LASSP Pizza Talk

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Cellular Automata & Pattern Formation

Siew-Ann Cheong

Prologue

- Not an advertisement for Wolfram. . .
- Not a Wolfite. . .
- Rather. . .
  - P681 Pattern Formation and Spatio-Temporal Chaos/Prof Eberhard Bodenschatz.
  - End-of-course poster. . .
  - . . . and adventures beyond.

Cellular Automata

- A collection of finite state machines. The state of the i^{th} machine at time t given by $s_i(t) \in \mathcal{A}$, where $\mathcal{A}$ is a finite set, also called the *alphabet*;
- A collection of neighborhoods. The neighborhood of the i^{th} machine is denoted by $\mathcal{N}_i$;
- A dynamical rule $\varphi : \mathcal{N}_i \longrightarrow \mathcal{A}$, such that $s_i(t+1) = \varphi(s_j(t) \mid j \in \mathcal{N}_i)$.

Classification of CAs

- Elementary and compound CAs. Examples are Game of Life (GOL) and the Nagel-Schreckenberg model of traffic flow respectively.
- Wolfram classified all 256 1-D elementary CAs (ECAs) by their dynamical properties. Types I, II and III.
- Wolfram naming convention: if the ECA is

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \end{array}$$

  then Wolfram rule number is $\sum_{j=0}^{7} \alpha_j 2^j$.
- No known attempts at classifying ECAs of higher dimensions.

From Pattern to ECA

- In P681, given PDE model, find what patterns form spontaneously. Can do the same for CA models.
- Ask the inverse question instead: given a pattern, what are all the possible CAs that spontaneously generate it?
- Two parts to this question:
  - what CA rules will have given pattern as fixed point; and
  - under which CA rules is the pattern stable?

Stripped Phase in 1-D

- Consider stripped phase in 1-D:

○●○●○●○●○●○●○●○

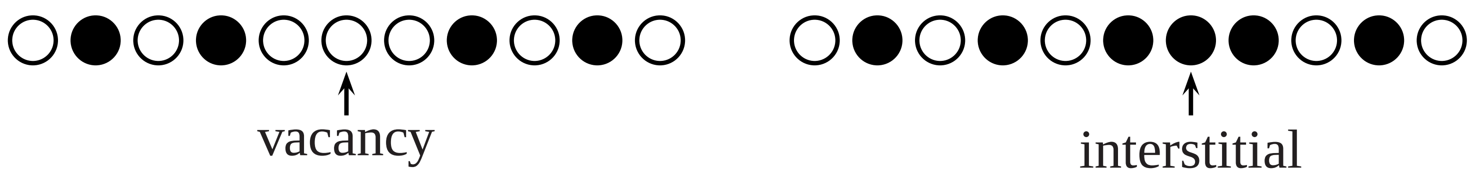
- Fixed point requirement implies the transition rules

●○● $\rightarrow$ ×○× and ○●○ $\rightarrow$ ×●×.

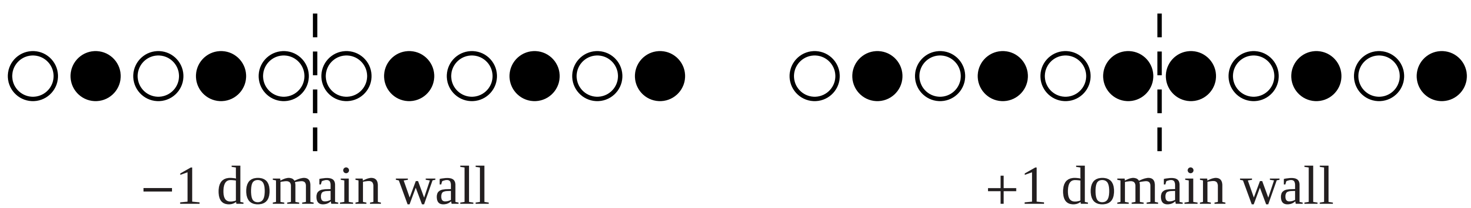
- Does not uniquely determine ECA rule, 6 more transition rules to specify.

