Graphs and Matrices with Integral Spectrum in Some Families

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Introduction

Notation

Given a finite family of matrices $\mathcal{M}$ we denote by

$$Z(\mathcal{M}) = \#\{M \in \mathcal{M} : \text{Spec } M \subseteq \mathbb{Z}\}$$

Given a finite family of graphs $\mathcal{G}$ we put

$$Z(\mathcal{G}) = Z(\mathcal{A}),$$

where $\mathcal{A}$ is the set of adjacency matrices of graphs $G \in \mathcal{G}$. 
Families of Matrices and Graphs

We are interested in estimating $Z(\mathcal{M})$ and $Z(\mathcal{G})$ for various interesting families $\mathcal{M}$ and $\mathcal{G}$, such as

- Arbitrary matrices in a box:
  \[ \mathcal{M}(K, h) = \{ M = (m_{ij})_{i,j=1}^n : |k_{ij} - m_{ij}| < h \} \]
  for a given $n \times n$ matrix $K = (k_{ij})_{i,j=1}^n$;

- Symmetric matrices in a box:
  \[ S(K, h) = \{ M = (m_{ij})_{i,j=1}^n : m_{ij} = m_{ji}, |k_{ij} - m_{ij}| < h \} \]
  for a given $n \times n$ matrix $K = (k_{ij})_{i,j=1}^n$;

- 0, 1-matrices;

- Circulant graphs;

- Regular graphs;

- Arbitrary graphs.
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Motivation

Top ten reasons to study these questions:

1. They are interesting

2. They are interesting

... ... ... ...

9. They are interesting

10. They are related to network topologies that support perfect quantum state transfer: *Christandal, Datta, Ekert, Kay and Landahl (2004)*
*Facer, Twamley and Cresser (2008)*
*Godsil (2008)*
Integral Graphs

Background

The notion introduced by *Harary and Schwenk (1974)*

Several explicit constructions of integral graphs of various types:

*Brouwer and Koolen (1993)*

*Wang, Li and Hoede (2005)*

*Lepović (2006)*

*So (2006)*

*Indulal and Vijayakumar (2007)*

*Stevanović, de Abreu, de Freitas and Del-Vecchio (2007)*

*Brouwer (2008)*

*Brouwer and Haemers (2009)*

*Liu and Wang (2010)*

*Carvalho and Rama (2010)*

... and many more
Circulant Graphs

Let \( S = \{s_1, s_2, \ldots, s_k\} \) be a set of \( k \) integers with
\[
1 \leq s_1, s_2, \ldots, s_k < n.
\]

We consider only undirected graphs \( \Rightarrow \) \( S \) is symmetric:

\[
s \in S \quad \text{iff} \quad n - s \in S.
\]

A circulant graph \( G(n; S) \) is a \( k \)-regular graph on \( n \) vertices \( \{v_1, \ldots, v_n\} \) such that \( v_i \) and \( v_j \) are incident whenever \( i - j \in S \).

\( G(n; S) \) is \( k \)-regular where \( k = \#S \).

\( G(n; S) \) is connected iff \( \gcd(\{s : s \in S\}) = 1 \).
Eigenvalues of $G(n; S)$ are explicitly given by:

$$\lambda_j = \sum_{s \in S} \exp\left(\frac{2\pi i j s}{n}\right), \quad j = 1, \ldots, n,$$

and not that hard to control.

Let $C_n = \{G(n; S) : \text{symmetric } S \subseteq \{1, \ldots, n-1\}\}$

*So (2006):*

A full description of circulant integral graphs (based on explicit formulas for the eigenvalues)

$$Z(C_n) \leq 2^{\tau(n)-1},$$

where $\tau(n)$ is divisors function.
Let
\[ C^*_{n,k} = \{ G(n; S) \in C_n : \#S = k, G(n; S) \text{ connected} \} \]

**Saxena, Severini and Shparlinski (2007):** For
\[ n \geq \exp \left( c \sqrt{k \log \log(k + 2) \log k} \right) \]
with some absolute constant \( c > 0 \), we have
\[ Z(C_{n,k}) = 0. \]

**Klin and Kovács (2010)**
Description of automorphism groups of integral circulant graphs.

**Open Question 1** What about more general Cayley graphs?
Integral Graphs with Few Cycles

Integral trees are fully classified by Watanabe and Schwenk (1979).

