \in \phi_i A$). In $\mathcal{O}/p\mathcal{O}$, we have $0 = \cap_{i=1}^g \phi_i(\theta)^{e_i}$, that is

$$p\mathcal{O} = \pi^{-1}(0) = \cap_{i=1}^g \pi^{-1}(\phi_i^{e_i}A) \supset \cap_{i=1}^g \mathfrak{p}_i^{e_i} = \prod_{i=1}^g \mathfrak{p}_i^{e_i}.$$

We then have that this last product is divided by $p\mathcal{O} = \prod \mathfrak{p}_i^{e_{\mathfrak{p}_i}}$, that is $e_i \geq e_{\mathfrak{p}_i}$.

Let $n = [K : \mathbb{Q}]$. To show that we have equality, that is $e_i = e_{\mathfrak{p}_i}$, we use the previous proposition:

$$n = [K : \mathbb{Q}] = \sum_{i=1}^g e_{\mathfrak{p}_i} f_{\mathfrak{p}_i} \leq \sum_{i=1}^g e_i \deg(\phi_i) = \dim_{\mathbb{F}_p}(A) = \dim_{\mathbb{F}_p} \mathbb{Z}^n/p\mathbb{Z}^n = n.$$

□

The above proposition gives a concrete method to compute the factorization of a prime $p\mathcal{O}_K$:

1. Choose a prime $p \in \mathbb{Z}$ whose factorization in $p\mathcal{O}_K$ is to be computed.

2. Let f be the minimal polynomial of θ such that $\mathcal{O}_K = \mathbb{Z}[\theta]$.

3. Compute the factorization of $\bar{f} = f \mod p$:

$$\bar{f} = \prod_{i=1}^{g} \phi_i(X)^{e_i}.$$

4. Lift each ϕ_i in a polynomial $f_i \in \mathbb{Z}[X]$.

5. Compute $\mathfrak{p}_i = (p, f_i(\theta))$ by evaluating f_i in θ.

6. The factorization of $p\mathcal{O}$ is given by

$$p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}.$$

Examples 3.4. 1. Let us consider $K = \mathbb{Q}(\sqrt[3]{2})$, with ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}]$. We want to factorize $5\mathcal{O}_K$. By the above proposition, we compute

$$\begin{aligned} X^3 - 2 &\equiv (X-3)(X^2+3X+4) \\ &\equiv (X+2)(X^2-2X-1) \mod 5. \end{aligned}$$

We thus get that

$$5\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2,\ \mathfrak{p}_1 = (5, 2+\sqrt[3]{2}),\ \mathfrak{p}_2 = (5, \sqrt[3]{4} - 2\sqrt[3]{2} - 1).$$

2. Let us consider $\mathbb{Q}(i)$, with $\mathcal{O}_K = \mathbb{Z}[i]$, and choose $p = 2$. We have $\theta = i$ and $f(X) = X^2 + 1$. We compute the factorization of $\bar{f}(X) = f(X) \mod 2$:

$$X^2 + 1 \equiv X^2 - 1 \equiv (X-1)(X+1) \equiv (X-1)^2 \mod 2.$$

We can take any lift of the factors to $\mathbb{Z}[X]$, so we can write

$$2\mathcal{O}_K = (2, i-1)(2, i+1) \text{ or } 2 = (2, i-1)^2$$

which is the same, since $(2, i-1) = (2, 1+i)$. Furthermore, since $2 = (1-i)(1+i)$, we see that $(2, i-1) = (1+i)$, and we recover the result of Example 3.3.

Definition 3.5. We say that p is inert if $p\mathcal{O}$ is prime, in which case we have $g = 1$, $e = 1$ and $f = n$. We say that p is totally ramified if $e = n$, $g = 1$, and $f = 1$.

The discriminant of K gives us information on the ramification in K.