Defects in Stripped Phase

- To analyze stability of stripped phase, need to investigate behaviour of departures from pattern, i.e. defects, under various ECA rules.
- Point defects:



- Domain walls:



Strips Stable in Presence of Point Defects

- Since ECA not completely specified, can choose remaining transition rules to stabilize stripped phase in presence of point defects.
- Demand that isolated vacancy 'heals': implies transition rules

$$\bigcirc\bigcirc\bigcirc \to \times\bullet\times,\ \ \bullet\bigcirc\bigcirc \to \times\bigcirc\times,\ \ \bigcirc\bigcirc\bullet \to \times\bigcirc\times.$$

- Demand that isolated interstitial 'heals': implies transition rules

$$\bullet\bullet\bullet \to \times\bigcirc\times,\ \ \bigcirc\bullet\bullet \to \times\bullet\times,\ \ \bullet\bullet\bigcirc \to \times\bullet\times.$$

- ECA completely specified by requirements that: (a) stripped phase is fixed point; (b) isolated vacancies 'heal'; and (c) isolated interstitials 'heal'.

Completed ECA Rule

ECA is Rule 77 in Wolfram's classification scheme:

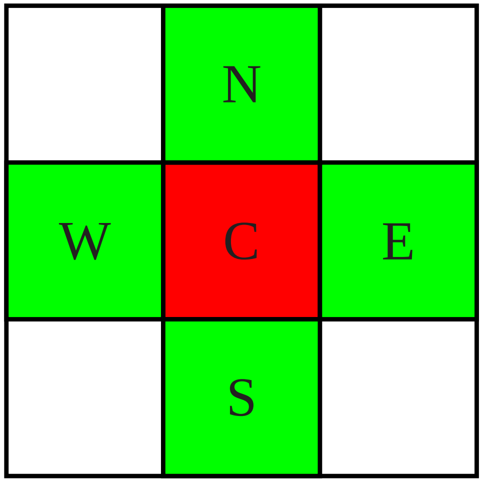
$s_{j-1}(t)$	$s_j(t)$	$s_{j+1}(t)$	$s_j(t+1)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

Further Considerations

- **Domain Wall Dynamics.** Both ±1 domain walls stationary under Rule 77, i.e. if start from random initial configuration, all domain walls initially present will be 'frozen in'.
- **Robustness of Stripped Phase.** By modifying some transition rules in Rule 77, can test genericity of stripped pattern. Found that:
  - Stripped phase *most* stable under Rule 77, but also stable under 6 other ECA rules derived from Rule 77, in which a single transition rule is modified.
  - Stripped phase *marginally* stable under 4 ECA rules derived from Rule 77, in which one or two transition rules are modified.
  - Stripped phase unstable once more than two transition rules are modified from Rule 77. Oscillatory phase nucleates.

2-D ECAs

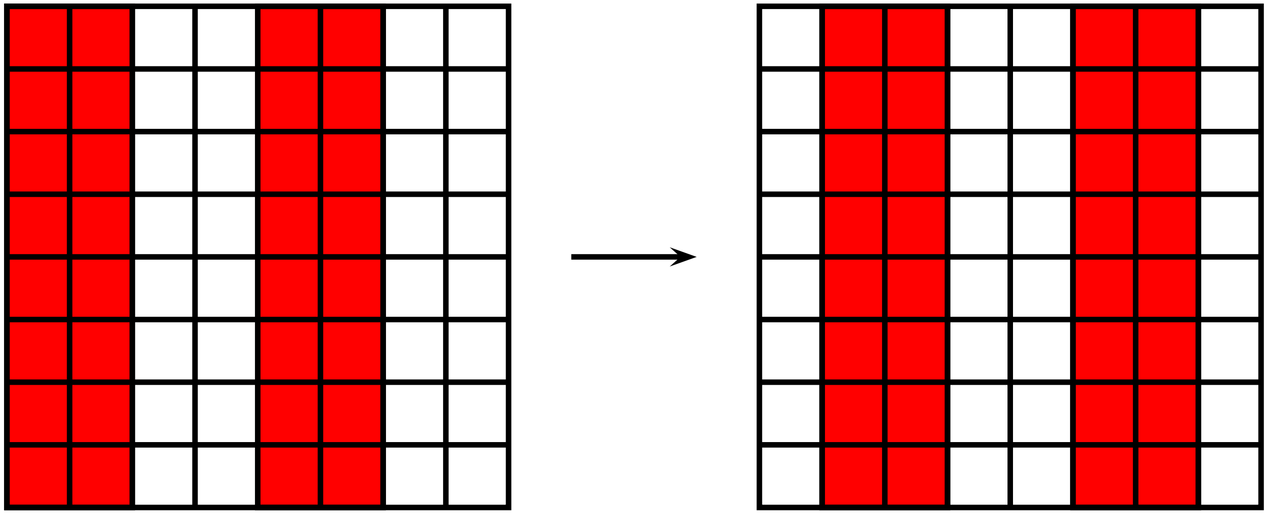
- In 1-D, neighborhood simple, unless one wants to go to next nearest neighbor.
- In 2-D, greater variety of neighborhoods. Simplest neighborhood for 2-D CA is von Neumann (VN) neighborhood:



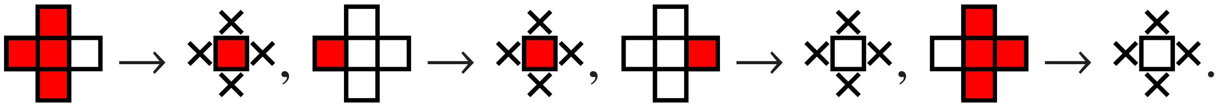
- With VN neighborhood, total of $2^5 = 32$ possible local configurations $\Longrightarrow$ total of $2^{32} = 4{,}294{,}967{,}296$ 2-D ECAs.

λ = 4, v = +1 Traveling Wave Phase

- A traveling wave phase with $\lambda = 4$ and $v = +1$ looks like

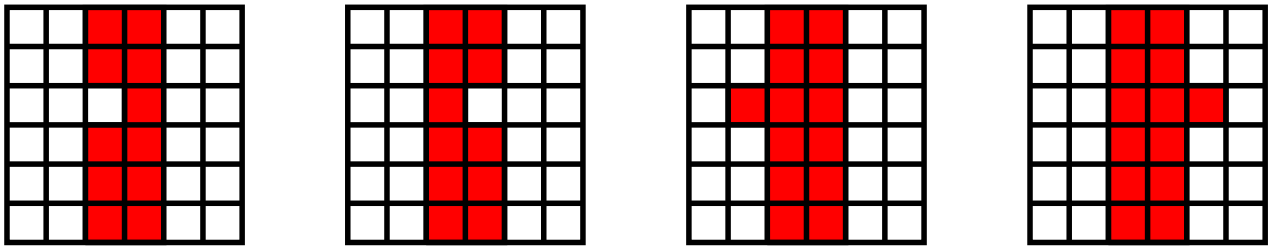


- The traveling wave transition rules are



Multiple Defect Analysis & Transition Rule Conflict

- Unlike in 1-D, point defect analysis alone cannot fully specify ECA. Need to do multiple defect analysis.
- Four types of point defect:



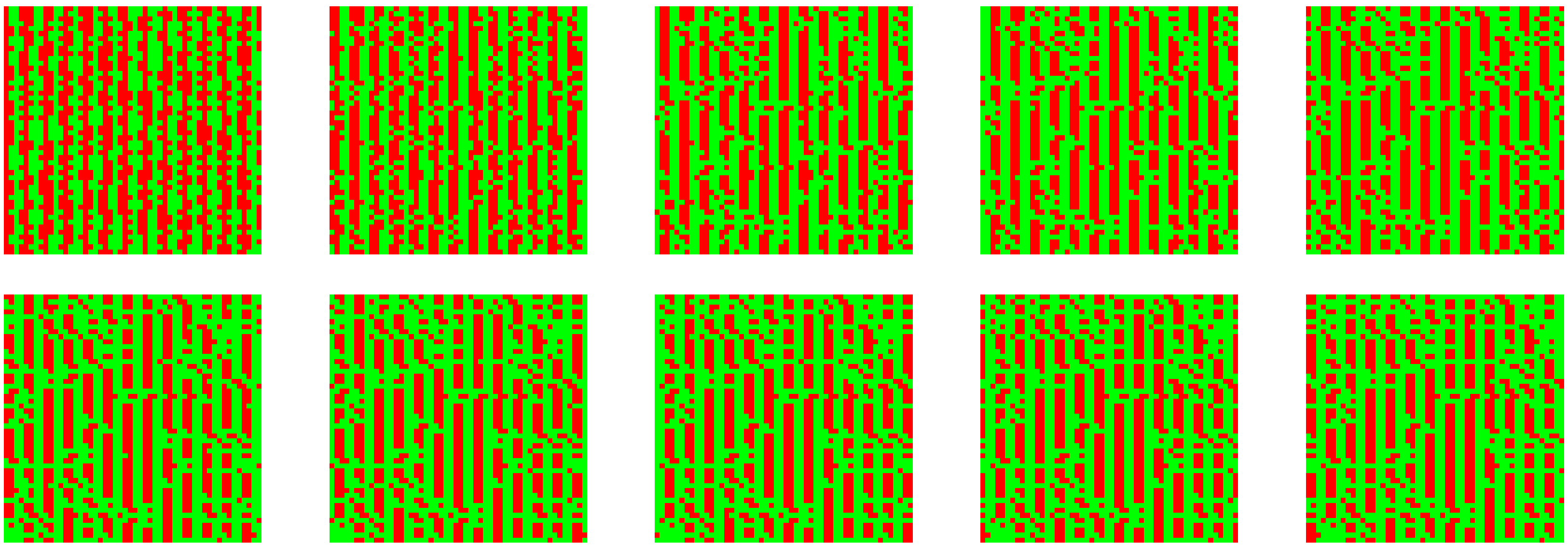
V_L V_R I_L I_R

- In this chosen pattern, transition rules implied by V_L conflicts with that implied by V_R, and transition rules implied by I_L conflicts with that implied by I_R.
- Generic problem.

Protocol for Conflict Resolution

- When transition rule implied by two configurations in conflict, give precedence to configuration with lower number of defects.
- When transition rule implied by leading edge configuration conflicts with that implied by trailing edge configuration, give precedence to trailing edge configuration.
- Can show that some multi-defect configurations whose implied transition rules are forfeited will still be ‘healed’.
- Compromise necessary because traveling wave breaks left-right symmetry.
- Completed CA rule is Rule 2,383,284,874.

Simulating Rule 2,383,284,874



Compound CAs

- Some patterns cannot be achieved using ECAs because conflict resolution protocol used cannot ensure stability of desired pattern.
- What to do?
  - Use larger neighborhoods — equivalent to a restricted class of compound ECAs.
  - Use larger state space, say $s_i(t) = 0, \frac{1}{2}, 1$.
- The main idea is to increase the number of transition rules available for pattern matching.
- Another way is to compound together ECAs.

How to Compound ECAs

- Enumerate all defect configurations that can be ‘healed’ in a few time steps.
- For each defect configuration, find the ECA that ‘heals’, while acting as identity map on other configurations, other than the desired patterned configurations.

λ = 4, v = +1 Traveling Wave Phase in 1-D

config			$V_L + I_L$	$V_L + I_R$	$V_R + I_L$	$V_R + I_R$
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	0	0	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	0
1	0	1	0	1	0	1
1	1	0	0	0	0	0
1	1	1	1	0	1	0
rule			139	43	142	46

Does It Work?

- Rules 43 and 142 by themselves most readily generate the desired pattern for initial density $\rho = \frac{1}{2}$. Not good away from half-filling.
- Rules 46 and 139 less readily generate desired pattern.
- Compounding 46 + 139 or 43 + 142 does not make desired pattern any more stable.
- Reason: competing fixed points. Back to square one — need to find fixed points or limit cycles of given ECA.

References

1. B. Chopard and M. Droz, *Cellular Automata Modeling of Physical Systems*, Cambridge University Press, 1998

2. R.J. Gaylord and K. Nishidate, *Modeling Nature: Cellular Automata Simulations with Mathematica*®, Springer-Verlag, 1996