Grassmann Varieties and Linear Codes

Sudhir R. Ghorpade

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India
http://www.math.iitb.ac.in/~srg/

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Grassmann Varieties
A Quick Introduction

\( V \) : vector space of dimension \( m \) over a field \( \mathbb{F} \)

For \( 1 \leq \ell \leq m \), we have the Grassmann variety:

\[
G_{\ell,m} = G_\ell(V) := \{ \ell\text{-dimensional subspaces of } V \}.
\]

Plücker embedding:

\[
G_{\ell,m} \hookrightarrow \mathbb{P}^{k-1} \quad \text{where} \quad k := \binom{m}{\ell}.
\]

Explicitly, \( \mathbb{P}^{k-1} = \mathbb{P}(\wedge^\ell V) \) and

\[
\mathcal{W} = \langle w_1, \ldots, w_\ell \rangle \longleftrightarrow [w_1 \wedge \cdots \wedge w_\ell] \in \mathbb{P}(\wedge^\ell V).
\]

For example, \( G_{1,m} = \mathbb{P}^{m-1} \). In terms of coordinates,

\[
\mathcal{W} = \langle w_1, \ldots, w_\ell \rangle \in G_\ell(V) \longleftrightarrow p(\mathcal{W}) = (p_\alpha(A_{\mathcal{W}}))_{\alpha \in \Lambda(\ell,m)}
\]

where \( A_{\mathcal{W}} = (a_{ij}) \) is a \( \ell \times m \) matrix whose rows are (the coordinates of) a basis of \( \mathcal{W} \) and \( p_\alpha(A_{\mathcal{W}}) \) is the \( \alpha \text{th} \) minor of \( A_{\mathcal{W}} \), viz., \( \det (a_{i\alpha_j})_{1 \leq i,j \leq \ell} \) and where
\[ l(\ell, m) := \left\{ \alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell : 1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m \right\}. \]

Facts:

- \( G_{\ell,m} \) is a projective algebraic variety given by the common zeros of certain quadratic homogeneous polynomials in \( k \) variables. As a projective algebraic variety \( G_{\ell,m} \) is nondegenerate, irreducible, nonsingular, and rational.

- There is a natural transitive action of \( GL_m \) on \( G_{\ell,m} \) and if \( P_\ell \) denotes the stabilizer of a fixed \( W_0 \in G_{\ell,m} \), then \( P_\ell \) is a maximal parabolic subgroup of \( GL_m \) and \( G_{\ell,m} \cong GL_m/P_\ell \).

- If \( F = \mathbb{R} \) or \( \mathbb{C} \), then \( G_{\ell,m} \) is a (real or complex) manifold, and its cohomology spaces and Betti numbers are explicitly known. In fact, \( b_\nu = \dim H^{2\nu}(G_{\ell,m}; \mathbb{C}) \) is precisely the number of partitions of \( \nu \) into at most \( \ell \) parts, each part \( \leq m - \ell \),
Suppose $\mathbb{F} = \mathbb{F}_q$ is the finite field with $q$ elements. Then $G_{\ell,m} = G_{\ell,m}(\mathbb{F}_q)$ is a finite set and its cardinality is given by the Gaussian binomial coefficient:

$$
\begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q) \cdots (q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q) \cdots (q^\ell - q^{\ell-1})}.
$$

This is a polynomial in $q$ of degree $\delta := \ell(m - \ell)$ and in fact,

$$|G_{\ell,m}(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \sum_{\nu=0}^{\delta} b_{\nu} q^\nu = q^\delta + q^{\delta-1} + 2q^{\delta-2} + \cdots + 1.$$

Note that

$$\lim_{q \to 1} \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \binom{m}{\ell}.$$
\( \mathbb{F}_q \): finite field with \( q \) elements.

\([n, k]_q\)-code: a \( k \)-dimensional subspace \( C \) of \( \mathbb{F}_q^n \).