Theorem 3.4. *Let K be a number field. If p is ramified, then p divides the discriminant Δ_K.*

Proof. Let $\mathfrak{p} \mid p\mathcal{O}$ be an ideal such that $\mathfrak{p}^2 \mid p\mathcal{O}$ (we are just rephrasing the fact that p is ramified). We can write $p\mathcal{O} = \mathfrak{p}I$ with I divisible by all the primes above p ($\mathfrak{p}$ is voluntarily left as a factor of I). Let $\alpha_1, \dots, \alpha_n \in \mathcal{O}$ be a $\mathbb{Z}$-basis of $\mathcal{O}$ and let $\alpha \in I$ but $\alpha \notin p\mathcal{O}$. We write

$$\alpha = b_1\alpha_1 + \ldots + b_n\alpha_n,\ b_i \in \mathbb{Z}.$$

Since $\alpha \notin p\mathcal{O}$, there exists a b_i which is not divisible by p, say b_1. Recall that

$$\Delta_K = \det \begin{pmatrix} \sigma_1(\alpha_1) & \ldots & \sigma_1(\alpha_n) \\ \vdots & & \vdots \\ \sigma_n(\alpha_1) & \ldots & \sigma_n(\alpha_n) \end{pmatrix}^2$$

where σ_i, $i = 1, \dots, n$ are the n embeddings of K into $\mathbb{C}$. Let us replace α_1 by α, and set

$$D = \det \begin{pmatrix} \sigma_1(\alpha) & \ldots & \sigma_1(\alpha_n) \\ \vdots & & \vdots \\ \sigma_n(\alpha) & \ldots & \sigma_n(\alpha_n) \end{pmatrix}^2.$$

Now D and Δ_K are related by

$$D = \Delta_K b_1^2,$$

since D can be rewritten as

$$D = \det \left(\begin{pmatrix} \sigma_1(\alpha_1) & \ldots & \sigma_1(\alpha_n) \\ \vdots & & \vdots \\ \sigma_n(\alpha_1) & \ldots & \sigma_n(\alpha_n) \end{pmatrix} \begin{pmatrix} b_1 & 0 & \ldots & 0 \\ b_2 & 1 & & 0 \\ & & \ddots & \\ b_n & & \ldots & 1 \end{pmatrix} \right)^2.$$

We are thus left to prove that $p \mid D$, since by construction, we have that p does not divide b_1^2.

Intuitively, the trick of this proof is to replace proving that $p|\Delta_K$ where we have no clue how the factor p appears, with proving that $p|D$, where D has been built on purpose as a function of a suitable α which we will prove below is such that all its conjugates are above p.

Let L be the Galois closure of K, that is, L is a field which contains K, and which is a normal extension of $\mathbb{Q}$. The conjugates of α all belong to L. We know that α belongs to all the primes of $\mathcal{O}_K$ above p. Similarly, $\alpha \in K \subset L$ belongs to all primes $\mathfrak{P}$ of $\mathcal{O}_L$ above p. Indeed, $\mathfrak{P} \cap \mathcal{O}_K$ is a prime ideal of $\mathcal{O}_K$ above p, which contains α.

We now fix a prime $\mathfrak{P}$ above p in $\mathcal{O}_L$. Then $\sigma_i(\mathfrak{P})$ is also a prime ideal of $\mathcal{O}_L$ above p ($\sigma_i(\mathfrak{P})$ is in L since $L/\mathbb{Q}$ is Galois, $\sigma_i(\mathfrak{P})$ is prime since $\mathfrak{P}$ is, and $p = \sigma_i(p) \in \sigma_i(\mathfrak{P})$). We have that $\sigma_i(\alpha) \in \mathfrak{P}$ for all σ_i, thus the first column of the matrix involves in the computation of D is in $\mathfrak{P}$, so that $D \in \mathfrak{P}$ and $D \in \mathbb{Z}$, to get

$$D \in \mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}.$$

□

We have just proved that if p is ramified, then $p|\Delta_K$. The converse is also true.

Examples 3.5. 1. We have seen in Example 3.2 that the discriminant of $K = \mathbb{Q}(\sqrt{5})$ is $\Delta_K = 5$. This tells us that only 5 is ramified in $\mathbb{Q}(\sqrt{5})$.

2. In Example 3.3, we have seen that 2 ramifies in $K = \mathbb{Q}(i)$. So 2 should appear in Δ_K. One can actually check that $\Delta_K = -4$.

