Wireless Networks: New Models and Results

Radhika Gowaikar
California Institute of Technology

Nanyang Technological University, Singapore
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A Quick History

- **Pre-1800s** – fires, drums, mirrors, pigeons
- **1800s** – telephone and telegraph cables
- **1900s** – wireless communication
- **Future** – multimedia over wireless
Outline

• Introduction
  – Wireless connection
  – Wireless networks

• Networks with Random Connections
  – Outline, Model, Result

• Wireless Erasure Networks
  – Outline, Model, Result

• Further directions
The Wireless Connection

Shared medium, Broadcast, Interference, Randomness due to fading
Advantages of the Wireless Connection

• Randomness leads to diversity
• Less investment in infrastructure
• Mobile systems can be connected temporarily
• Can operate very different systems on the same platform
Wireless Networks

- **Types and functions**
  - Ad hoc, sensor, cellular, Local Area (LAN), Wide Area (WAN), hybrid

- **Models**
  - Geometric models, fading models, combinations

- **Performance measures**
  - Capacity, Throughput, Rate-Distortion, Delay, Robustness

- **Constraints**
  - Decentralized operation, real-time operation

General multi-terminal network problem is very much open.
Simplifications and Techniques

- **Simplifications**
  - Simple models – wireline, interference-free
  - Asymptotic regimes of parameters – low power, large networks
  - Scaling laws rather than exact analysis

- **Techniques**
  - Routing, treat information like “flow”
  - Co-operation, relaying
  - Combine and code different data streams – network coding
  - Game theoretic, random graphs
In this talk: Two network problems

- Network with Random Connections
  - Ad hoc network with fading connections
  - Throughput scaling laws
  - Relaying information
  - Random graphs

- Wireless Erasure Network
  - Erasure links
  - Capacity region
  - Network coding
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Achievable Throughputs for Random Networks


Outline for this section

• Problem Statement
  – Successful Communication, Multihop Protocol

• Existing Models and Results
  – Kumar-Gupta Model – Scaling Law of $O(\sqrt{n})$

• New network model – Random connections
  – Random Model – Scaling Law of upto $O(n/\log^c n)$

• Operation and analysis
  – Scheduling, Ensuring Error-Free Communication

• Main result

• Examples and Simulations

• More general models

• Remarks and Summary
• $n$ nodes, channel $h_{i,j}$ between nodes $i$ and $j$, $k$ source-destination pairs

• Nodes can relay messages

• Communication using (multiple) hops
A node can decode only if the signal-to-interference-plus-noise-ratio (SINR) exceeds some threshold $\rho_0$. Node 5 can decode only if

$$\frac{P|h_{1,5}|^2}{\sigma^2 + P(|h_{2,5}|^2 + |h_{3,5}|^2 + |h_{4,5}|^2)} = \frac{P\gamma_{1,5}}{\sigma^2 + P(\gamma_{2,5} + \gamma_{3,5} + \gamma_{4,5})} \geq \rho_0.$$
Scheduling Multiple Hops

- Relay nodes decode and retransmit message
- Nodes cannot transmit and receive simultaneously
  - Therefore non-colliding schedule required
- \( h \) hops are used for each communication
• SINR condition must be fulfilled at every relay node.

• Want to **maximize throughput**

\[
T = (1 - \epsilon) \frac{\text{Number of source-destination pairs}}{\text{Number of hops}} \log(1 + \rho_0)
\]

\[
T = (1 - \epsilon) \frac{k}{h} \log(1 + \rho_0).
\]
Kumar and Gupta (2000) used a decay law.
\[ \gamma_{i,j} \propto \frac{1}{(\text{distance}_{i,j})^m}, m > 2 \] gave Throughput \( \propto \sqrt{n} \).

- Simultaneous transmission can take place between \( O(n) \) nodes and their nearest neighbors.
- Two arbitrary nodes are \( O(\sqrt{n}) \) hops apart.
Other Results

Discouraging since per-user throughput goes down as $\frac{1}{\sqrt{n}}$

- Several other results reinforce this e.g. Léveque and Telatar, Dousse and Thiran, Franceschetti et al.

