Grassmann Varieties and Linear Codes

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V : vector space of dimension m over a field $\mathbb{F}$
For $1 \leq \ell \leq m$, we have the Grassmann variety:

$$G_{\ell,m} = G_\ell(V) := \{\ell\text{-dimensional subspaces of } V\}.$$

Plücker embedding:

$$G_{\ell,m} \hookrightarrow \mathbb{P}^{k-1} \quad \text{where} \quad k := \binom{m}{\ell}.$$

Explicitly, $\mathbb{P}^{k-1} = \mathbb{P}(\wedge^\ell V)$ and

$$W = \langle w_1, \dots, w_\ell \rangle \longleftrightarrow [w_1 \wedge \cdots \wedge w_\ell] \in \mathbb{P}(\wedge^\ell V).$$

For example, $G_{1,m} = \mathbb{P}^{m-1}$. In terms of coordinates,

$$W = \langle w_1, \dots, w_\ell \rangle \in G_\ell(V) \longleftrightarrow p(W) = (p_\alpha(A_W))_{\alpha \in I(\ell,m)},$$

where $A_W = (a_{ij})$ is a $\ell \times m$ matrix whose rows are (the coordinates of) a basis of W and $p_\alpha(A_W)$ is the α^{th} minor of A_W, viz., $\det \left(a_{i\alpha_j}\right)_{1 \leq i,j \leq \ell}$, and where

$$I(\ell, m) := \left\{\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell : 1 \le \alpha_1 < \cdots < \alpha_\ell \le m\right\}.$$

Facts:

- $G_{\ell,m}$ is a projective algebraic variety given by the common zeros of certain quadratic homogeneous polynomials in k variables. As a projective algebraic variety $G_{\ell,m}$ is nondegenerate, irreducible, nonsingular, and rational.
- There is a natural transitive action of GL_m on $G_{\ell,m}$ and if P_ℓ denotes the stabilizer of a fixed $W_0 \in G_{\ell,m}$, then P_ℓ is a maximal parabolic subgroup of GL_m and $G_{\ell,m} \simeq \mathrm{GL}_m/P_\ell$.
- If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then $G_{\ell,m}$ is a (real or complex) manifold, and its cohomology spaces and Betti numbers are explicitly known. In fact, $b_\nu = \dim H^{2\nu}(G_{\ell,m}; \mathbb{C})$ is precisely the number of partitions of ν into at most ℓ parts, each part $\le m - \ell$,

Grassmannian Over Finite Fields

Suppose $\mathbb{F} = \mathbb{F}_q$ is the finite field with q elements. Then $G_{\ell,m} = G_{\ell,m}(\mathbb{F}_q)$ is a finite set and its cardinality is given by the Gaussian binomial coefficient:

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q)\cdots(q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q)\cdots(q^\ell - q^{\ell-1})}.$$

This is a polynomial in q of degree $\delta := \ell(m-\ell)$ and in fact,

$$|G_{\ell,m}(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \sum_{\nu=0}^{\delta} b_\nu q^\nu = q^\delta + q^{\delta-1} + 2q^{\delta-2} + \cdots + 1.$$

Note that

$$\lim_{q\to 1} \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \binom{m}{\ell}.$$

(Linear) Codes

- $\mathbb{F}_q$: finite field with q elements.
- $[n, k]_q$-code: a k-dimensional subspace C of $\mathbb{F}_q^n$.
- Hamming weight of $c = (c_1, \ldots, c_n) \in \mathbb{F}_q^n$:

$$w_H(c) := \#\{i : c_i \neq 0\}.$$

- Hamming weight of a subcode D of C:

$$w_H(D) := \#\{i : \exists\, c = (c_1, \ldots, c_n) \in D \text{ with } c_i \neq 0\}.$$

- Minimum distance of a (linear) code C:

$$d(C) := \min\{w_H(c) : c \in C,\ c \neq 0\}.$$

- The r^{th} higher weight of C $(1 \leq r \leq k)$:

$$d_r(C) := \min\{w_H(D) : D \subseteq C,\ \dim D = r\}.$$

- C is nondegenerate if $C \not\subseteq$ coordinate hyperplane of $\mathbb{F}_q^n$, or equivalently, if $d_k(C) = n$.

