Iterated Space-Time Code Constructions from Quaternion Algebras

Nadya Markin, Frédérique Oggier

School of Physical and Mathematical Sciences, Nanyang Technological University

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Talk Outline

1 MIMO Channel
   - MIDO codes

2 Codes from Algebras

3 Decoding

4 Fast Decodability

5 Iterated Construction
   - Diversity Criteria
   - Examples of Iterated MIDO Codes
   - Performance of Iterated MIDO Codes
Multiple Input Multiple Output

- Space-time codes: used in multiple-antennae systems for higher data rate and reliability over fading channels
Multiple Input Double Output

- **Application**: Broadcasting to a portable device
- **Consider**: 4 Transmit, 2 Receive antennas \( R_1, R_2 \), perfect CSIR

At each time interval \( j \),

- \( R_1 \) receives a superposition of signals \((x_{1j}, x_{2j}, x_{3j}, x_{4j})\) plus noise
Multiple Input Double Output

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\[(h_{11}x_{1j} + \ldots + h_{14}x_{4j}) + n_{1j}\]
System Model: MIDO codes

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At each time interval $j$,

- $R_1$ receives a superposition of signals
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- $R_2$ receives
  \[(h_{21}x_{1j} + \ldots + h_{24}x_{4j}) + n_{2j}\]
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- $R_2$ receives
  
  $$(h_{21}x_{1j} + \ldots + h_{24}x_{4j}) + n_{2j}$$

- To transmit 8 complex symbols, need coherence time of at least 4.

Putting this into matrix form we obtain:

$$Y_{2 \times 4} = H_{2 \times 4} X_{4 \times 4} + N_{2 \times 4}$$

$H$ = channel matrix

$X$ = space-time codeword

$N$ = noise matrix

$H, N$ are i.i.d. complex Gaussian
**Full Diversity:** Want a collection of matrices so that

$$\min\{\det(X - Y) : X, Y \in \mathcal{C}\} \neq 0$$

this bounds pairwise probability of error (Tarokh et al)

- Algebras give rise to space-time codes
  - linearity
  - full diversity (in case algebra is division)
  - nice criteria for diversity
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  - full diversity (in case algebra is division)
  - nice criteria for diversity

- Many examples of codes with good performance already known.
- Problem: Maximum Likelihood (ML) decoding complexity too high
- Need algebraic codes which are fast decodable
ML Decoding

- A $4 \times 4$ MIDO code can transmit up to 8 complex information symbols, so 16 real (say PAM) information symbols.
- Space-time code $C$ is a vector space generated by matrices $B_i$.
- In our case $B_i \in \text{Mat}_{4 \times 4}$ and $X \in C$ has the form

\[
X = \sum_{i=1}^{16} g_i B_i,
\]

where $g_i$ are the information symbols from a real constellation.

ML Decoding

$Y =$ received matrix. Minimize the distance

\[
\{ d(X) = \| \| Y - HX \|_F^2 \} \forall X \in C \}
\]

(1)
Each $4 \times 4$ matrix $B_i \mapsto HB_i \mapsto b_i \in \mathbb{R}^{16} \cong \mathbb{C}^8$.

A 16-dimensional code gives rise to the matrix

$$B = (b_1, b_2, \ldots, b_{16}) \in M_{16 \times 16}(\mathbb{R}),$$

(2)

**QR decomposition:** $B = QR$, with $Q^\dagger Q = I$.

**ML decoding** reduces to minimizing

$$d(X) = \|y - QRg\|_E^2 = \|Q^\dagger y - Rg\|_E^2$$

(3)

- Complexity of exhaustive search is $O(|S|^{16})$, where $S$ = the constellation.
- **Fast decodable codes:** those which offer improvement on decoding complexity.
Fast Decodability

Complexity can be reduced if $R$ is guaranteed to have a nice zero-structure.

**Definition (Fast-decodable Codes)**

A space-time code is said to be *fast-decodable* if its $R$ matrix has the following form:

$$R = \begin{bmatrix} \Delta & B_1 \\ 0 & R_2 \end{bmatrix},$$

where $\Delta$ is a diagonal matrix and $R_2$ is upper-triangular.