**Hamming weight** of \( c = (c_1, \ldots, c_n) \in \mathbb{F}_q^n \):

\[
w_H(c) := \# \{ i : c_i \neq 0 \}.
\]

**Hamming weight** of a subcode \( D \) of \( C \):

\[
w_H(D) := \# \{ i : \exists c = (c_1, \ldots, c_n) \in D \text{ with } c_i \neq 0 \}.
\]

**Minimum distance** of a (linear) code \( C \):

\[
d(C) := \min \{ w_H(c) : c \in C, c \neq 0 \}.
\]

The \( r^{th} \) higher weight of \( C \) (\( 1 \leq r \leq k \)):

\[
d_r(C) := \min \{ w_H(D) : D \subseteq C, \dim D = r \}.
\]

\( C \) is **nondegenerate** if \( C \not\subseteq \) coordinate hyperplane of \( \mathbb{F}_q^n \), or equivalently, if \( d_k(C) = n \).
Write $A^\delta(\mathbb{F}_q) := \mathbb{F}_q^\delta = \{P_1, P_2, \ldots, P_{q^\delta}\}$. Consider the evaluation map of the polynomial ring in $\delta$ variables:

$$\text{Ev} : \mathbb{F}_q[X_1, \ldots, X_\delta] \rightarrow \mathbb{F}_q^{q^\delta}$$

$$f \mapsto (f(P_1), \ldots, f(P_{q^\delta})),$$

The $r^{\text{th}}$ order generalized Reed-Muller code of length $q^\delta$:

$$\text{RM}(r, \delta) := \text{Ev}(\mathbb{F}_q[X_1, \ldots, X_\delta]_{\leq r}) \quad \text{for } r < q.$$ 

This has dimension $\binom{\delta+r}{r}$ and minimum distance $(q - r)q^{\delta-1}$.

More generally, one can consider $\text{RM}(r, \delta)$ for $r \leq \delta(q - 1)$.

$G_{k,n}(\mathbb{F}_q)$ may be viewed as the ‘moduli space’ of all $[n, k]_q$-linear codes. An optimal class of codes, called MDS codes [these are $[n, k]_q$-linear codes for which the minimum distance $d$ is the maximum possible, viz., $d = n - k + 1$] correspond to certain open strata of $G_{k,n}$. This viewpoint is useful for the following:

**Problem**

Given a prime power $q$ and integers $k, n$ with $1 \leq k \leq n$, determine

$$\gamma(q) = \gamma(q; k, n) = \# \left( [n, k]_q\text{-MDS codes} \right).$$

This is equivalent to determining

$$\# \left( \text{inequivalent representations over } \mathbb{F}_q \text{ of } U_{k,n} \right)$$

where $U_{k,n}$ is the so-called uniform matroid.

The problem is open, in general.
Matroid $M$ on a finite set $S$: determined by the rank function $r = r_M : \mathcal{P}(S) \to \mathbb{Z}$ satisfying
(i) $0 \leq r(I) \leq |I|$,  
(ii) $I \subseteq J \Rightarrow r(I) \leq r(J)$, and
(iii) $r(I \cup J) + r(I \cap J) \leq r(I) + r(J)$.

- $I \subseteq S$ is independent if $r(I) = |I|$.
- base: maximal independent subset of $S$.
- uniform matroid $U_{k,n}$: the matroid on $n$ elements s.t. any $k$ elements form a base.
- representation of $M$ over a field $F$: map $f : S \to V$, where $V$ is a vector space over $F$, such that
  \[
  \dim(\text{span}\{f(x) : x \in I\}) = r_M(I) \quad \forall I \subseteq S.
  \]
- Representations $f_1 : S \to V_1$ and $f_2 : S \to V_2$ are equivalent if \(\exists\) an isomorphism $\phi : V_1 \to V_2$ such that $\phi \circ f_1 = f_2$.

A \([n, k]_q\)-projective system is a collection \(\mathcal{P}\) of \(n\) not necessarily distinct points in \(\mathbb{P}^{k-1}\); this is nondegenerate if it is not contained in a hyperplane.

\[
[n, k]_q\text{-code } C \leadsto [n, k]_q\text{-projective system } \mathcal{P}
\]

Conversely a nondegenerate \([n, k]_q\)-projective system gives rise to a nondegenerate \([n, k]_q\)-code, and the resulting correspondence is a bijection, up to equivalence. Note that

\[
d(C) = n - \max\{\#\mathcal{P} \cap H : H \text{ hyperplane of } \mathbb{P}^{k-1}\}.
\]

and for \(r = 1, \ldots, k\),

\[
d_r(C) = n - \max\{\#\mathcal{P} \cap E : E \text{ linear subvariety of codim } r \text{ in } \mathbb{P}^{k-1}\}.
\]
Grassmann Codes