Corollary 3.5. *There is only a finite number of ramified primes.*

Proof. The discriminant only has a finite number of divisors. □

3.3 Relative Extensions

Most of the theory seen so far assumed that the base field is $\mathbb{Q}$. In most cases, this can be generalized to an arbitrary number field K, in which case we consider a number field extension L/K. This is called a relative extension. By contrast, we may call absolute an extension whose base field is $\mathbb{Q}$. Below, we will generalize several definitions previously given for absolute extensions to relative extensions.

Let K be a number field, and let L/K be a finite extension. We have correspondingly a ring extension $\mathcal{O}_K \to \mathcal{O}_L$. If $\mathfrak{P}$ is a prime ideal of $\mathcal{O}_L$, then $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$ is a prime ideal of $\mathcal{O}_K$. We say that $\mathfrak{P}$ is above $\mathfrak{p}$. We have a factorization

$$\mathfrak{p}\mathcal{O}_L = \prod_{i=1}^{g} \mathfrak{P}_i^{e_{\mathfrak{P}_i|\mathfrak{p}}},$$

where $e_{\mathfrak{P}_i/\mathfrak{p}}$ is the relative ramification index. The relative inertial degree is given by

$$f_{\mathfrak{P}_i|\mathfrak{p}} = [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}].$$

We still have that

$$[L : K] = \sum e_{\mathfrak{P}|\mathfrak{p}} f_{\mathfrak{P}|\mathfrak{p}}$$

where the summation is over all $\mathfrak{P}$ above $\mathfrak{p}$.

Let $M/L/K$ be a tower of finite extensions, and let $\mathcal{P}, \mathfrak{P}, \mathfrak{p}$ be prime ideals of respectively M, L, and K. Then we have that

$$\begin{aligned} f_{\mathcal{P}|\mathfrak{p}} &= f_{\mathcal{P}|\mathfrak{P}} f_{\mathfrak{P}|\mathfrak{p}} \\ e_{\mathcal{P}|\mathfrak{p}} &= e_{\mathcal{P}|\mathfrak{P}} e_{\mathfrak{P}|\mathfrak{p}}. \end{aligned}$$

Let I_K, I_L be the groups of fractional ideals of K and L respectively. We can also generalize the application norm as follows:

$$\begin{aligned} \mathrm{N} : \quad I_L &\to \quad I_K \\ \mathfrak{P} &\mapsto \quad \mathfrak{p}^{f_{\mathfrak{P}|\mathfrak{p}}}, \end{aligned}$$

which is a group homomorphism. This defines a relative norm for ideals, which is itself an ideal!

In order to generalize the discriminant, we would like to have an $\mathcal{O}_K$-basis of $\mathcal{O}_L$ (similarly to having a $\mathbb{Z}$-basis of $\mathcal{O}_K$), however such a basis does not exist in general. Let $\alpha_1, \dots, \alpha_n$ be a K-basis of L where $\alpha_i \in \mathcal{O}_L$, $i = 1, \dots, n$. We set

$$disc_{L/K}(\alpha_1, \dots, \alpha_n) = \det \begin{pmatrix} \sigma_1(\alpha_1) & \dots & \sigma_n(\alpha_1) \\ \vdots & & \vdots \\ \sigma_1(\alpha_n) & \dots & \sigma_n(\alpha_n) \end{pmatrix}^2$$

where $\sigma_i : L \to \mathbb{C}$ are the embeddings of L into $\mathbb{C}$ which fix K. We define $\Delta_{L/K}$ as the ideal generated by all $disc_{L/K}(\alpha_1, \dots, \alpha_n)$. It is called relative discriminant.

3.4 Normal Extensions

Let L/K be a Galois extension of number fields, with Galois group $G = \mathrm{Gal}(L/K)$. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$. If $\mathfrak{P}$ is a prime above $\mathfrak{p}$ in $\mathcal{O}_L$, and $\sigma \in G$, then $\sigma(\mathfrak{P})$ is a prime ideal above $\mathfrak{p}$. Indeed, $\sigma(\mathfrak{P}) \cap \mathcal{O}_K \subset K$, thus $\sigma(\mathfrak{P}) \cap \mathcal{O}_K = \mathfrak{P} \cap \mathcal{O}_K$ since K is fixed by σ.