Write $\mathbb{A}^\delta(\mathbb{F}_q) := \mathbb{F}_q^\delta = \{P_1, P_2, \dots, P_{q^\delta}\}$. Consider the evaluation map of the polynomial ring in δ variables:

$$\begin{aligned} \mathrm{Ev} : \mathbb{F}_q[X_1, \dots, X_\delta] &\to \mathbb{F}_q^{q^\delta} \\ f &\longmapsto \left(f(P_1), \dots, f(P_{q^\delta})\right), \end{aligned}$$

The r^{th} order generalized Reed-Muller code of length q^δ:

$$\mathrm{RM}(r,\delta) := \mathrm{Ev}\left(\mathbb{F}_q[X_1, \dots, X_\delta]_{\le r}\right) \quad \text{for } r < q.$$

This has dimension $\binom{\delta+r}{r}$ and minimum distance $(q-r)q^{\delta-1}$.

More generally, one can consider $\mathrm{RM}(r,\delta)$ for $r \le \delta(q-1)$.
Side Remark: An interesting new variant of R-M codes, called affine Grassmann codes, has recently been studied; cf. Beelen, Ghorpade, and Høoholdt, *IEEE Trans. Inform. Theory*, **56** (2010), 3166-3176 and **58** (2012), 3843-3855.

Role of Grassmann Varieties in Coding Theory

$G_{k,n}(\mathbb{F}_q)$ may be viewed as the 'moduli space' of all $[n,k]_q$-linear codes. An optimal class of codes, called MDS codes [these are $[n,k]_q$-linear codes for which the minimum distance d is the maximum possible, viz., $d = n - k + 1$] correspond to certain open strata of $G_{k,n}$. This viewpoint is useful for the following:

Problem

Given a prime power q and integers k, n with $1 \le k \le n$, determine

$$\gamma(q) = \gamma(q; k, n) = \#\left(\ [n,k]_q\text{-MDS codes}\right).$$

This is equivalent to determining

$$\#\,(\text{inequivalent representations over } \mathbb{F}_q \text{ of } U_{k,n})$$

where $U_{k,n}$ is the so-called uniform matroid.

The problem is open, in general.

A Quick Recap of Matroids

Matroid M on a finite set S: determined by the rank function $r = r_M : \mathcal{P}(S) \to \mathbb{Z}$ satisfying
(i) $0 \le r(I) \le |I|$, (ii) $I \subseteq J \Rightarrow r(I) \le r(J)$, and
(iii) $r(I \cup J) + r(I \cap J) \le r(I) + r(J)$.

- $I \subseteq S$ is independent if $r(I) = |I|$.
- base: maximal independent subset of S.
- uniform matroid $U_{k,n}$: the matroid on n elements s.t. any k elements form a base.
- representation of M over a field F: map $f : S \to V$, where V is a vector space over F, such that

$$\dim\left(\mathrm{span}\{f(x) : x \in I\}\right) = r_M(I) \quad \forall I \subseteq S.$$

- Representations $f_1 : S \to V_1$ and $f_2 : S \to V_2$ are equivalent if $\exists$ an isomorphism $\phi : V_1 \to V_2$ such that $\phi \circ f_1 = f_2$.

Reference: S. R. Ghorpade and G. Lachaud, Hyperplane sections of Grassmannians and the number of MDS linear codes, *Finite Fields Appl.*, **7**, (2001), 468–506.