**Definition (g-group Decodable Codes)**

A space-time code of dimension $K$ is called *g-group decodable* if the matrix $R$ has the form $R = diag(R_1, \ldots, R_g)$, where each $R_i$ is a square upper triangular matrix.
Orthogonality Relations on Basis Elements

Matrix $R$ can be tricky to calculate, because it depends on channel matrix $H$. 

Definition

Given an ordering on $B_1, \ldots, B_K$, let $M$ be a matrix capturing information about orthogonality relations of the basis elements of $B_i$:

$$M_{k,l} = ||B_k B^*_l + B_l B^*_k||_F. \quad (4)$$

Nice zero structure of $M \Rightarrow$ nice zero structure of $R$. (Rajan et al.)

$M = \Delta B_1 B_2 B_3 \cdots \Rightarrow R = \Delta R_1 R_2 \cdots$.
Matrix $R$ can be tricky to calculate, because it depends on channel matrix $H$.

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Nice zero structure of $M \implies$ nice zero structure of $R$. (Rajan et al.)

- $M = \begin{bmatrix} \Delta & B_1 \\ B_2 & B_3 \end{bmatrix}$, where $\Delta$ is diagonal $\implies R = \begin{bmatrix} \Delta & B_1 \\ 0 & R_1 \end{bmatrix}$.
- $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \implies R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$.  

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Iterated Space-Time Code Constructions from Quaternion Algebras
• Codes arising from algebras.

• Full Diversity:

\[ \min \{ \det(X - Y) : X, Y \in \mathcal{C} \} \neq 0 \]

\[ \downarrow \text{ by linearity} \]

\[ \min \{ \det(X) : X \in \mathcal{C} \} \neq 0 \]

• Fast Decodability: Matrix \( R \) must have a nice zero-block structure

Sufficient to find an ordering on basis elements of the code, so that \( M \) has a nice zero-structure.
We propose an **iterated construction** of space-time codes

- Full-rate $4 \times 4$ MIMO codes
- Fast-decodable: complexity reduction from $O(|S|^{16})$ to $O(|S|^{13}), O(|S|^{10}), O(|S|^{8})$.
- Criteria for full diversity
Recall Hamiltonian Quaternions.

- $\mathbb{H} =$ vector space of dimension 4 over $\mathbb{R}$, with basis $\{1, i, j, k\}$. 
Recall Hamiltonian Quaternions.

- $\mathbb{H}$=vector space of dimension 4 over $\mathbb{R}$, with basis
  \[ \{1, i, j, k\}. \]

- Rules: $i^2 = -1$, $j^2 = -1$, $k = ij = -ji$.

An element $x \in \mathbb{H}$ can be written as

\[ x = c + jd, \text{ where } c, d \in \mathbb{C}. \]

The resulting code $C$ corresponds to the celebrated Alamouti code [1]

\[ C = \left\{ \begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix} \mid c, d \in \mathbb{Z}[i] \right\}. \]
Generalized Quaternion Algebras and $2 \times 2$ Codes

Similarly

- $Q = (a, \gamma)_F$
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Similarly

- $Q = (a, \gamma)_F$
- $Q =$ vector space of dimension 4 over $F$, with basis $\{1, i, j, k\}$.
- Rules: $i^2 = a, j^2 = \gamma, k = ij = -ji$.

So $\mathbb{H} = (\mathbb{R}, -1, -1)$

$$c + jd \mapsto \begin{bmatrix} c & \gamma \sigma(d) \\ d & \sigma(c) \end{bmatrix}.$$

- Code $C = \{\lambda(x) : x \in \Lambda\}$, where $\Lambda$ is an order of $Q$.
Codes from Quaternion Algebras