Theorem 3.6. *Let*

$$\mathfrak{p}\mathcal{O}_L = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i}$$

be the factorization of $\mathfrak{p}\mathcal{O}_L$ in $\mathcal{O}_L$. Then G acts transitively on the set $\{\mathfrak{P}_1, \dots, \mathfrak{P}_g\}$. Furthermore, we have that

$$e_1 = \dots = e_g = e \text{ where } e_i = e_{\mathfrak{P}_i|\mathfrak{p}}$$
$$f_1 = \dots = f_g = f \text{ where } f_i = f_{\mathfrak{P}_i|\mathfrak{p}}$$

and

$$[L : K] = efg.$$

Proof. G **acts transitively.** Let $\mathfrak{P}$ be one of the $\mathfrak{P}_i$. We need to prove that there exists $\sigma \in G$ such that $\sigma(\mathfrak{P}_j) = \mathfrak{P}$ for $\mathfrak{P}_j$ any other of the $\mathfrak{P}_i$. In the proof of Corollary 2.10, we have seen that there exists $\beta \in \mathfrak{P}$ such that $\beta\mathcal{O}_L\mathfrak{P}^{-1}$ is an integral ideal coprime to $\mathfrak{p}\mathcal{O}_L$. The ideal

$$I = \prod_{\sigma \in G} \sigma(\beta\mathcal{O}_L\mathfrak{P}^{-1})$$

is an integral ideal of $\mathcal{O}_L$ (since $\beta\mathcal{O}_L\mathfrak{P}^{-1}$ is), which is furthermore coprime to $\mathfrak{p}\mathcal{O}_L$ (since $\sigma(\beta\mathcal{O}_L\mathfrak{P}^{-1})$ and $\sigma(\mathfrak{p}\mathcal{O}_L)$ are coprime and $\sigma(\mathfrak{p}\mathcal{O}_L) = \sigma(\mathfrak{p})\sigma(\mathcal{O}_L) = \mathfrak{p}\mathcal{O}_L$).

Thus I can be rewritten as

$$\begin{aligned} I &= \frac{\prod_{\sigma\in G}\sigma(\beta)\mathcal{O}_L}{\prod_{\sigma\in G}\sigma(\mathfrak{P})} \\ &= \frac{N_{L/K}(\beta)\mathcal{O}_L}{\prod_{\sigma\in G}\sigma(\mathfrak{P})} \end{aligned}$$

and we have that

$$I\prod_{\sigma\in G}\sigma(\mathfrak{P}) = N_{L/K}(\beta)\mathcal{O}_L.$$

Since $N_{L/K}(\beta) = \prod_{\sigma\in G}\sigma(\beta)$, $\beta\in\mathfrak{P}$ and one of the σ is the identity, we have that $N_{L/K}(\beta)\in\mathfrak{P}$. Furthermore, $N_{L/K}(\beta)\in\mathcal{O}_K$ since $\beta\in\mathcal{O}_L$, and we get that $N_{L/K}(\beta)\in\mathfrak{P}\cap\mathcal{O}_K = \mathfrak{p}$, from which we deduce that $\mathfrak{p}$ divides the right hand side of the above equation, and thus the left hand side. Since I is coprime to $\mathfrak{p}$, we get that $\mathfrak{p}$ divides $\prod_{\sigma\in G}\sigma(\mathfrak{P})$. In other words, using the factorization of $\mathfrak{p}$, we have that

$$\prod_{\sigma\in G}\sigma(\mathfrak{P}) \text{ is divisible by } \mathfrak{p}\mathcal{O}_L = \prod_{i=1}^{g}\mathfrak{P}_i^{e_i}$$

and each of the $\mathfrak{P}_i$ has to be among $\{\sigma(\mathfrak{P})\}_{\sigma\in G}$.