A $[n,k]_q$-projective system is a collection $\mathcal{P}$ of n not necessarily distinct points in $\mathbb{P}^{k-1}$; this is nondegenerate if it is not contained in a hyperplane.

$$[n,k]_q\text{-code } \mathcal{C} \rightsquigarrow \; [n,k]_q\text{-projective system } \mathcal{P}$$

Conversely a nondegenerate $[n,k]_q$-projective system gives rise to a nondegenerate $[n,k]_q$-code, and the resulting correspondence is a bijection, up to equivalence. Note that

$$d(\mathcal{C}) = n - \max\{\#\mathcal{P}\cap H : H \text{ hyperplane of } \mathbb{P}^{k-1}\}.$$

and for $r = 1, \ldots, k$,

$$d_r(\mathcal{C}) = n - \max\{\#\mathcal{P}\cap E : E \text{ linear subvariety of codim } r \text{ in } \mathbb{P}^{k-1}\}.$$

Thanks to the Plücker embedding,

$$G_{\ell,m}(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k-1} \rightsquigarrow [n,k]_q\text{-code } C(\ell,m).$$

Length n is the Gaussian binomial coefficient:

$$n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{\ell-1})}{(q^\ell-1)(q^\ell-q)\cdots(q^\ell-q^{\ell-1})}.$$

and the dimension k is the binomial coefficient:

$$k = \binom{m}{\ell}.$$

Theorem (Ryan (1990, $q=2$), Nogin (1996, any q))

$$d\left(C(\ell,m)\right) = q^{\delta} \quad \textit{where} \quad \delta := \ell(m-\ell).$$

It may be noted that δ is the dimension of $G_{\ell,m}$ as a projective variety.

Higher Weights of Grassmann Codes

The first result in this direction is:

Theorem (Nogin (1996), Ghorpade-Lachaud(2000))

More generally, for $1 \le r \le \mu$ *we have*

$$d_r\left(C(\ell, m)\right) = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-r+1},$$

where $\mu := \max\{\ell, m-\ell\} + 1$.

The alternative proofs in Ghorpade-Lachaud(2000) used a characterization of the so-called close families of ℓ-subsets of the finite set $\{1, \ldots, m\}$. This can be viewed as an analogue for uniform hypergraphs of the elementary result in graph theory that

> *A simple graph in which any two edges are incident is either a star or a triangle.*

Note: $\mu = \max\{\ell, m-\ell\} + 1$ is usually much smaller than $k = \binom{m}{\ell}$ and so the above theorem doesn't give all the higher weights.

Schubert Codes

Let α be in $I(\ell, m)$, that is,

$$\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell, \ 1 \le \alpha_1 < \cdots < \alpha_\ell \le m.$$

Consider the corresponding Schubert variety

$$\Omega_\alpha := \{W \in G_{\ell,m} : \dim(W \cap A_{\alpha_i}) \ge i \ \forall i\},$$

where A_j is the span of the first j vectors in a fixed basis of our m-space. We have

$$\Omega_\alpha(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k_\alpha - 1} \rightsquigarrow [n_\alpha, k_\alpha]_q\text{-code } C_\alpha(\ell, m)$$

where

$$n_\alpha = |\Omega_\alpha(\mathbb{F}_q)| \text{ and } k_\alpha = |\{\beta \in I(\ell, m) : \beta \le \alpha\}|,$$

with $\le$ being the componentwise partial order:

$$\beta = (\beta_1, \ldots, \beta_\ell) \le \alpha = (\alpha_1, \ldots, \alpha_\ell) \Leftrightarrow \beta_i \le \alpha_i \ \forall i.$$

Minimum Distance of Schubert Codes

Proposition (Ghorpade-Lachaud(2000))

For any $\alpha \in I(\ell, m)$,

$$d\left(C_\alpha(\ell, m)\right) \leq q^{\delta_\alpha} \quad \text{where} \quad \delta_\alpha := \sum_{i=1}^{\ell} (\alpha_i - i).$$