Thanks to the Plücker embedding,

\[ G_{\ell,m}(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k-1} \leadsto [n, k]_q\text{-code } C(\ell, m). \]

Length \( n \) is the Gaussian binomial coefficient:

\[ n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q) \cdots (q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q) \cdots (q^\ell - q^\ell-1)}. \]

and the dimension \( k \) is the binomial coefficient:

\[ k = \binom{m}{\ell}. \]

Theorem (Ryan (1990, \( q = 2 \)), Nogin (1996, any \( q \)))

\[ d \left( C(\ell, m) \right) = q^\delta \text{ where } \delta := \ell(m - \ell). \]

It may be noted that \( \delta \) is the dimension of \( G_{\ell,m} \) as a projective variety.
The first result in this direction is:

**Theorem (Nogin (1996), Ghorpade-Lachaud(2000))**

*More generally, for $1 \leq r \leq \mu$ we have*

$$d_r (C(\ell, m)) = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1},$$

*where $\mu := \max\{\ell, m - \ell\} + 1$.*

The alternative proofs in Ghorpade-Lachaud(2000) used a characterization of the so-called close families of $\ell$-subsets of the finite set $\{1, \ldots, m\}$. This can be viewed as an analogue for uniform hypergraphs of the elementary result in graph theory that

*A simple graph in which any two edges are incident is either a star or a triangle.*

**Note:** $\mu = \max\{\ell, m - \ell\} + 1$ is usually much smaller than $k = \binom{m}{\ell}$ and so the above theorem doesn’t give all the higher weights.
Let $\alpha$ be in $I(\ell, m)$, that is,

$$\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell, \ 1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m.$$  

Consider the corresponding Schubert variety

$$\Omega_\alpha := \{ W \in G_{\ell, m} : \dim(W \cap A_{\alpha_i}) \geq i \ \forall i \},$$

where $A_j$ is the span of the first $j$ vectors in a fixed basis of our $m$-space. We have

$$\Omega_\alpha(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k_\alpha-1} \curvearrowright [n_\alpha, k_\alpha]_q\text{-code } C_\alpha(\ell, m)$$

where

$$n_\alpha = |\Omega_\alpha(\mathbb{F}_q)| \text{ and } k_\alpha = |\{ \beta \in I(\ell, m) : \beta \leq \alpha \}|,$$

with $\leq$ being the componentwise partial order:

$$\beta = (\beta_1, \ldots, \beta_\ell) \leq \alpha = (\alpha_1, \ldots, \alpha_\ell) \iff \beta_i \leq \alpha_i \ \forall i.$$

Proposition (Ghorpade-Lachaud(2000))

For any $\alpha \in I(\ell, m)$,

$$d \left( C_\alpha(\ell, m) \right) \leq q^{\delta_\alpha} \quad \text{where} \quad \delta_\alpha := \sum_{i=1}^{\ell} (\alpha_i - i).$$

It may be noted that when $\alpha$ is the “maximal element” $(m - \ell + 1, \ldots, m - 1, m)$ of $I(\ell, m)$, then $\Omega_\alpha = G_{\ell, m}$ while $\delta_\alpha = \delta = \ell(m - \ell)$ and so the above inequality is an equality. In fact, the following conjecture was made in the same paper:

Minimum Distance Conjecture (MDC)

For any $\alpha \in I(\ell, m)$,

$$d \left( C_\alpha(\ell, m) \right) = q^{\delta_\alpha}.$$
If $\ell = 2$ and $\alpha = (m - h - 1, m)$, then

$$n_\alpha = \frac{(q^m - 1)(q^{m-1} - 1)}{(q^2 - 1)(q - 1)} - \sum_{j=1}^{h} \sum_{i=1}^{j} q^{2m-j-2-i}$$

and

$$k_\alpha = \frac{m(m - 1)}{2} - \frac{h(h + 1)}{2}.$$ 

[Hao Chen (2000)]

In general,

$$n_\alpha = \sum_{\ell-1} \prod_{i=0}^{\ell-1} \left[ \frac{\alpha_{i+1} - \alpha_i}{k_{i+1} - k_i} \right] q^{(\alpha_i - k_i)(k_{i+1} - k_i)}$$

where the sum is over $(k_1, \ldots, k_{\ell-1}) \in \mathbb{Z}^{\ell}$ satisfying $i \leq k_i \leq \alpha_i$ and $k_i \leq k_{i+1}$ for $1 \leq i \leq \ell - 1$; by convention, $\alpha_0 = 0 = k_0$ and $k_\ell = \ell$. 