All the ramification indices are equal. By the first part, we know that there exists $\sigma\in G$ such that $\sigma(\mathfrak{P}_i) = \mathfrak{P}_k$, $i\neq k$. Now, we have that

$$\begin{aligned} \sigma(\mathfrak{p}\mathcal{O}_L) &= \prod_{i=1}^{g}\sigma(\mathfrak{P}_i)^{e_i} \\ &= \mathfrak{p}\mathcal{O}_L \\ &= \prod_{i=1}^{g}\mathfrak{P}_i^{e_i} \end{aligned}$$

where the second equality holds since $\mathfrak{p}\in\mathcal{O}_K$ and L/K is Galois. By comparing the two factorizations of $\mathfrak{p}$ and its conjugates, we get that $e_i = e_k$.

All the inertial degrees are equal. This follows from the fact that σ induces the following field isomorphism

$$\mathcal{O}_L/\mathfrak{P}_i \simeq \mathcal{O}_L/\sigma(\mathfrak{P}_i).$$

Finally we have that

$$|G| = [L:K] = efg.$$

□

For now on, let us fix $\mathfrak{P}$ above $\mathfrak{p}$.

Definition 3.6. The stabilizer of $\mathfrak{P}$ in G is called the decomposition group, given by

$$D = D_{\mathfrak{P}/\mathfrak{p}} = \{\sigma\in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\} < G.$$

The index $[G : D]$ must be equal to the number of elements in the orbit $G\mathfrak{P}$ of $\mathfrak{P}$ under the action of G, that is $[G : D] = |G\mathfrak{P}|$ (this is the orbit-stabilizer theorem).

By the above theorem, we thus have that $[G : D] = g$, where g is the number of distinct primes which divide $\mathfrak{p}\mathcal{O}_L$. Thus

$$\begin{aligned} n &= efg \\ &= ef\frac{|G|}{|D|} \end{aligned}$$

and

$$|D| = ef.$$

If $\mathfrak{P}'$ is another prime ideal above $\mathfrak{p}$, then the decomposition groups $D_{\mathfrak{P}/\mathfrak{p}}$ and $D_{\mathfrak{P}'/\mathfrak{p}}$ are conjugate in G via any Galois automorphism mapping $\mathfrak{P}$ to $\mathfrak{P}'$ (in formula, we have that if $\mathfrak{P}' = \tau(\mathfrak{P})$, then $\tau D_{\mathfrak{P}/\mathfrak{p}}\tau^{-1} = D_{\tau(\mathfrak{P})/\mathfrak{p}}$).

Proposition 3.7. *Let $D = D_{\mathfrak{P}/\mathfrak{p}}$ be the decomposition group of $\mathfrak{P}$. The subfield*

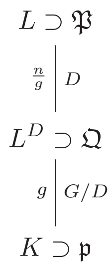
$$L^D = \{\alpha \in L \mid \sigma(\alpha) = \alpha,\ \sigma \in D\}$$

is the smallest subfield M of L such that $(\mathfrak{P} \cap \mathcal{O}_M)\mathcal{O}_L$ does not split. It is called the decomposition field of $\mathfrak{P}$.

Proof. We first prove that L/L^D has the property that$(\mathfrak{P} \cap \mathcal{O}_{L^D})\mathcal{O}_L$ does not split. We then prove its minimality.

We know by Galois theory that $\mathrm{Gal}(L/L^D)$ is given by D. Furthermore, the extension L/L^D is Galois since L/K is. Let $\mathfrak{Q} = \mathfrak{P} \cap \mathcal{O}_{L^D}$ be a prime below $\mathfrak{P}$. By Theorem 3.6, we know that D acts transitively on the set of primes above $\mathfrak{Q}$, among which is $\mathfrak{P}$. Now by definition of $D = D_{\mathfrak{P}/\mathfrak{p}}$, we know that $\mathfrak{P}$ is fixed by D. Thus there is only $\mathfrak{P}$ above $\mathfrak{Q}$.