- Quaternion algebras are a special case of a cyclic algebra.
- Quaternion algebra $Q = (a, \gamma)_F$ is of degree 2 over its maximal subfield $K = F(\sqrt{a})$.
- $Q \cong K \oplus jK$ with $j^2 = \gamma$. It is a right vector space over $K$.
- Left regular representation gives matrices

$$\lambda : Q \rightarrow M_{2 \times 2}(K)$$

$$c + jd \mapsto \begin{bmatrix} c & \gamma \sigma(d) \\ d & \sigma(c) \end{bmatrix}.$$ 

- Code $\mathcal{C} = \{\lambda(x) : x \in \Lambda\}$, where $\Lambda$ is an order of $Q$.

$$Q = (a, \gamma)_F \supset \Lambda$$

$$K = F(\sqrt{a})$$

$$F \ni \gamma$$
In general
- Cyclic algebras are constructed from a number field extension $K/F$,
  $$
  \mathcal{A} = (K/F, \sigma, \gamma)
  $$
- If $\mathcal{A}$ division, then resulting matrices are full rank, i.e., the code has full diversity
- Easy criterion for full diversity in terms of $\gamma$
  $$
  \mathcal{A} \text{ is division } \iff \gamma^i \notin N_{K/F}(K) \quad 1 \leq i < [K : F]
  $$
- Quaternion algebra $Q = (a, \gamma)_F$ is division $\iff \gamma \notin N_{K/F}(K)$
- Quaternion algebras give fast decodable $2 \times 2$ codes.
Iterated Code Construction

- Start with a generalized quaternion algebra $Q(a, \gamma)_F$. We have $\sigma : \sqrt{a} \mapsto \sqrt{a}$.
- $Q$ gives rise to a $2 \times 2$ space-time code.
- Iterated construction maps a pair of $2 \times 2$ algebraic space-time codewords to a $4 \times 4$ MIDO space-time codeword.
- Write $\sigma$ for the map acting componentwise by
  \[
  \sigma : \begin{bmatrix} c & \gamma \sigma(d) \\ d & \sigma(c) \end{bmatrix} \mapsto \begin{bmatrix} \sigma(c) & \gamma d \\ \sigma(d) & c \end{bmatrix}.
  \]

Iterated Construction

For $\theta$ of $K$, define $\alpha_\theta : M_2(K) \times M_2(K) \to M_4(K)$

\[
\alpha_\theta : (A, B) \mapsto \begin{bmatrix} A & \theta \sigma(B) \\ B & \sigma(A) \end{bmatrix},
\]
Iterated Code Construction

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\[
\alpha_\theta : \left( \begin{bmatrix} c & \gamma \sigma(d) \\ d & \sigma(c) \end{bmatrix}, \begin{bmatrix} e & \gamma \sigma(f) \\ f & \sigma(e) \end{bmatrix} \right) \mapsto \begin{bmatrix} c & \gamma \sigma(d) & \theta \sigma(e) & \theta \gamma f \\ d & \sigma(c) & \theta \sigma(f) & \theta e \\ e & \gamma \sigma(f) & \sigma(c) & \gamma d \\ f & \sigma(e) & \sigma(d) & c \end{bmatrix}. \tag{5}
\]
Lemma (Division Condition)

Let $K = F(\sqrt{a}), \gamma, \theta \in F$ and $Q = (a, \gamma)_F$. Let $\mathcal{A}$ denote the image of $\alpha_\theta$. The algebra $\mathcal{A}$ is division if and only if $\theta \neq z\sigma(z)$ for all $z \in Q$. 
Full Diversity of the Iterated Construction

Lemma (Division Condition)

Let \( K = F(\sqrt{a}), \gamma, \theta \in F \) and \( Q = (a, \gamma)_F \). Let \( \mathcal{A} \) denote the image of \( \alpha \theta \). The algebra \( \mathcal{A} \) is division if and only if \( \theta \neq z \sigma(z) \) for all \( z \in Q \).

Lemma (Concrete Criteria)

Let \( K = F(\sqrt{a}), \gamma, \theta \in F \). We have the following equivalence:

1. \( \theta \neq \det \begin{bmatrix} u & \gamma \sigma(v) \\ v & \sigma(u) \end{bmatrix} \), for any \( u, v \in K \) such that \( v \in \sqrt{a}F \), and
2. \( \theta \neq \gamma (\text{mod } K^{\times 2}) \), where \( K^{\times 2} \) denotes the squares in \( K \)

\[ \Leftrightarrow \]

\( \theta \neq z \sigma(z) \) for any \( z \in Q = (a, \gamma)_F \).