- Grossglauser and Tse get a constant per-user throughput, but with mobile nodes

- **Randomness** provides a ray of hope

- Recent work by Miorandi and Altman, Balister et al., Franceschetti et al, Hekmat and van Mieghem shows that randomness introduces properties that are missing in a purely geometric model

Therefore we propose an entirely random model.
New Model – Random Links

- Strength of connection does not depend on distance
- All connection strengths are identically and independently distributed according to $f_n(\gamma)$

- Model is justified over small area with many obstructions
- Line-of-sight is absent, multipath dominates
Concept of “Good” Edges

- Introduce a parameter $\beta_n$. Let $P(\gamma \geq \beta_n) = Q_n(\beta_n) = p_n$.
- Edges with $\gamma \geq \beta_n$ are “good.”
- Obtain $G(n, p)$ by keeping only these and eliminating the rest.
  - $G(n, p)$ is a well-studied random graph model.
  - e.g. diameter is known to be $\frac{\log n}{\log np}$.

Now find schedule of non-colliding paths in this graph.
Theorem 1 (Broder et al. 1996) Suppose that $G = G(n, p)$ and $p \geq \frac{\log n + \omega n}{n}$, where $\omega n \rightarrow \infty$. Then, provided $k$ is not greater than $\alpha n \frac{\log np}{\log n}$, there are vertex-disjoint paths connecting $s_i$ to $d_i$ for any set of $k$ randomly chosen source-destination pairs.

- This is guaranteed with probability going to 1 for a positive constant $\alpha$.
- We can thus establish upto $k = \alpha n \frac{\log np}{\log n}$ non-colliding (in fact, vertex-disjoint) paths. Theorem is tight upto constant.
- Can show that every message makes around $h = \frac{\log n}{\alpha \log np}$ hops.
- Now need to ensure that the probability of error can be made to go to zero.
Probability of Error

\[ P(\text{Message } i \text{ fails}) = P(\bigcup_{j=1}^{h} \text{SINR at hop } j \leq \rho_0) \]
\[ \leq \# \text{ hops} \times P(\text{SINR at hop } 1 \leq \rho_0) \]

\[
\epsilon_n = P(\text{Message } i \text{ fails}) \leq \frac{\log n}{\alpha \log np} \frac{\sigma^2_{\gamma}/(k-1)}{(P\beta_n - \rho_0\sigma^2_{\gamma}/(k-1)P\rho_0 - \mu_{\gamma}))^2}.
\]

Can use Chebyshev bound if \( \rho_0 \leq \frac{P\beta_n}{\sigma^2 + P(k-1)\mu_{\gamma}}. \)

Put Scheduling and Probability of Error results together to get Main Result.
Main Result

Theorem 2 Let $Q_n(x) = P(\gamma \geq x)$. Choose any $\beta_n$ such that $Q_n(\beta_n) = \frac{\log n + \omega_n}{n}$, where $\omega_n \to \infty$. Let $\mu_\gamma = E\gamma$, $\sigma^2_\gamma = E(\gamma - \mu_\gamma)^2$. Then there exists a positive constant $\alpha$ such that a throughput of

$$T = (1 - \epsilon_n) k_n(\beta_n) \alpha \frac{\log(nQ_n(\beta_n))}{\log n} \log \left(1 + \frac{a_n \beta_n}{\frac{\sigma^2_P}{P} + (k_n(\beta_n) - 1)\mu_\gamma} \right)$$

is achievable for any positive $a_n$ such that $a_n \leq 1$ and any $k_n(\beta_n)$ that satisfy the conditions:

1. $k_n(\beta_n) \leq \alpha n \frac{\log(nQ_n(\beta_n))}{\log n}$

2. $\epsilon_n \leq \frac{a_n^2}{\alpha(1 - a_n)^2} \frac{(k_n(\beta_n) - 1)\sigma^2_\gamma}{\left(\frac{\sigma^2_P}{P} + (k_n(\beta_n) - 1)\mu_\gamma\right)^2} \frac{\log n}{\log(nQ_n(\beta_n))} \to 0$
### Heavy Dependence on $f_n(\gamma)$

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$f_n(\gamma)$</th>
<th>Throughput</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shadow</td>
<td>$(1 - p_n)\delta(\gamma) + p_n\delta(\gamma - 1)$</td>
<td>$\frac{1}{w_n} \frac{\log^2(\log n)}{\log^3 n} n$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$e^{-\gamma}$</td>
<td>$\log n$</td>
</tr>
<tr>
<td>Decay</td>
<td>$\frac{4\pi \Delta}{nm} \frac{1}{\gamma^{1+\frac{2}{m}}}, m &gt; 2$</td>
<td>$\frac{1}{w_n} \frac{\log^2(\log n)}{(\log n)^{2+m/2}} n$</td>
</tr>
<tr>
<td>Heavy Tail</td>
<td>$\frac{c}{1+\gamma^4}$</td>
<td>$\frac{\log \log n}{\log^{4/3} n} n^{1/3}$</td>
</tr>
</tbody>
</table>
\[ f_n(\gamma) = (1 - p_n)\delta(\gamma) + p_n\delta(\gamma - 1). \] Natural choice of \( \beta_n = 1. \)