It may be noted that when α is the "maximal element" $(m-\ell+1, \ldots, m-1, m)$ of $I(\ell, m)$, then $\Omega_\alpha = G_{\ell,m}$ while $\delta_\alpha = \delta = \ell(m-\ell)$ and so the above inequality is an equality. In fact, the following conjecture was made in the same paper:

Minimum Distance Conjecture (MDC)

For any $\alpha \in I(\ell, m)$,

$$\boxed{d\left(C_\alpha(\ell, m)\right) = q^{\delta_\alpha}.}$$

Length of Schubert Codes

- If $\ell = 2$ and $\alpha = (m - h - 1, m)$, then

$$n_\alpha = \frac{(q^m - 1)(q^{m-1} - 1)}{(q^2 - 1)(q - 1)} - \sum_{j=1}^{h} \sum_{i=1}^{j} q^{2m-j-2-i}$$

and

$$k_\alpha = \frac{m(m-1)}{2} - \frac{h(h+1)}{2}.$$

[Hao Chen (2000)]

- In general,

$$n_\alpha = \sum \prod_{i=0}^{\ell-1} \begin{bmatrix} \alpha_{i+1} - \alpha_i \\ k_{i+1} - k_i \end{bmatrix}_q q^{(\alpha_i - k_i)(k_{i+1} - k_i)}$$

where the sum is over $(k_1, \ldots, k_{\ell-1}) \in \mathbb{Z}^\ell$ satisfying $i \le k_i \le \alpha_i$ and $k_i \le k_{i+1}$ for $1 \le i \le \ell - 1$; by convention, $\alpha_0 = 0 = k_0$ and $k_\ell = \ell$. [Vincenti (2001)]

Length of Schubert Codes (Contd.)

- $n_\alpha = \sum_{\beta \le \alpha} q^{\delta_\beta}$, where $\delta_\beta = \sum_{i=1}^{\ell} (\beta_i - i)$.
  Ehresmann (1934); Ghorpade-Tsfasman (2005)
- Suppose α has $u+1$ consecutive blocks:
  $\alpha = (\alpha_1, \ldots, \alpha_{p_1}, \ \ldots, \ \alpha_{p_u+1}, \ldots, \alpha_{p_{u+1}})$. Then

  $$n_\alpha = \sum_{s_1=p_1}^{\alpha_{p_1}} \cdots \sum_{s_u=p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \lambda(\alpha_{p_i}, \alpha_{p_{i+1}}; s_i, s_{i+1})$$

  where, $s_0 = p_0 = 0$; $s_{u+1} = p_{u+1} = \ell$, and
  $\lambda(a, b; s, t) := \sum_{r=s}^{t} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} a-s \\ r-s \end{bmatrix}_q \begin{bmatrix} b-r \\ t-r \end{bmatrix}_q$.
  [Ghorpade-Tsfasman (2005)]
- $n_\alpha = \det \left(q^{(j-i)(j-i-1)/2} \begin{bmatrix} \alpha_j - j + 1 \\ i - j + 1 \end{bmatrix}_q \right)_{1 \le i,j \le \ell}$.
  [Ghorpade-Krattenthaler]

Dimension of Schubert Codes [G-Tsfasman (2005)]

- Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in I(\ell, m)$. The dimension of $C_\alpha(\ell, m)$ is the $\ell \times \ell$ determinant:

$$k_\alpha = \begin{vmatrix} \binom{\alpha_1}{1} & 1 & 0 & \ldots & 0 \\ \binom{\alpha_1}{2} & \binom{\alpha_2-1}{1} & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ \binom{\alpha_1}{\ell} & \binom{\alpha_2-1}{\ell-1} & \binom{\alpha_3-2}{\ell-2} & \ldots & \binom{\alpha_\ell-\ell+1}{1} \end{vmatrix}.$$

- If $\alpha_1, \ldots, \alpha_\ell$ are in arithmetic progression, i.e., $\alpha_i = c(i-1) + d \ \forall i$ for some $c, d \in \mathbb{Z}$, then

$$k_\alpha = \frac{\alpha_1}{\ell!} \prod_{i=1}^{\ell-1} (\alpha_{\ell+1} - i) = \frac{\alpha_1}{\alpha_{\ell+1}} \binom{\alpha_{\ell+1}}{\ell}$$

where $\alpha_{\ell+1} = c\ell + d = \ell\alpha_2 + (1-\ell)\alpha_1$.