Let us now prove the minimality of L^D. Assume that there exists a field M with $L/M/K$, such that $\mathfrak{Q} = \mathfrak{P} \cap \mathcal{O}_M$ has only one prime ideal of $\mathcal{O}_L$ above it. Then this unique ideal must be $\mathfrak{P}$, since by definition $\mathfrak{P}$ is above $\mathfrak{Q}$. Then $\mathrm{Gal}(L/M)$ is a subgroup of D, since its elements are fixing $\mathfrak{P}$. Thus $M \supset L^D$. ∎



terminology	e	f	g
inert	1	n	1
totally ramified	n	1	1
(totally) split	1	1	n

Table 3.1: Different prime behaviors

The next proposition uses the same notation as the above proof.

Proposition 3.8. *Let $\mathfrak{Q}$ be the prime of L^D below $\mathfrak{P}$. We have that*

$$f_{\mathfrak{Q}/\mathfrak{p}} = e_{\mathfrak{Q}/\mathfrak{p}} = 1.$$

If D is a normal subgroup of G, then $\mathfrak{p}$ is completely split in L^D.

Proof. We know that $[G : D] = g(\mathfrak{P}/\mathfrak{p})$ which is equal to $[L^D : K]$ by Galois theory. The previous proposition shows that $g(\mathfrak{P}/\mathfrak{Q}) = 1$ (recall that g counts how many primes are above). Now we compute that

$$\begin{aligned} e(\mathfrak{P}/\mathfrak{Q})f(\mathfrak{P}/\mathfrak{Q}) &= \frac{[L : L^D]}{g(\mathfrak{P}/\mathfrak{Q})} \\ &= [L : L^D] \\ &= \frac{[L : K]}{[L^D : K]}. \end{aligned}$$

Since we have that

$$[L : K] = e(\mathfrak{P}/\mathfrak{p})f(\mathfrak{P}/\mathfrak{p})g(\mathfrak{P}/\mathfrak{p})$$

and $[L^D : K] = g(\mathfrak{P}/\mathfrak{p})$, we further get

$$\begin{aligned} e(\mathfrak{P}/\mathfrak{Q})f(\mathfrak{P}/\mathfrak{Q}) &= \frac{e(\mathfrak{P}/\mathfrak{p})f(\mathfrak{P}/\mathfrak{p})g(\mathfrak{P}/\mathfrak{p})}{g(\mathfrak{P}/\mathfrak{p})} \\ &= e(\mathfrak{P}/\mathfrak{p})f(\mathfrak{P}/\mathfrak{p}) \\ &= e(\mathfrak{P}/\mathfrak{Q})f(\mathfrak{P}/\mathfrak{Q})e(\mathfrak{Q}/\mathfrak{p})f(\mathfrak{Q}/\mathfrak{p}) \end{aligned}$$

where the last equality comes from transitivity. Thus

$$e(\mathfrak{Q}/\mathfrak{p})f(\mathfrak{Q}/\mathfrak{p}) = 1$$

and $e(\mathfrak{Q}/\mathfrak{p}) = f(\mathfrak{Q}/\mathfrak{p}) = 1$ since they are positive integers.

If D is normal, we have that L^D/K is Galois. Thus

$$[L^D : K] = e(\mathfrak{Q}/\mathfrak{p})f(\mathfrak{Q}/\mathfrak{p})g(\mathfrak{Q}/\mathfrak{p}) = g(\mathfrak{Q}/\mathfrak{p})$$

and $\mathfrak{p}$ completely splits. □

Let σ be in D. Then σ induces an automorphism of $\mathcal{O}_L/\mathfrak{P}$ which fixes $\mathcal{O}_K/\mathfrak{p} = \mathbb{F}_\mathfrak{p}$. That is we get an element $\phi(\sigma) \in \mathrm{Gal}(\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p})$. We have thus constructed a map

$$\phi : D \to \mathrm{Gal}(\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}).$$

This is a group homomorphism. We know that $\mathrm{Gal}(\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p})$ is cyclic, generated by the Frobenius automorphism defined by

$$\mathrm{Frob}_\mathfrak{P}(x) = x^q,\ \ q = |\mathbb{F}_\mathfrak{p}|.$$

Definition 3.7. The inertia group $I = I_{\mathfrak{P}/\mathfrak{p}}$ is defined as being the kernel of ϕ.