The nonnorm condition on \( \theta \) is easily satisfied when \( F \) is real, \( K \) is imaginary.
Lemma

Let $\theta = \gamma = -1$, let $F = \mathbb{Q}(\sqrt{b})$, $Q = (a, \gamma)_F$ for $a < 0, b > 0$.

The complexity of the iterated MIDO code arising from $\alpha_{\theta}(Q, Q)$ is $O(|S|^8)$.

Remarks

Since $[F : \mathbb{Q}] = 2$, this code carries 16 real information symbols. This code does not have full diversity, however. Have full diversity for when $\theta = -k$, where $k$ is any nonsquare.
Lemma

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Proof.

We show this code is 2-group decodable. Subdivide the basis of the code into two groups $\Gamma_1 \cup \Gamma_2$ so that $AB^* + BA^* = 0$ for all $A \in \Gamma_1$, $B \in \Gamma_2$.

\[ \Gamma_1 = \{\alpha_\theta(D, 0), \alpha_\theta(0, J)\}, \Gamma_2 = \{\alpha_\theta(J, 0), \alpha_\theta(0, D)\}, \]

where

\[ D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{b} & 0 \\ 0 & \sqrt{b} \end{bmatrix}, \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix}, \begin{bmatrix} \sqrt{a}\sqrt{b} & 0 \\ 0 & -\sqrt{a}\sqrt{b} \end{bmatrix} \right\} \]

and

\[ J = \left\{ \begin{bmatrix} 0 & \gamma \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \gamma\sqrt{b} \\ \sqrt{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\gamma\sqrt{a} \\ \sqrt{a} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\gamma\sqrt{a}\sqrt{b} \\ \sqrt{a}\sqrt{b} & 0 \end{bmatrix} \right\}. \]

In this case iterated code inherits orthogonality properties of generalized Alamouti code.
Example (Iterated MIDO Alamouti Code)

Pick $F = \mathbb{Q}(\sqrt{b})$, $b > 0$, and $K = F(i)$, with $\sigma : i \mapsto -i$. Then

$\alpha_\theta : \left( \begin{bmatrix} c & \gamma d^* \\ d & c^* \end{bmatrix}, \begin{bmatrix} e & \gamma f^* \\ f & e^* \end{bmatrix} \right) \mapsto \begin{bmatrix} c & \gamma d^* & \theta e^* & \theta \gamma f \\ d & c^* & \theta f^* & \theta e \\ e & \gamma f^* & c^* & \gamma d \\ f & e^* & d^* & c \end{bmatrix}$.

Since $c \in F(i)$, $c = c_0 + ic_1$, now $F = \mathbb{Q}(\sqrt{b})$, and thus

$c = c_0 + ic_1 = (c_{00} + \sqrt{b}c_{01}) + i(c_{10} + \sqrt{b}c_{11}),$

with $c_{00}, c_{01}, c_{10}, c_{11} \in \mathbb{Q}$. Alternatively

$c = (c_{00} + ic_{10}) + \sqrt{b}(c_{01} + i\sqrt{b}c_{11}),$
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with $c_{00}, c_{01}, c_{10}, c_{11} \in \mathbb{Q}$. Alternatively

$$c = (c_{00} + ic_{10}) + \sqrt{b}(c_{01} + i\sqrt{b}c_{11}),$$

$c$ can encode 2 QAM symbols $c_{00} + ic_{10}$ and $c_{01} + i\sqrt{b}c_{11}$. The whole MIDO code contains 8 QAM symbols.
Quasi-orthogonal code proposed by Jafarkhani[4] of the Lemma above with $F = \mathbb{Q}$. Note that it transmits only 8 real symbols.