Classification of integral graphs with at most two cycles: Omidi (2010)
Let $A_n$ be the set of all adjacency matrices of graphs on $n$ vertices.

Ahmadi, Alon, Blake and Shparlinski (2008):

$$Z(A_n) \leq 2^{n(n-1)/2-n/400}.$$ 

Not tight!

**Conjecture:** $Z(A_n) = \exp(O(n))$.

For $n = 2^m$ any Cayley graph of $(\mathbb{Z}_2)^m$ is integral:

$$Z(A_n) \geq 2^{\Omega(n)}.$$
Ideas Behind the Proof

• ‘Circular Law’

• For most of adjacency matrices have most of eigenvalues are \( \leq Cn^{1/2} \)

• Since eigenvalues are integral, at least one is of multiplicity \( \geq cn^{1/2} \).

• In matrix with an eigenvalue of multiplicity \( s \) there is an symmetric \( s \times s \) minor defined by other entries.

• Bound.
More Details

Distribution of Eigenvalues

As $A \in A_n$ is symmetric, its eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

are real.


**Lemma 2** For any $c > 1$ and large enough $n$

$$\Pr_{A \in A_n} \left[ -c\sqrt{n} < \lambda_i < c\sqrt{n}, \ i = 2, \ldots, n \right] \geq 1 - n^{-10}.$$
Let $E_i$ denote the expected value of $\lambda_i$ of $A \in \mathcal{A}_n$ chosen uniformly at random.

**Corollary 3** For $i > 1$ and large enough $n$

$$|E_i| < 2\sqrt{n}$$

**Proof.** Using Lemma 2 with $c = 3/2$, we have that with probability $1 - n^{-10}$, $\lambda_i$ is at most $3\sqrt{n}/2$ and with probability $n^{-10}$ it is at most $n$:

$$E_i < \left(1 - \frac{1}{n^{10}}\right) \frac{3}{2} \sqrt{n} + \frac{1}{n^{10}} n < 2\sqrt{n}$$

for large enough values of $n$. Similarly we have

$$E_i > \left(1 - \frac{1}{n^{10}}\right) \frac{-3}{2} \sqrt{n} + \frac{1}{n^{10}} (-n) > -2\sqrt{n}.$$  

$\square$
Let $M_i$ denote the median of $\lambda_i$ of $A \in \mathcal{A}_n$: for at least $0.5 \# \mathcal{A}_n$ matrices $A \in \mathcal{A}_n$ we have $\lambda_i \geq M_i$.

**Corollary 4** We have

$$|M_i| < 6\sqrt{n}, \quad i = 2, \ldots, n.$$  

**Proof.** Suppose that $M_i \geq 6\sqrt{n}$. Using Lemma 2 with $c = 3/2$ we obtain

$$E_i \geq \frac{1}{2}(6\sqrt{n}) + \frac{1}{2} \left(\frac{-3}{2}\sqrt{n}\right) + \frac{1}{n^{10}}(-n) \geq 2\sqrt{n},$$

which contradicts Cor. 3. Similarly $M_i \leq -6\sqrt{n}$.

\[\square\]

*Alon, Krivelevich and Vu (2002)*

**Lemma 5** We have

$$\Pr_{A \in \mathcal{A}_n} [\vert \lambda_s - M_s \vert > t] \leq 4e^{-t^2/8r^2}$$

where $r = \min\{s, n - s + 1\}$. 