Example 3.6. Let $K = \mathbb{Q}(i)$ and $\mathcal{O}_K = \mathbb{Z}[i]$. We have that $K/\mathbb{Q}$ is a Galois extension, with Galois group $G = \{1, \sigma\}$ where $\sigma : a + ib \mapsto a - ib$.

- We have that
$$(2) = (1 + i)^2\mathbb{Z}[i],$$
thus the ramification index is $e = 2$. Since $efg = n = 2$, we have that $f = g = 1$. The residue field is $\mathbb{Z}[i]/(1 + i)\mathbb{Z}[i] = \mathbb{F}_2$. The decomposition group D is G since $\sigma((1+i)\mathbb{Z}[i]) = (1+i)\mathbb{Z}[i]$. Since $f = 1$, $\mathrm{Gal}(\mathbb{F}_2/\mathbb{F}_2) = \{1\}$ and $\phi(\sigma) = 1$. Thus the kernel of ϕ is $D = G$ and the inertia group is $I = G$.

- We have that
$$(13) = (2 + 3i)(2 - 3i),$$
thus the ramification index is $e = 1$. Here $D = 1$ for $(2 \pm 3i)$ since $\sigma((2+3i)\mathbb{Z}[i]) = (2-3i)\mathbb{Z}[i] \neq (2+3i)\mathbb{Z}[i]$. We further have that $g = 2$, thus $efg = 2$ implies that $f = 1$, which as for 2 implies that the inertia group is $I = G$. We have that the residue field for $(2\pm3i)$ is $\mathbb{Z}[i]/(2\pm3i)\mathbb{Z}[i] = \mathbb{F}_{13}$.

- We have that $(7)\mathbb{Z}[i]$ is inert. Thus $D = G$ (the ideal belongs to the base field, which is fixed by the whole Galois group). Since $e = g = 1$, the inertial degree is $f = 2$, and the residue field is $\mathbb{Z}[i]/(7)\mathbb{Z}[i] = \mathbb{F}_{49}$. The Galois group $\mathrm{Gal}(\mathbb{F}_{49}/\mathbb{F}_7) = \{1, \tau\}$ with $\tau : x \mapsto x^7$, $x \in \mathbb{F}_{49}$. Thus the inertia group is $I = \{1\}$.

We can prove that ϕ is surjective and thus get the following *exact sequence*:

$$1 \to I \to D \to \mathrm{Gal}(\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}) \to 1.$$

The decomposition group is so named because it can be used to decompose the field extension L/K into a series of intermediate extensions each of which has a simple factorization behavior at $\mathfrak{p}$. If we denote by L^I the fixed field of I, then the above exact sequence corresponds under Galois theory to the following tower of fields:

$$\begin{array}{c} L \supset \mathfrak{P} \\ {\scriptstyle e}\big| \\ L^I \\ {\scriptstyle f}\big| \\ L^D \\ {\scriptstyle g}\big| \\ K \supset \mathfrak{p} \end{array}$$

Intuitively, this decomposition of the extension says that L^D/K contains all of the factorization of $\mathfrak{p}$ into distinct primes, while the extension L^I/L^D is the source of all the inertial degree in $\mathfrak{P}$ over $\mathfrak{p}$. Finally, the extension L/L^I is responsible for all of the ramification that occurs over $\mathfrak{p}$.

Note that the map ϕ plays a special role for further theories, including reciprocity laws and class field theory.

The main definitions and results of this chapter are

- Definition of discriminant, and that a prime ramifies if and only if it divides the discriminant.
- Definition of signature.
- The terminology relative to ramification: prime above/below, inertial degree, ramification index, residue field, ramified, inert, totally ramified, split.
- The method to compute the factorization if $\mathcal{O}_K = \mathbb{Z}[\theta]$.
- The formula $[L:K] = \sum_{i=1}^{g} e_i f_i$.
- The notion of absolute and relative extensions.
- If L/K is Galois, that the Galois group acts transitively on the primes above a given $\mathfrak{p}$, that $[L:K] = efg$, and the concepts of decomposition group and inertia group.