Classical and Quantum Entropy

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Equilibrium Entropy: A Static Picture

In dimensionless form, with $k_B = 1$,

$$\text{Microcanonical:} \quad S = \log \Omega \tag{1}$$

$$\text{Canonical:} \quad S = -\log Z - \beta \langle E \rangle = \langle \log p_i \rangle = \sum_i p_i \log p_i \tag{2}$$

where $\beta = T^{-1}$, Z is the canonical partition function and $\langle E \rangle$ the expectation value of the internal energy E.

In study of equilibrium at $t \to \infty$, dynamical details deliberately discarded. How to incorporate dynamical details into a "dynamical entropy"?

Notes on Ensemble Picture

1. No equilibrium for each member of Gibbs ensemble, for example, ideal gas in cube of edge L.

2. Microscopic and macroscopic properties of each member fluctuates from time to time.

3. Macroscopic measurements = "coarse-graining" in time, i.e. time averaging.

4. Gibbs ensemble "equates" time average to phase space average, provided system ergodic.

5. Collisionless gas in a cube not ergodic. Need binary or higher-order collisions.

Motivating Dynamical Entropy

KEY: Entropy = Information Loss

Consider time evolution of (micro)state from $t = 0$ to $t = T$,

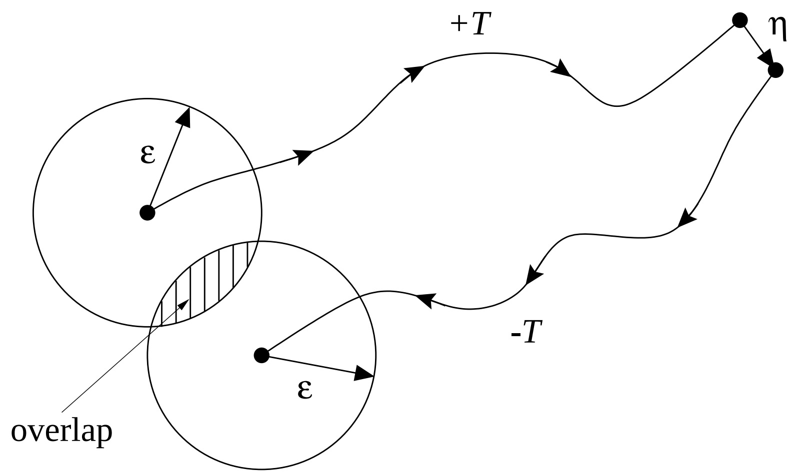
$$\begin{array}{ccc} \{x_i, v_i\} & \rightleftarrows & \{x_f, v_f\} \\ t = 0 & & t = T \end{array} \tag{3}$$

If dynamics time-reversal invariant, then $\{x_f, v_f\} \to \{x_i, v_i\}$ possible if **ALL** information known about final state.

What if we don't? If mistake made in determining final state, how close can we get back to initial state?

The Home-Coming . . .

Attach ϵ-ball to $\{x_i, v_i\}$, evolve it forward in time by $\Delta t = T$, make a small mistake ϵ', reverse time and evolve back:



$$\text{overlap\%} \sim \text{overlap}/\epsilon^D \tag{4}$$

where D is dimension of phase space, ignoring factors of $O(1)$.

Entropy & Information

Intuitively expect that overlap% $\downarrow$ as $T \uparrow$. Thus should seek meaningful quantity in the limit

$$\lim_{\epsilon,\eta \to 0} \lim_{T \to \infty} \frac{\text{overlap\%}}{T} = \text{rate of information loss} \tag{5}$$

Truly dynamical because time evolution taken into account.

Also, $0 \leq \text{overlap\%} \leq 1$ interpret as some sort of probability p_i, and deduced the *dynamical entropy* as