**Example (Iterated Alamouti MIDO Code)**

Take $F = \mathbb{Q}$ and $K = \mathbb{Q}(i)$, with $\sigma : i \mapsto -i$ the complex conjugation, and $\gamma = -1$.

$$\begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix}.$$

Now, for $\theta = -1$, we get

$$\alpha_{\theta} : \left( \begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix}, \begin{bmatrix} e & -f^* \\ f & e^* \end{bmatrix} \right) \mapsto \begin{bmatrix} c & -d^* & -e^* & f \\ d & c^* & -f^* & -e \\ e & -f^* & c^* & -d \\ f & e^* & d^* & c \end{bmatrix}.$$
Iterated Silver MIDO code

Silver Code

The Silver code, discovered in [3], and re-discovered in [6], is given by codewords of the form

$$\begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{bmatrix},$$

where

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{\sqrt{7}} \begin{bmatrix} 1 + i & -1 + 2i \\ 1 + 2i & 1 - i \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

and $x_1, x_2, x_3, x_4 \in \mathbb{Z}[i]$ are the information symbols.

Alternatively [2], it can be viewed as scaled matrices $\begin{bmatrix} c & -\sigma(d) \\ d & \sigma(c) \end{bmatrix}$, coming from $(-1, -1)_F$, where $F = \mathbb{Q}(\sqrt{-7})$ and $K = F(i)$, with $\sigma : i \mapsto -i$. 

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Iterated Silver MIDO code

Iterated Silver Code

For $\theta \in F$ we have

$$\alpha_\theta : \left( \begin{bmatrix} c & -\sigma(d) \\ d & \sigma(c) \end{bmatrix}, \begin{bmatrix} e & -\sigma(f) \\ f & \sigma(e) \end{bmatrix} \right) \mapsto \begin{bmatrix} c & -\sigma(d) & \theta \sigma(e) & -\theta f \\ d & \sigma(c) & \theta \sigma(f) & \theta e \\ e & -\sigma(f) & \sigma(c) & -d \\ f & \sigma(e) & \sigma(d) & c \end{bmatrix}.$$ 

Lemma

The complexity of an iterated Silver MIDO code is at most $O(|S|^{13})$, no matter the choice of $\theta$. 
Iterated Silver MIDO code

Lemma (Iterated Silver MIDO Code)

The complexity of the iterated MIDO Silver code with $\theta = -1$ is $O(|S|^{10})$.

It can be verified by direct computation that the $R$ matrix of the iterated MIDO Silver code when $\theta = -1$ has the shape

\[
\begin{bmatrix}
    t & t & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t \\
    0 & t & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t \\
    0 & 0 & t & t & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t \\
    0 & 0 & 0 & t & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t \\
    0 & 0 & 0 & 0 & t & t & 0 & 0 & t & t & t & t & t & t & t \\
    0 & 0 & 0 & 0 & 0 & t & 0 & 0 & t & t & t & t & t & t & t \\
    0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t & t & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & t & t & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & t & t & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & t & 0 & t & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & t & 0 & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & t & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The iterated Silver code is conditionally 4-group decodable.
## Summary of Complexity

<table>
<thead>
<tr>
<th>Iterated Code</th>
<th>Parameters</th>
<th>Max Complexity</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alamouti</td>
<td>$\gamma = 1, \theta = 1$</td>
<td>$O(</td>
<td>S</td>
</tr>
<tr>
<td>Silver</td>
<td>$\gamma = 1, \theta \in F$</td>
<td>$O(</td>
<td>S</td>
</tr>
<tr>
<td>Silver</td>
<td>$\gamma = 1, \theta = -1$</td>
<td>$O(</td>
<td>S</td>
</tr>
</tbody>
</table>
Compare performance of Iterated Silver code with $\theta = i$ and complexity $O(|S^{13}|)$.

Figure: Comparison among codes with decoding complexity $O(|S|^{12})$ and $O(|S|^{13})$. 

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Iterated Space-Time Code Constructions from Quaternion Algebras
Silver $\theta = -1$ vs Crossed Product Algebra Codes

Figure: Comparison among codes with decoding complexity $O(|S|^{10})$. 