• Gives \( T \) in terms of \( p_n \). For fixed \( n \), can find optimum

\[ p_n = \frac{\log n + \omega_n}{n}. \]

1,000 nodes. Simulations \( p^* \approx 0.008 \). Theory \( \frac{\log(1000)}{1000} \approx 0.0069 \).
Shadow Fading Distribution – Scaling Law

- For $p_n = \frac{\log n + \omega_n}{n}$, $h = \frac{\log n}{\log(\log n + \omega_n)}$, $T = \frac{1}{w_n} \frac{\log^2(\log n)}{\log^3 n} n$

$p = \frac{2\log n}{n}$, 100 – 1200 nodes.
Density obtained from Decay Law

Consider a single node transmitting in a network with a distance-based decay law, say \( \frac{1}{(\text{distance})^m} \). The marginal distribution of received powers can be shown to be

\[
f_n(\gamma) = \frac{4\pi\Delta}{nm} \frac{1}{\gamma^{1+\frac{2}{m}}} \gamma \in \left[ \left( \frac{2\pi\Delta}{n + 2\pi\Delta d^2} \right)^{m/2}, \frac{1}{d^m} \right], m > 0.
\]

Achievable throughputs scale like

\[
T = \begin{cases} 
\frac{\log(\log n + \omega_n)}{\log n(\log n + \omega_n)^{m/2}} n^{m/2} & m < 2 \\
\frac{\log(\log n + \omega_n)}{\log^2 n(\log n + \omega_n)} n & m = 2 \\
\frac{1}{w_n} \frac{\log^2(\log n + \omega_n)}{\log^2 n(\log n + \omega_n)^{m/2}} n & m > 2.
\end{cases}
\]

Significantly better than \( O(\sqrt{n}) \) predicted by Kumar-Gupta (2000) for \( m > 2 \).
Density obtained from Decay Law

\[ f_n(\gamma) = \frac{4\pi\Delta}{nm} \frac{1}{\gamma^{1+\frac{2}{m}}} \]

Optimum choice of \( \beta_n \) is \( \frac{2\pi\Delta}{(\log n + \omega_n)^{m/2}} \)

- \( p_n = \frac{\log n + \omega_n}{n} \), \( h = \frac{\log n}{\log(n + \omega_n)} \), \( T = \frac{1}{\omega_n} \frac{\log^2(n + \omega_n)}{\log^2 n (\log(n + \omega_n)^{3/2}} \ n \)

\[ m = 3, \ d = 1, \ \Delta = 1. \] Throughput increases almost linearly.
More general models

The random model does not incorporate distance effects.

- **Globally**: channel strength decays with distance
- **Locally**: channel strengths are drawn i.i.d. from a distribution

\[
p_x(\gamma) = \begin{cases} 
  f(\gamma) & \text{if } x \leq r \\
  \frac{\mu \gamma r^m}{x^m} \exp\left(-\gamma \frac{x^m}{\mu \gamma r^m}\right) & \text{if } x > r
\end{cases}
\]
Throughput for the Two-Scale Model

Example:

Decay law: $\frac{1}{\text{dist}^m}$  
Distribution: $\frac{1}{(1+\gamma)^t}, t > 2$

Throughput: $\frac{1}{\log^2 n} n^{\frac{1}{t-1}}$
Multiscale and mixture models

• Three-scale model – can use ideas similar to the two-scale model

\[ p_x(\gamma) = \begin{cases} 
  f(\gamma) & \text{if } x \leq r_1 \\
  \frac{r_2-x}{r_2-r_1} f(\gamma) + \frac{x-r_1}{r_2-r_1} \frac{\mu \gamma r_2^m}{x^m} \exp(-\gamma \frac{x^m}{\mu \gamma r_2^m}) & \text{if } r_1 < x \leq r_2 \\
  \frac{\mu \gamma r_2^m}{x^m} \exp(-\gamma \frac{x^m}{\mu \gamma r_2^m}) & \text{if } x > r_2
\end{cases} \]

• Mixture model – need to develop new approach

\[ p_x(\gamma) = \frac{R-x}{R} f(\gamma) + \frac{x}{R} \frac{1}{x^m} \exp(-\gamma x^m) \]
Remarks and future work

- Developed decentralized implementation schemes

- Upperbounds on the achievable throughput using multihop protocol
  - Information-theoretic upperbounds?