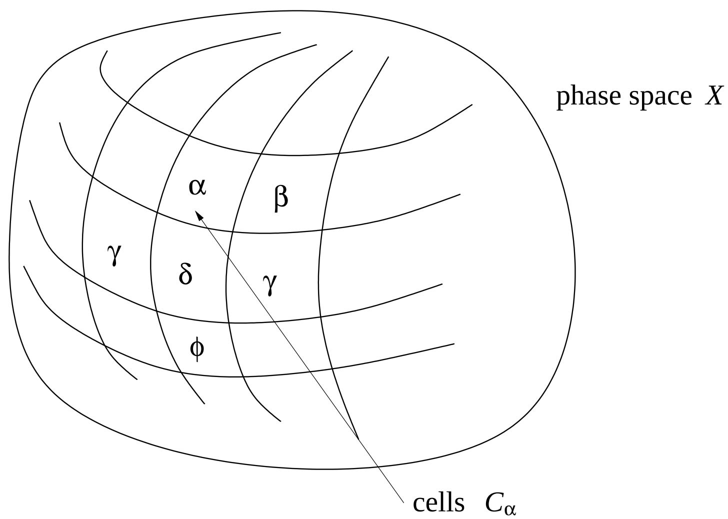
$$H = -\sum_i p_i \log p_i \leq S \tag{6}$$

i.e. condition of equilibrium is condition of maximization of H or state of maximum loss of information about system.

Road To Classical Dynamical Entropy

In totality, must consider entire phase space.

But {phase space($t = 0$)} $\equiv$ {phase space($t = T$)}, must find way of introducing dynamics $\Longrightarrow$ concept of a *partition*.



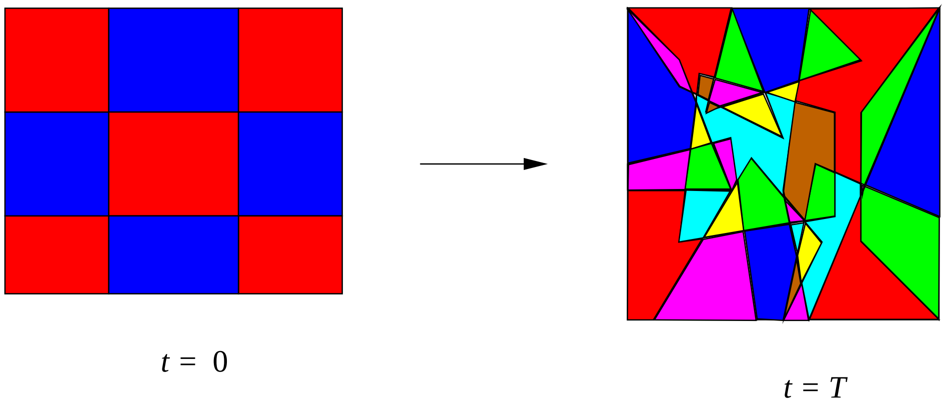
partition $\xi = \{$countable # of subsets C_α of $X \mid \cup_\alpha C_\alpha = X, C_\alpha \cap C_\beta = \emptyset$ if $\alpha \neq \beta\}$ up to sets of measure zero with respect to some measure μ, e.g. phase space volume.

Dynamical Refinement of a Partition

Can define

$$H = \sum_{\alpha} \mu(C_\alpha) \log \mu(C_\alpha) \tag{7}$$

but not dynamical. Need concept of refinement of a partition.



This refinement can be brought about by a (measure-preserving) dynamical map: $\varphi^T : X(t=0) \to X(t=T)$, and denote by ξ^{φ_T} the dynamically refined partition of ξ.

The Kolmogorov-Sinai Metric Entropy

An important proposition:

$$h_\mu(\varphi, \xi) = \lim_{T\to\infty} \frac{H(\xi^{\varphi_T})}{T} = \lim_{T\to\infty} \sum_\alpha \frac{\mu(C_\alpha)\log \mu(C_\alpha)}{T}, \quad C_\alpha \in \xi^{\varphi_T} \tag{8}$$

exists for all measurable partition.

The *Kolmogorov-Sinai metric entropy* of φ with respect to μ is

$$h_\mu(\varphi) = \sup\{h_\mu(\varphi, \xi) \mid \text{all } \xi \text{ such that } H(\xi) < \infty\} \tag{9}$$

which is a topological quantity.

Can be recasted as an algebraic entropy (A. Connes and E. Størmer, Acta Math. **134**, 289–306 (1975); A. Connes, H. Narnhoffer and W. Thirring, Commun. Math. Phys. **112**, 691 (1987)), important when we consider quantum mechanics.