From Cor. 4 and Lemma 5 we derive:

**Corollary 6** We have,

\[
\Pr_{A \in \mathcal{A}_n} \left[ |\lambda_i| < 7\sqrt{n}, \ i = 2, \ldots, n \right] \geq 1 - 8e^{-n/32}.
\]

**Proof.** By Cor. 4 we have

\[
\Pr_{A \in \mathcal{A}_n} \left[ \lambda_2 > 7\sqrt{n} \right] = \Pr_{A \in \mathcal{A}_n} \left[ \lambda_2 - 6\sqrt{n} > \sqrt{n} \right] \\
\leq \Pr_{A \in \mathcal{A}_n} \left[ \lambda_2 - M_2 > \sqrt{n} \right].
\]

Applying Lemma 5 with \( t = \sqrt{n} \) we have

\[
\Pr_{A \in \mathcal{A}_n} \left[ \lambda_2 > 7\sqrt{n} \right] \leq 4e^{-n/32}.
\]

Similarly

\[
\Pr_{A \in \mathcal{A}_n} \left[ \lambda_n < -7\sqrt{n} \right] \leq 4e^{-n/32}.
\]

\( \square \)
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Multiplicities of Eigenvalues

Let $\lambda$ be an eigenvalue of a matrix $M$:

1. algebraic multiplicity is its order as a root of the characteristic polynomial of $M$;

2. geometric multiplicity is the rank of the null-space of $M - \lambda I$

$M$ is symmetric:
   algebraic multiplicity = geometric multiplicity

A principal submatrix of order $r$ of $n \times n$ matrix $M$ is a submatrix of $M$ obtained by deleting rows $R_{i_1}, R_{i_2}, \ldots, R_{i_{n-r}}$ and columns $C_{i_1}, C_{i_2}, \ldots, C_{i_{n-r}}$ where $1 \leq i_1 < i_2 < \ldots < i_{n-r} \leq n$.

Principal submatrices of a symmetric matrix are symmetric too.
Well-known result:

**Lemma 7** A symmetric matrix $M$ is of rank $r$ iff $M$ has a nonsingular principal submatrix of order $r$ and has no larger principal submatrix which is nonsingular.

Not so well-known result:

**Lemma 8** Let $\lambda$ be a real number. Then the number $N_\lambda(n, s)$ of adjacency matrices of order $n$ having $\lambda$ as an eigenvalue of algebraic multiplicity $s$ is at most

$$N_\lambda(n, s) \leq \binom{n}{s} 2^{n(n-1)/2-s(s-1)/2}.$$
Concluding the Proof

By Cor. 6 the number of matrices with eigenvalue outside of $[-7 \sqrt{n}, 7 \sqrt{n}]$ is at most $8e^{-n/32}2^{n(n-1)/2}$.

The remaining matrices have one eigenvalue of multiplicity at least

$$t = \frac{n - 1}{14\sqrt{n} + 1}.$$  

By Lemma 8, there are at most

$$\sum_{-7 \sqrt{n} \leq \lambda \leq 7 \sqrt{n}} \sum_{t \leq s \leq n} N_{\lambda}(n, s)$$

$$\leq (14\sqrt{n} + 1) \sum_{t \leq s \leq n} \binom{n}{s} 2^{n(n-1)/2-s(s-1)/2+1}$$

$$\leq (14\sqrt{n} + 1)n \binom{n}{\lfloor n/2 \rfloor} 2^{n(n-1)/2-t(t-1)/2+1}$$

graphs having an integral spectrum.
Dumitriu and Pal (2009)

Analogue of the result of Alon, Krivelevich and Vu (2002) about the distribution of eigenvalues.

Ahamdi and Shparlinski (???)

Nothing yet, but it’s in our plans
Integral Matrices

Arbitrary matrices in a box

For a $n \times n$ matrix $K = (k_{ij})_{i,j=1}^n$ we define

$$\mathcal{M}(K, h) = \{ M = (m_{ij})_{i,j=1}^n : |k_{ij} - m_{ij}| < h \}$$

Let $O_n$ be the $n \times n$ zero matrix.