- $k_\alpha = \sum_{s_1=p_1}^{\alpha_{p_1}} \sum_{s_2=p_2}^{\alpha_{p_2}} \cdots \sum_{s_u=p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \binom{\alpha_{p_{i+1}} - \alpha_{p_i}}{s_{i+1} - s_i}$,

What do we know about the MDC?

Recall that the MDC states that $d\left(C_\alpha(\ell, m)\right) = q^{\delta_\alpha}$, where $\delta_\alpha := (\alpha_1 - 1) + \cdots + (\alpha_\ell - \ell)$.

- True if $\alpha = (m - \ell + 1, \ldots, m - 1, m)$. [Nogin]
- True if $\ell = 2$. [Hao Chen (2000)]; and independently, [Guerra-Vincenti (2002)].
- Lower bound for $d\left(C_\alpha(\ell, m)\right)$ [G-V (2002)]:

$$\frac{q^{\alpha_1}(q^{\alpha_2} - q^{\alpha_1}) \cdots (q^{\alpha_\ell} - q^{\alpha_{\ell-1}})}{q^{1+2+\cdots+\ell}} \geq q^{\delta_\alpha - \ell}.$$

- MDC is true for $C_{(2,4)}(2,4)$. [Vincenti (2001)]
- MDC is true for all Schubert divisors in $G_{\ell,m}$. [Ghorpade-Tsfasman (2005)]
- MDC is true, in general! [Xu Xiang (2008)]

Back to Higher Weights of Grassmann Codes

Recall: $\mu := \max\{\ell, m - \ell\} + 1$ and we had:

Theorem (Nogin (1996), Ghorpade-Lachaud(2000))

For $1 \le r \le \mu$, *we have*

$$d_r\left(C(\ell, m)\right) = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1}.$$

One has the following counterpart from the other end.

Theorem (Hansen-Johnsen-Ranestad (2007))

On the other hand, for $0 \le r \le \mu$,

$$d_{k-r}\left(C(\ell, m)\right) = n - (1 + q + \cdots + q^{r-1}).$$

These results cover several initial and terminal elements of the weight hierarchy of $C(\ell, m)$. Yet, a considerable gap remains.

Examples:

- $(\ell, m) = (2, 5)$. Here $k = 10$, $\mu = 4$ and we know:

$$d_1, \ldots, d_4 \quad \text{as well as} \quad d_6, \ldots, d_{10}.$$

  But d_5 seems to be unknown.
- $(\ell, m) = (2, 6)$. Here $k = 15$, $\mu = 5$ and $d_6, \ldots, d_9$ are not known.
- For $C(2, m)$ with $m \geq 2$, the values of d_r for $m \leq r < \binom{m-1}{2}$ do not seem to be known.

Theorem (Hansen-Johnsen-Ranestad (2007))

$$d_5(C(2,5)) = q^6 + q^5 + 2q^4 + q^3 = d_4 + q^4.$$

Conjecture (Hansen-Johnsen-Ranestad (2007))

$d_r - d_{r-1}$ is always a power of q.

One step forward

Theorem (Ghorpade-Patil-Pillai (2009))

Assume that $\ell = 2$ and $m \geq 4$ so that

$$\mu = \max\{2, m-2\} + 1 = m - 1 \text{ and } k = \binom{m}{2}.$$

Then

$$d_{\mu+1}(C(2,m)) = d_\mu + q^{\delta-2}$$

and

$$d_{k-\mu-1}(C(2,m) = n - \left(1 + q + \cdots + q^\mu + q^2\right).$$

Corollary. Complete weight hierarchy of $C(2,6)$.