[Vincenti (2001)]
\[ n_\alpha = \sum_{\beta \leq \alpha} q^{\delta_\beta}, \quad \text{where} \quad \delta_\beta = \sum_{i=1}^{\ell} (\beta_i - i). \]

Ehresmann (1934); Ghorpade-Tsfasman (2005)

Suppose \( \alpha \) has \( u + 1 \) consecutive blocks:

\[
\alpha = (\alpha_1, \ldots, \alpha_{p_1}, \ldots, \alpha_{p_u+1}, \ldots, \alpha_{p_u+1}).
\]

Then

\[
n_\alpha = \sum_{s_1 = p_1}^{\alpha_{p_1}} \cdots \sum_{s_u = p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \lambda(\alpha_{p_i}, \alpha_{p_i+1}; s_i, s_{i+1})
\]

where, \( s_0 = p_0 = 0; \) \( s_{u+1} = p_{u+1} = \ell, \) and

\[
\lambda(a, b; s, t) := \sum_{r=s}^{t} (-1)^{r-s} q^{r-s\choose 2} \left[ a - s \right] q^{b - r \choose r - s} q^{t - r \choose t - r}. \]

[Ghorpade-Tsfasman (2005)]

\[
n_\alpha = \det \left( q^{(j-i)(j-i-1)/2} \left[ \alpha_j - j + 1 \over i - j + 1 \right] q \right)_{1 \leq i, j \leq \ell}.
\]

[Ghorpade-Krattenthaler]
Let \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \in I(\ell, m) \). The dimension of \( C_{\alpha}(\ell, m) \) is the \( \ell \times \ell \) determinant:

\[
k_\alpha = \begin{vmatrix}
(\alpha_1) & 1 & 0 & \cdots & 0 \\
(\alpha_2 - 1) & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(\alpha_\ell - \ell + 1) & (\alpha_\ell - \ell) & (\alpha_\ell - 2) & \cdots & 1
\end{vmatrix}.
\]

If \( \alpha_1, \ldots, \alpha_\ell \) are in arithmetic progression, i.e., \( \alpha_i = c(i - 1) + d \ \forall i \) for some \( c, d \in \mathbb{Z} \), then

\[
k_\alpha = \frac{\alpha_1}{\ell!} \prod_{i=1}^{\ell-1} (\alpha_{i+1} - i) = \frac{\alpha_1}{\alpha_{\ell+1}} \binom{\alpha_{\ell+1}}{\ell}
\]

where \( \alpha_{\ell+1} = c\ell + d = \ell\alpha_2 + (1 - \ell)\alpha_1 \).

\[
k_\alpha = \sum_{s_1=p_1}^{\alpha_{p_1}} \sum_{s_2=p_2}^{\alpha_{p_2}} \cdots \sum_{s_u=p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \left( \frac{\alpha_{p_{i+1}} - \alpha_{p_i}}{s_{i+1} - s_i} \right),
\]
What do we know about the MDC?

Recall that the MDC states that $d(C_\alpha(\ell, m)) = q^{\delta_\alpha}$, where

$$\delta_\alpha := (\alpha_1 - 1) + \cdots + (\alpha_\ell - \ell).$$

- True if $\alpha = (m - \ell + 1, \ldots, m - 1, m)$. [Nogin]
- True if $\ell = 2$. [Hao Chen (2000)]; and independently, [Guerra-Vincenti (2002)].
- Lower bound for $d(C_\alpha(\ell, m))$ [G-V (2002)]:

$$\frac{q^{\alpha_1}(q^{\alpha_2} - q^{\alpha_1}) \cdots (q^{\alpha_\ell} - q^{\alpha_{\ell-1}})}{q^{1+2+\cdots+\ell}} \geq q^{\delta_\alpha - \ell}.$$

- MDC is true for $C_{(2,4)}(2, 4)$. [Vincenti (2001)]
- MDC is true for all Schubert divisors in $G_{\ell,m}$. [Ghorpade-Tsfasman (2005)]
- MDC is true, in general! [Xu Xiang (2008)]
Recall: \( \mu := \max \{ \ell, m - \ell \} + 1 \) and we had:

**Theorem (Nogin (1996), Ghorpade-Lachaud(2000))**

For \( 1 \leq r \leq \mu \), we have

\[
d_r (C(\ell, m)) = q^\delta + q^{\delta - 1} + \cdots + q^{\delta - r + 1}.
\]

One has the following counterpart from the other end.