**Martin and Wong (2009):**

$$Z(\mathcal{M}(O_n, h)) \leq h^{n^2-2+o(1)}$$

**Shparlinski (2009):** Improvement and generalisation:

$$Z(\mathcal{M}(K, h)) \leq (h + \|K\|)h^{n^2-n+o(1)}$$

The result is based on a new upper bound

$$\# \{ M \in \mathcal{M}(K, h) : \det M = 0 \} \ll h^{n^2-n} \log h \quad (1)$$

(uniformly over $K$) applied to $K - \lambda I$ instead of $K$ and the bound $\lambda = O(h + \|K\|)$ on eigenvalues $\lambda$ of $M \in \mathcal{M}(K, h)$.

The proof of (1) is **elementary**.
For a $n \times n$ matrix $K = (k_{ij})_{i,j=1}^{n}$ we define

$$S(K, h) = \{ M = (m_{ij})_{i,j=1}^{n} : m_{ij} = m_{ji}, \ |k_{ij} - m_{ij}| < h \}$$

*Shparlinski (2009):*

$$Z(S(K, h)) \leq h^{n(n+1)/2-1+o(1)}$$

The bound is uniform over $K$.

The result is based on a new upper bound

$$\# \{ S \in S(K, h) : \det S = 0 \} \ll h^{n(n+1)/2-1+o(1)}$$

(2)

(uniformly over $K$) and a result of *Weyl (1912)* on the **stability** of eigenvalues of symmetric matrices:

For every eigenvalue $\lambda$ of $M \in S(K, h)$ there exists and an eigenvalue $\eta$ of $K$ with

$$\lambda = \eta + O(h)$$
Matrices with a given determinant

The case of $K = O_n$: very well studied

*Duke, Rudnick and Sarnak (1993)* for $a \neq 0$ and *Katznelson (1993)* when $a = 0$ we immediately obtain

$$\#\{M \in \mathcal{M}(O_n, h) : \det M = a\} \ll \begin{cases} h^{n^2-n}, & a \neq 0, \\ h^{n^2-n} \log h, & a = 0, \end{cases}$$

where the implied constant may depend on $n$.

*Wigman (2005):*
A variant of the result of *Katznelson (1993)* for matrices with *primitive rows*
Similarly, Duke, Rudnick and Sarnak (1993) for $a \neq 0$ and Eskin and Katznelson (1994) when $a = 0$

$$\# \{S \in \mathcal{S}(O_n, h) : \det S = a\} \ll \begin{cases} h^n(n-1)/2, & a \neq 0, \\ h^n(n-1)/2 \log h, & a = 0. \end{cases}$$

In fact they all give **asymptotic formulas** but for matrices ordered w.r.t. a different norm.

*Bourgain, Costelo, Tao, Vu and Wood (2006–???)*

A series of results about singular 0, 1-matrices and matrices with entries in $\{-k, \ldots, k\}$ for a fixed $k$ and growing dimension $k$.

They also lead to various estimates on the number of integral matrices in some families. One example is given in Bourgain, Vu and Wood (2009).
Arbitrary $K$:

The proof of (1) is elementary:

Assume that $X, Y \in \mathcal{M}(K, h)$ are obtained from an $n \times (n - 1)$-matrix $R$ by augmenting it by $x, y \in \mathbb{Z}^n$:

$$X = (R|x) \quad \text{and} \quad Y = (R|y).$$

If $\det(X) = \det(Y)$ then putting

$$z = x - y \quad \text{and} \quad Z = (R|z)$$

we get $\det Z = 0$ (expand $Z$ with respect to the last column).

Therefore,

$$\#\{x = (x_1, \ldots, x_n) \in \mathbb{Z}^n : \det(R|x) = a, |k_{i,n} - x_i| < h, i = 1, \ldots, n\}$$

$$\leq \#\{z = (z_1, \ldots, z_n) \in \mathbb{Z}^n : \det(R|z) = 0, |z_i| < 2h, i = 1, \ldots, n\}.$$

If $x_1, \ldots, x_J$ are the elements from set on the LHS then the vectors $z_j = x_1 - x_j, j = 1, \ldots, J$ are distinct and belong to the set on the RHS.
Summing this over all \((2h - 1)^{n^2-n}\) integral matrices \(R = (r_{ij})_{i,j=1}^{n,n-1}\) with 
\[ |k_{ij} - r_{ij}| < h, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n - 1, \]
we obtain
\[
\#\{M \in \mathcal{M}(K, h) : \det M = a\} 
\leq \#\{M \in \mathcal{M}(K_1, 2h) : \det M = 0\},
\]
where \(K_1\) is obtained from \(K\) by replacing its \(n\)th column by a zero vector.