Remark. The proof of the above theorem uses a characterization of decomposable subspaces of $\wedge^\ell V$ where V is an m-dimensional vector space, and this auxiliary result can be viewed as an algebraic analogue of the structure theorem for close families of ℓ-subsets of an m-set.

Complete weight hierarchy of $C(2, m)$

Consider the (strict) Young tableau $Y = Y_m$ corresponding to the partition $(m-1, m-2, \ldots, 2, 1)$ of $k = \binom{m}{2}$ with

$$Y_{ij} = 2i + j - 3 \quad \text{for } 1 \le i \le m-1 \text{ and } 1 \le j \le m-i.$$

Then

$$n = \mathrm{C}_q(Y) = \sum_{\nu \ge 0} c_\nu(Y) q^\nu,$$

where $c_\nu(Y)$= # of times ν appears in Y.

Theorem (Ghorpade-Patil-Pillai)

If $T_1, \ldots, T_h$ are strict subtableaux of Y of area $r = k - s$, and

$$g_s(2, m) := \max\{\mathrm{C}_q(T_1), \ldots, \mathrm{C}_q(T_h)\},$$

then

$$d_s\left(C(2, m)\right) = n - g_s(2, m) \quad \text{for } 1 \le s \le k.$$

Codes associated to flag varieties

Fix an m-dimensional vector space V and a sequence

$$\underline{\ell} = (\ell_1, \ldots, \ell_s) \in \mathbb{Z}^s \quad \text{with} \quad 0 < \ell_1 < \cdots < \ell_s < m.$$

We can consider the partial flag variety

$$\mathcal{F}_{\underline{\ell}}(V) := \{(V_1, \ldots, V_s) : V_1 \subset \cdots \subset V_s, \ \dim V_i = \ell_i\}.$$

Thanks to Plücker and Segre,

$$\mathcal{F}_{\underline{\ell}}(V) \hookrightarrow \prod_{i=1}^{s} G_{\ell_i,m} \hookrightarrow \prod_{i=1}^{s} \mathbb{P}^{k_i - 1} \hookrightarrow \prod_{i=1}^{s} \mathbb{P}^{(k_1 \cdots k_s)-1}$$

where $k_i = \binom{m}{\ell_i}$. As before,

$$\mathcal{F}_{\underline{\ell}}(V)\,(\mathbb{F}_q) \rightsquigarrow [n_{\underline{\ell}}, k_{\underline{\ell}}]_q\text{-code } C(\underline{\ell}; m).$$

Special partial flags

The parameters of the code $C(\underline{\ell}; m)$ are known in a special case.

Theorem (Rodier (2003))

If $s = 2$ *and* $\underline{\ell} = (1, m-1)$, *then*

$$n_{\underline{\ell}} = \frac{(q^m - 1)(q^{m-1} - 1)}{(q-1)^2} \quad \textit{and} \quad k_{\underline{\ell}} = m^2 - 1.$$

Moreover,

$$d(C(\underline{\ell}; m)) = q^{2m-3} - q^{m-2}.$$

The length $n_{\underline{\ell}}$ of $C(\underline{\ell}, m)$

$$n_{\underline{\ell}} = \begin{bmatrix} m \\ \ell_1,\ \ell_2 - \ell_1, \ldots, \ell_{s+1} - \ell_s \end{bmatrix} = \prod_{i=1}^{s} \begin{bmatrix} m - \ell_{i-1} \\ \ell_i - \ell_{i-1} \end{bmatrix}$$

where, by convention, $\ell_0 := 0$ and $\ell_{s+1} := m$.
Equivalently, the length $n_{\underline{\ell}}$ is given by