**Theorem (Hansen-Johnsen-Ranestad (2007))**

On the other hand, for \( 0 \leq r \leq \mu \),

\[
d_{k-r} (C(\ell, m)) = n - (1 + q + \cdots + q^{r-1}).
\]

These results cover several initial and terminal elements of the weight hierarchy of \( C(\ell, m) \). Yet, a considerable gap remains.
Narrowing the gap

Examples:

- \((\ell, m) = (2, 5)\). Here \(k = 10\), \(\mu = 4\) and we know:
  
  \[d_1, \ldots, d_4\] as well as \(d_6, \ldots, d_{10}\).

  But \(d_5\) seems to be unknown.

- \((\ell, m) = (2, 6)\). Here \(k = 15\), \(\mu = 5\) and \(d_6, \ldots, d_9\) are not known.

- For \(C(2, m)\) with \(m \geq 2\), the values of \(d_r\) for \(m \leq r < \binom{m-1}{2}\) do not seem to be known.

Theorem (Hansen-Johansen-Ranestad (2007))

\[d_5(C(2, 5)) = q^6 + q^5 + 2q^4 + q^3 = d_4 + q^4.\]

Conjecture (Hansen-Johansen-Ranestad (2007))

\[d_r - d_{r-1}\] is always a power of \(q\).
Theorem (Ghorpade-Patil-Pillai (2009))

Assume that $\ell = 2$ and $m \geq 4$ so that

$$\mu = \max\{2, m - 2\} + 1 = m - 1 \quad \text{and} \quad k = \binom{m}{2}.$$

Then

$$d_{\mu+1}(C(2, m)) = d_\mu + q^{\delta-2}$$

and

$$d_{k-\mu-1}(C(2, m)) = n - (1 + q + \cdots + q^\mu + q^2).$$

Corollary. Complete weight hierarchy of $C(2, 6)$.

Remark. The proof of the above theorem uses a characterization of decomposable subspaces of $\wedge^\ell V$ where $V$ is an $m$-dimensional vector space, and this auxiliary result can be viewed as an algebraic analogue of the structure theorem for close families of $\ell$-subsets of an $m$-set.
Consider the (strict) Young tableau $Y = Y_m$ corresponding to the partition $(m - 1, m - 2, \ldots, 2, 1)$ of $k = \binom{m}{2}$ with $Y_{ij} = 2i + j - 3$ for $1 \leq i \leq m - 1$ and $1 \leq j \leq m - i$.

Then

$$n = C_q(Y) = \sum_{\nu \geq 0} c_\nu(Y) q^\nu,$$

where $c_\nu(Y) = \#$ of times $\nu$ appears in $Y$.

**Theorem (Ghorpade-Patil-Pillai)**

If $T_1, \ldots, T_h$ are strict subtableaux of $Y$ of area $r = k - s$, and

$$g_s(2, m) := \max\{C_q(T_1), \ldots, C_q(T_h)\},$$

then

$$d_s(C(2, m)) = n - g_s(2, m) \quad \text{for } 1 \leq s \leq k.$$
Fix an $m$-dimensional vector space $V$ and a sequence

$$\ell = (\ell_1, \ldots, \ell_s) \in \mathbb{Z}^s \text{ with } 0 < \ell_1 < \cdots < \ell_s < m.$$ 

We can consider the partial flag variety

$$\mathcal{F}_\ell(V) := \{ (V_1, \ldots, V_s) : V_1 \subset \cdots \subset V_s, \dim V_i = \ell_i \}.$$ 

Thanks to Plücker and Segre,

$$\mathcal{F}_\ell(V) \hookrightarrow \prod_{i=1}^{s} G_{\ell_i, m} \hookrightarrow \prod_{i=1}^{s} \mathbb{P}^{k_i-1} \hookrightarrow \prod_{i=1}^{s} \mathbb{P}^{(k_1 \cdots k_s)-1}$$

where $k_i = \binom{m}{\ell_i}$. As before,

$$\mathcal{F}_\ell(V)(\mathbb{F}_q) \sim \left[ n_\ell, k_\ell \right]_q \text{-code } C(\ell; m).$$
The parameters of the code $C(\ell; m)$ are known in a special case.