Quantum Static Entropy

Shannon-Rényi Entropy: $H(\psi) = \sum_i \left|\langle i\,|\psi|\,i\rangle\right|^2 \log\left|\langle i\,|\psi|\,i\rangle\right|^2$

von Neumann Entropy: $S(\varrho) = -\,\mathrm{tr}\,\varrho \log \varrho$

Quantum Relative Entropy: $S(\psi, \chi) = \mathrm{tr}\,\varrho_\psi\,(\log \varrho_\psi - \log \varrho_\chi)$

Effective State Entropy: $S(\varrho) = -\int dE\ \mathrm{tr}\,\widetilde{\varrho}_E \log \widetilde{\varrho}_E$

Problems: von Neumann entropy zero for pure states, infinite for mixed states, and the conditional von Neumann entropy *negative* for entangled states (N.J. Cerf and C. Adami, in *New Developments on Fundamental Problems in Quantum Physics* edited by M. Ferrero and A. van der Merwe, Kluwer, 1997).

Quantum Dynamical Entropy

1. Connes-Størmer-Narnhoffer-Thirring entropy ✓

2. Alicki-Fannes entropy

3. Coherent State entropy ✓

Connes-Størmer-Narnhoffer-Thirring Algebraic Entropy: The Essentials

1. C$*$-algebra $L^\infty(M)$ of infinitely integrable complex-valued functions on compact phase space M with Borel probability measure μ, equipped with faithful normal trace

$$\tau(f) = \int_M f\, d\mu < \infty, \qquad f \in L^\infty(M) \tag{10}$$

   take the place of minimal σ-algebra on M.

2. Finite-dimensional subalgebras $\mathcal{N}$ generated by complete set of minimal projection operators $\{p_i\}$ such that $p_i \cdot p_j = \delta_{ij}$ takes the place of partition ξ.

3. Define

$$H(\mathcal{N}) = \sum_i \tau\left(p_i \log p_i\right) = H(\xi) \tag{11}$$

4. μ-preserving dynamical map φ on M induces τ-preserving dynamical map Φ on $L^\infty(M)$. Use Φ to generate dynamical refinement.

5. Define Connes-Størmer-Narnhoffer-Thirring entropy as

$$\begin{aligned} h(\mathcal{N}) &= \sup_\Phi h(\mathcal{N}, \Phi) \\ &= \sup_\Phi \lim_{n\to\infty} n^{-1} H(\mathcal{N} \vee \Phi(\mathcal{N}) \vee \cdots \vee \Phi^{n-1}(\mathcal{N})) \end{aligned} \tag{12}$$

6. C$*$-algebra $L^\infty(M)$ commutative, can be shown to equal Kolmogorov-Sinai metric entropy.

Connes-Størmer-Narnhoffer-Thirring Quantum Dynamical Entropy

von Neumann algebra of projection operators used in quantum mechanics is C∗-algebra, i.e. basic recipe same as classical mechanics.

In defining procedure of Connes-Størmer-Narnhoffer-Thirring entropy, as in Kolmogorov-Sinai entropy, partition must remain finite. However, quantum algebra not commutative $\Longrightarrow$ dynamical refinement of $\mathcal{N}$ problematic. Quantum dynamically refined partition infinite (R. Alicki and M. Fannes, Lett. Math. Phys. **32**, 75 (1994)).

Trick: Perform abelianized refinement of $\mathcal{N}$, i.e. redundant physical information of incompatible observables removed.

Coherent State Entropy

Follows same cookbook recipe as Kolmogorov-Sinai and Connes-Størmer-Narnhoffer-Thirring entropies, but measure-theoretic rather than algebraic.

Define measures on quantum-mechanical phase space whose density functions are Husimi functions, defined as

$$\Phi(q,p,t) = \frac{1}{\pi\hbar}\int dQ\,dP\,e^{-\left(\frac{(q-Q)^2}{\hbar w^2}+w^2\frac{(p-P)^2}{\hbar}\right)}\Psi(Q,P,t) \qquad (13)$$

where

$$\Psi(q,p,t) = \frac{1}{(2\pi\hbar)^N}\int dQ\,\psi(q-Q,t)\psi^*(q+Q,t)e^{-2ipQ/\hbar} \qquad (14)$$

are the Wigner functions.

Coherent States and Measurement

Coherent states are *a posteriori* states associated with measurement, i.e. states we *want* to see.

Second Postulate of QM – after exact measurement, state collapses to one of eigenspaces of observable, but in experiment with uncertainty w, state collapses to group of eigenspaces centered around dominant eigenspace $\implies$ wavepackets of width w.

Gaussians are instantaneously minimum uncertainty wavepackets, i.e. what an experimenter want to see.

Only such states projected out of quantum state $\psi(q, t)$ to incorporate into Husimi functions.