$$n_{\underline{\ell}} = \sum_{\sigma \in W_{\underline{\ell}}} q^{\mathrm{inv}(\sigma)} = \sum_{\tau \in M_{\underline{\ell}}} q^{\mathrm{inv}(\tau)}$$

where $W_{\underline{\ell}}$: permutations $\sigma \in S_m$ satisfying

$$\sigma(\ell_{i-1} + 1) < \sigma(\ell_{i-1} + 2) < \cdots < \sigma(\ell_i),$$

for $i = 1, \ldots, s+1$, and $M_{\underline{\ell}}$: permutations of the multiset

$$\{1^{\ell_1}, 2^{\ell_2 - \ell_1}, \ldots, s^{\ell_s - \ell_{s-1}}, (s+1)^{m - \ell_s}\}$$

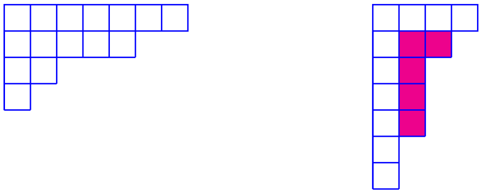
and inv denotes the number of inversions.
See, for example, [Ghorpade-Lachaud(2002)].

Dimension $k_{\underline{\ell}}$ of $C(\underline{\ell}, m)$

Given $\underline{\ell} = (\ell_1, \ldots, \ell_s)$, consider the partition

$$m - \ell_1 > m - \ell_2 > \cdots > m - \ell_s \geq 1$$

and let λ be the conjugate partition.
For example if $m = 8$ and $\underline{\ell} = (1, 3, 6, 7)$, then the associated partition is $(7, 5, 2, 1)$ and the conjugate partition is $\lambda = (4, 3, 2, 2, 2, 1, 1)$. These partitions can be viewed as folllows.



Description of $k_{\underline{\ell}}$ Contd.

For each box (i, j) in the (Young diagram of) λ, let $h_{(i,j)}$ be the hooklength at (i, j), that is, the number of boxes in the hook at (i, j). For example the hook at $(2, 2)$ in the partition $\lambda = (4, 3, 2, 2, 2, 1, 1)$ is shown by shaded boxes and we have $h_{(2,2)} = 5$.

Theorem

The dimension $k_{\underline{\ell}}$ of $C(\underline{\ell}, m)$ is given by

$$k_{\underline{\ell}} = \prod_{(i,j)\in\lambda} \frac{m + j - i}{h_{(i,j)}}.$$

Idea: Use the connection between flag varieties and representations of linear groups together with classical results from Combinatorial Representation Theory.

Rodier recovered!

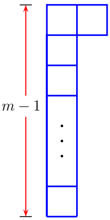
Example: If $\underline{\ell} = (1, m-1)$, then

$$\lambda = \text{conjugate of } (m-1, 1) = (2, \underbrace{1, 1, \ldots, 1}_{m-2 \text{ times}}).$$

Hence by the above formula

$$\begin{aligned} k_{\underline{\ell}} &= \frac{m(m+1)(m-1)(m-2)\cdots(m-(m-2))}{m(1)(m-2)(m-3)\cdots 1} \\ &= (m+1)(m-1) = m^2 - 1, \end{aligned}$$

as is to be expected.



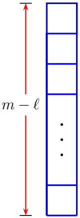
Another Example (Grassmann codes): If $\underline{\ell} = (\ell)$, then

$$\lambda = \text{conjugate of } (m - \ell) = (\underbrace{1, 1, \ldots, 1}_{m-\ell \text{ times}}).$$

Hence by the above formula

$$\begin{aligned} k_{\underline{\ell}} &= \frac{m(m-1)(m-2)\cdots(m-(m-\ell)+1)}{(m-\ell)(m-\ell-1)\cdots 1} \\ &= \frac{m!}{\ell!(m-\ell)!} = \binom{m}{\ell} \end{aligned}$$

as is to be expected.



Thank you!