**Theorem (Rodier (2003))**

If $s = 2$ and $\ell = (1, m - 1)$, then

$$n_\ell = \frac{(q^m - 1)(q^{m-1} - 1)}{(q - 1)^2} \quad \text{and} \quad k_\ell = m^2 - 1.$$  

Moreover,

$$d(C(\ell; m)) = q^{2m-3} - q^{m-2}.$$
The length $n_\ell$ of $C(\ell, m)$

$$n_\ell = \begin{bmatrix} m \\ \ell_1, \ell_2 - \ell_1, \ldots, \ell_{s+1} - \ell_s \end{bmatrix} = \prod_{i=1}^{s} \left[ m - \ell_i - 1 \right]$$

where, by convention, $\ell_0 := 0$ and $\ell_{s+1} := m$. Equivalently, the length $n_\ell$ is given by

$$n_\ell = \sum_{\sigma \in W_\ell} q^{\text{inv}(\sigma)} = \sum_{\tau \in M_\ell} q^{\text{inv}(\tau)}$$

where $W_\ell$: permutations $\sigma \in S_m$ satisfying

$$\sigma(\ell_{i-1} + 1) < \sigma(\ell_{i-1} + 2) < \cdots < \sigma(\ell_i),$$

for $i = 1, \ldots, s + 1$, and $M_\ell$: permutations of the multiset

$$\{1^{\ell_1}, 2^{\ell_2 - \ell_1}, \ldots, s^{\ell_{s+1} - \ell_s - 1}, (s + 1)^{m - \ell_s} \}$$

and $\text{inv}$ denotes the number of inversions.
See, for example, [Ghorpade-Lachaud(2002)].
Given $\ell = (\ell_1, \ldots, \ell_s)$, consider the partition

$$m - \ell_1 > m - \ell_2 > \cdots > m - \ell_s \geq 1$$

and let $\lambda$ be the conjugate partition.

For example if $m = 8$ and $\ell = (1, 3, 6, 7)$, then the associated partition is $(7, 5, 2, 1)$ and the conjugate partition is $\lambda = (4, 3, 2, 2, 2, 1, 1)$. These partitions can be viewed as follows.
For each box \((i, j)\) in the (Young diagram of) \(\lambda\), let \(h_{(i,j)}\) be the hooklength at \((i, j)\), that is, the number of boxes in the hook at \((i, j)\). For example the hook at \((2, 2)\) in the partition \(\lambda = (4, 3, 2, 2, 2, 1, 1)\) is shown by shaded boxes and we have \(h_{(2,2)} = 5\).

**Theorem**

The dimension \(k_{\ell}\) of \(C(\ell, m)\) is given by

\[
k_{\ell} = \prod_{(i,j) \in \lambda} \frac{m + j - i}{h_{(i,j)}}.
\]

**Idea:** Use the connection between flag varieties and representations of linear groups together with classical results from Combinatorial Representation Theory.
**Example:** If $\ell = (1, m-1)$, then

$$\lambda = \text{conjugate of } (m-1,1) = (2,1,1,\ldots,1)_{m-2 \text{ times}}.$$

Hence by the above formula

$$k_\ell = \frac{m(m+1)(m-1)(m-2)\cdots(m-(m-2))}{m(1)(m-2)(m-3)\cdots1} = (m+1)(m-1) = m^2 - 1,$$

as is to be expected.
Another Example (Grassmann codes): If $\ell = (\ell)$, then
\[ \lambda = \text{conjugate of } (m - \ell) = (1, 1, \ldots, 1). \]

Hence by the above formula
\[
k_{\ell} = \frac{m(m-1)(m-2)\cdots(m-(m-\ell)+1)}{(m-\ell)(m-\ell-1)\cdots1}
\]
\[
= \frac{m!}{\ell!(m-\ell)!} = \binom{m}{\ell}
\]
as is to be expected.
Thank you!