- How do adhoc networks compare with other (e.g. cellular) networks?
  - Spectral efficiency, power efficiency, operation

Summary

Incorporating randomness gives significantly more encouraging scaling laws than distance-based model.
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Outline for this section

• The Multicast Problem and known results
• New network model and operation
• Cuts and Cut Capacities
• Main Result – Capacity Result for Erasure Wireless Networks
• Sketch of the proof
  – Achievability and Converse
• Remarks and practical implementation
• Future work and Summary
The Multicast Problem

- Multiple sources with information
- Multiple destinations requesting subsets of information
- Each link is a channel
- Can all requests be satisfied? What is the capacity region?
- What strategies get you to capacity? What delays are expected?
Review of Existing Results

- General multi-terminal networks are unsolved though min-cut outer bounds are well known.

- For a wireline network, having a single sender and a single receiver, a max-flow min-cut capacity result exists.

- In addition, we now know how to achieve these min-cut bounds in some special cases of the multicast problem. (Ahlswede, Cai, Li, Yeung ’00; Koetter, Médard ’03)
Missing features

The wireline model does not incorporate broadcast at transmitting nodes and interference at nodes with multiple access – both of these are part of many practical systems.

Need to come up with a model that incorporates these features but is still tractable.
A simple yet realistic model

- Assume that all the links are erasure channels
  - Valid in systems with ARQ-like mechanisms

- **Broadcast** – same message on all outgoing links

- **Interference** – erasures patterns may be correlated
  - Errors on one link cause errors on another
Examples – Some simple networks

1. $\epsilon_1$

$\epsilon_2$

$s \rightarrow d$

$C = 1 - \epsilon_1 + 1 - \epsilon_2$

$C = 1 - \epsilon_1 \epsilon_2$

2. $\epsilon_{1,2}$

$\epsilon_{2,3}$

$s \rightarrow d$

$v_2$

$C = \min\{1 - \epsilon_{1,2} + 1 - \epsilon_{1,3}, 1 - \epsilon_{2,3} + 1 - \epsilon_{1,3}\}$

$C = ?$
Model for Erasure Wireless Network

- $G = (V, E)$ directed, acyclic graph
- Edges are memoryless, erasure channels with instantaneous transmission
- **Broadcast** at transmission

source $s = v_1$, destination $d = v_{|V|}$
transmitted signals $X_i$, received signals $Y_i$
Capacity of the network: Preliminaries

- Communication in blocks of $n$ packets
- Node $v_i$ uses encoding function $f_i$ on what it receives to determine what it transmits
- **Capacity** – maximum rate for which average probability of error goes to zero
\textbf{\textit{s} – \textit{d} cut and Cutset}

- An \textit{s} – \textit{d} cut: a partition of \( V \) into subsets \( V_s \ni s \) and \( V_d \ni d \)
- Cutset \( E(V_s) \): the set of edges going across the cut

\[
E(V_s) = \{(v_i, v_j) | (v_i, v_j) \in E, v_i \in V_s, v_j \in V_d\}
\]

\begin{figure}
\centering
\begin{tikzpicture}
\node at (0,0) [circle,draw] (s) {\( s \)};
\node at (2,0) [circle,draw] (v2) {\( v_2 \)};
\node at (4,0) [circle,draw] (v3) {\( v_3 \)};
\node at (2,-2) [circle,draw] (v4) {\( v_4 \)};
\node at (4,-2) [circle,draw] (v5) {\( v_5 \)};
\node at (4,2) [circle,draw] (d) {\( d \)};
\draw [->] (s) to [bend right] (v2);
\draw [->] (s) to [bend right] (v4);
\draw [->] (v2) to (v3);
\draw [->] (v3) to (v5);
\draw [->] (v4) to (v5);
\draw [->,dashed] (s) to (v3);
\draw [->,dashed] (v3) to (d);
\draw [->,dashed] (v4) to (d);
\draw [->] (v4) to (v5);
\node at (1.5,0) {$\epsilon_{2,3}$};
\node at (3,0) {$\epsilon_{4,3}$};
\node at (2.5,-1) {$\epsilon_{4,5}$};
\end{tikzpicture}
\end{figure}