Repeating the same argument with respect to the \((n - 1)\)th column of the matrix \(K_1\), we obtain
\[
\#\{M \in \mathcal{M}(K, h) : \det M = a\} 
\leq \#\{M \in \mathcal{M}(K_2, 4h) : \det M = 0\},
\]
where \(K_2\) now has two zero column.

After \(n\) steps we arrive to
\[
\#\{M \in \mathcal{M}(K, h) : \det M = a\} 
\leq \#\{M \in \mathcal{M}(K_n, 2^n h) : \det M = 0\},
\]
where \(K_n = O_n\) is the zero matrix. \(\square\)
The above argument does not work for symmetric matrices (the structure disappears immediately).

The proof of (2) used deep results on integral points on surfaces due to Browning, Heath-Brown and Salberger (2006)

To apply we need to prove absolute irreducibility of the highest form $H_k$ of the polynomial

$$F_K \left( \{X_{ij}\}_{1 \leq i \leq j \leq n} \right) = \det \left( \{k_{ij} + X_{ij}\} \right)_{i,j=1}^n$$

of degree $n$ in $n(n+1)/2$ variables $X_{ij}$, $1 \leq i \leq j \leq n$, where we also define $X_{ij} = X_{ji}$ for $i > j$:

- Remark that $H_K = F_{On}$.

- Then use induction.
Comments and Open Questions

A common weakness of the above results: They essentially deal with matrices having several (sometimes one!) integral eigenvalues rather than all integral eigenvalues.

Exploiting that fact may lead to an improvement.

Open Question 9 Find a way to make use of all eigenvalues.

A related problem:

Open Question 10 Given a set of $n$ real numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ estimate the number of graphs having $\Lambda$ as their spectrum.
Our approaches do not work for algebraic eigenvalues of low degree.

**Open Question 11** Given a number field $\mathbb{K}$, estimate the number of graphs/matrices in the above families with all eigenvalues in $\mathbb{K}$.

**Open Question 12** Given an integer $d$, estimate the number of graphs/matrices in the above families with all eigenvalues being algebraic numbers of degree at most $d$.

**Open Question 13** Obtain tight uniform bounds on the number of integral matrices $M \in \mathcal{M}(K, h)$ and $S \in \mathcal{S}(K, h)$ of a given rank $r$. 
Open Question 14 Obtain bounds on the number of integral matrices $M \in \mathcal{M}(K, h)$ and $S \in S(K, h)$ with a characteristic polynomials of a certain type (for example, reducible over $\mathbb{Z}$).

There are some results of Rivin (2008) but the do not give what is really expected for Question 14.

Open Question 15 For a given matrix $K = (k_{ij})_{i,j=1}^n$ and $n^2$ polynomials $f_{ij}(X) \in \mathbb{Z}[X]$, $i, j = 1, \ldots, n$, obtain bounds on the number of integral matrices in the set:

$$\left\{ \left( f_{ij}(x_{ij}) \right)_{i,j=1}^n : k_{ij} \leq x_{ij} < k_{ij} + h \right\}.$$
Back to our Motivation

Integral graphs and perfect quantum state transfer?

– Not quite so . . .

What do we really need?
For every quadruple $\lambda_h, \lambda_i, \lambda_j, \lambda_k$ of eigenvalues (with $\lambda_j \neq \lambda_k$), we need

$$\frac{\lambda_h - \lambda_i}{\lambda_j - \lambda_k} \in \mathbb{Q}.$$ 

Saxena, Severini and Shparlinski (2007):
For circulant graphs this property is essentially equivalent to integrality.

Open Question 16 What about other graphs?

Open Question 17 If this property differs from integrality, can we estimate the number of such graphs in some interesting families?