\( s – d \) cut given by \( V_s = \{s, v_2, v_4\} \)
\[
E(V_s) = \{(v_2, v_3), (v_4, v_3), (v_4, v_5)\}
\]
Cut capacity $W(V_s)$: the ‘value’ of a cut

$$W(V_s) = \sum_{i:(v_i,v_j)\in E(V_s)} \left( 1 - \prod_{j:(v_i,v_j)\in E(V_s)} \epsilon_{i,j} \right)$$

$$W(V_s) = 1 - \epsilon_{2,3} + 1 - \epsilon_{4,3}\epsilon_{4,5}$$
Main Result

Assume that the decoders know erasure locations from across the network.

**Theorem 3**  The capacity of the erasure wireless network with a single source and a single destination is given by the value of the cut with the minimum value.

\[ C = \min_{V_s} W(V_s) \]

**Theorem 4**  The capacity of the erasure wireless network with a single source and multiple destinations (say \( d_1, d_2, \ldots, d_r \)) is the smallest of the capacities to the individual destinations.

\[ C = \min_{d_i} \min_{V_s,i: \text{an } s-d_i \text{ cut}} W(V_s,i) \]
Back to the original examples

1. 

\[ C = 1 - \epsilon_1 + 1 - \epsilon_2 \]

\[ C = 1 - \epsilon_1 \epsilon_2 \]

2. 

\[ C = \min \{1 - \epsilon_{1,2} + 1 - \epsilon_{1,3}, 1 - \epsilon_{2,3} + 1 - \epsilon_{1,3}\} \]

\[ C = \min \{1 - \epsilon_{1,2}\epsilon_{1,3}, 1 - \epsilon_{2,3} + 1 - \epsilon_{1,3}\} \]
Sketch of Proof

- **Encoding functions** $f_i$ are chosen randomly and are known to destinations.
- Destinations know **erasure locations**, can simulate the network for every codeword.
- The message for which simulated output matches what the destination actually observed is the decoder output.

\[
X_2 = f_2(Y_{1,2}, Y_{4,2}) \\
X_5 = f_5(Y_{4,5})
\]

The network appears deterministic to the destinations.
Achievability and Converse

Invoke ideas of typical sets, randomness of encoding functions etc.

Probability of error across this cut = $2^n(R-(1-\epsilon_{2,3}+1-\epsilon_{4,3}\epsilon_{4,5}))$

Therefore define cut capacity as $W(V_s) = 1 - \epsilon_{2,3} + 1 - \epsilon_{4,3}\epsilon_{4,5}$

Now, if $R < W(V_s)$, the probability of error goes to zero.

Converse: Similar to standard outer-bound results
Tools used: Fano’s Inequality, Data Processing Inequality
Some Remarks

- Some more multicast results go through.
  - Multiple sources, multiple destinations, all destinations want all sources
  - Single source with multiple processes, multiple destinations needing disjoint subsets of processes from source

- Note that we do not have to perform channel and network coding separately to reach capacity. In fact, making each link or sub-network error-free is demonstrably sub-optimal.

- Perhaps the correct approach is to ask “What sort of side-information should the destinations have?”
Linear encoding and practical implementation

- Choosing all encoding functions to be linear also takes us to capacity
- Now the decoder only has to solve a linear systems of equations
- This takes only polynomial time (for a full-rank system)
- Practical implementations involving rateless codes have been proposed (Lun et al.)
Future work

- General multicast problems e.g. \( k \)-pairs problem
- **Multiple**, possibly **correlated** information sources
- Capacity when the decoder does not know erasure locations
- Networks with other channels – DMCs, AWGN etc.

Summary

Obtained capacity for a class of wireless erasure networks for several multicast settings
Other contributions

- Practical operating schemes for erasure and Gaussian networks
  - Nodes are permitted to only forward or decode
  - Proposed a (decentralized) greedy algorithm that determines the optimal operation for each node


- Decoding in multiple antenna systems
  - Proposed the Increasing Radii Algorithm that allows a trade-off between computational complexity of decoding and error performance
  - Speeds up the decoder by a factor of 50 in some cases, with negligible loss of performance

Future Directions

- **Real-time communications** esp. for control applications
  - Noisy measurements have to be transmitted over noisy channels
  - Highly accurate information has to be received in a very short amount of time

- **Computations** in a network
  - Nodes have partial information about some underlying process
  - Computing and communicating functions of noisy and distributed data

- **World without wires**
  - Local and global repercussions