The Iterated Random Function Problem

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Joint work with Ritam Bhaumik, Nilanjan Datta, Avijit Dutta, Ashwin Jha, Avradip Mandal, Nicky Mouha.

Outline of the Talk

- Iterated random function

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- Collision Probabilties and PRF analysis

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- We show upper bound using Coefficients H Technique

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- Unfortunately this is not true for random function (collision probability for a pair can be $O(rq/2^n)$)

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- All trails eventually lead to cycles

Collision Attack on f

Two main approaches:

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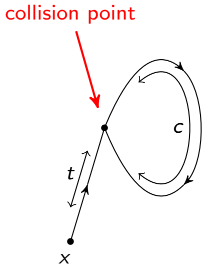

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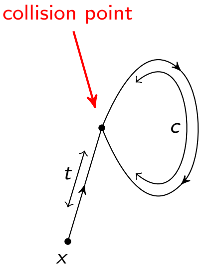

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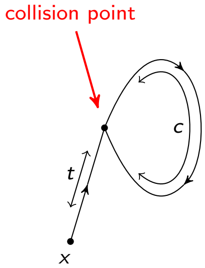

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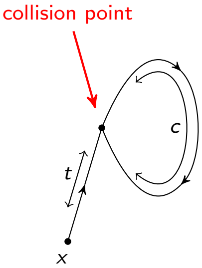

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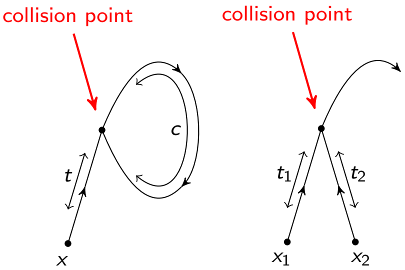

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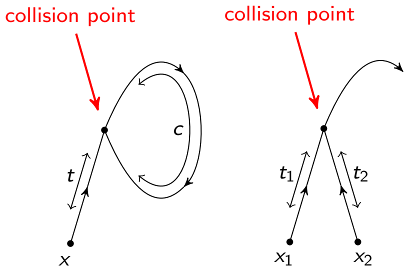

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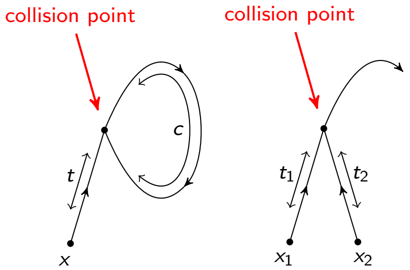

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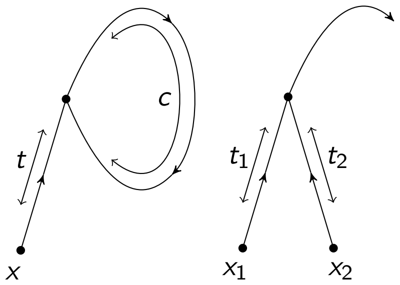

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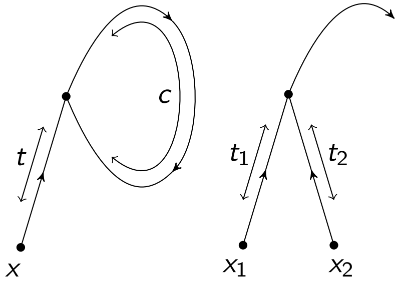

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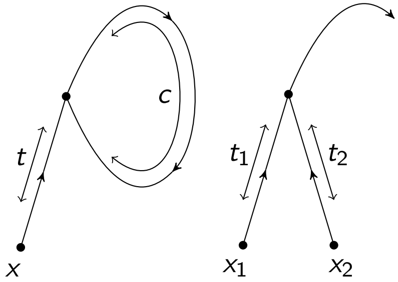

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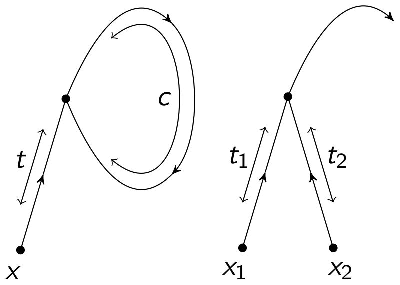

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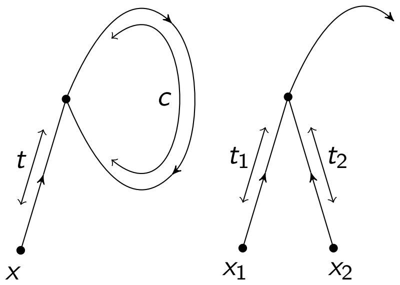

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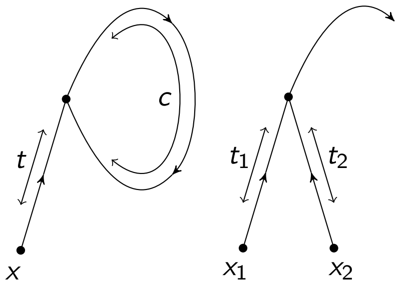

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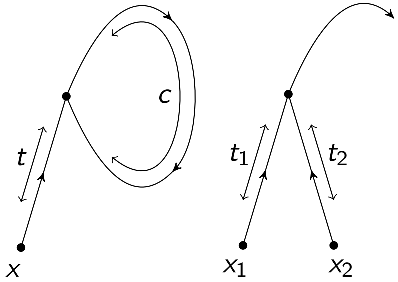

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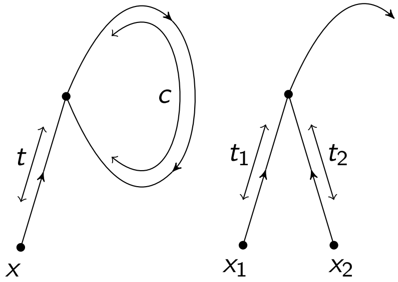

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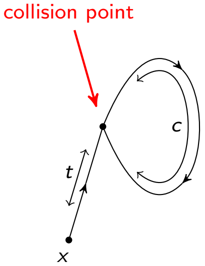

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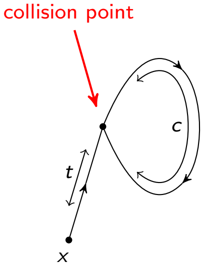

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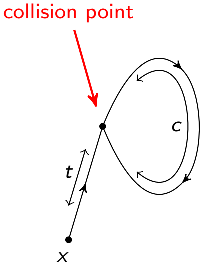

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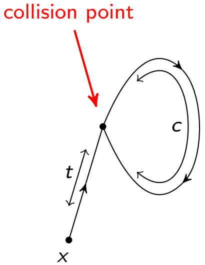

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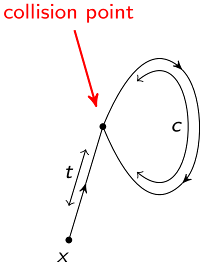

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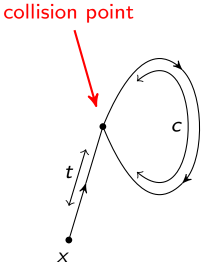

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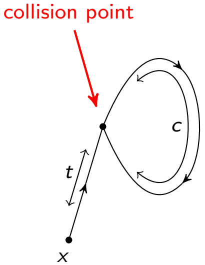

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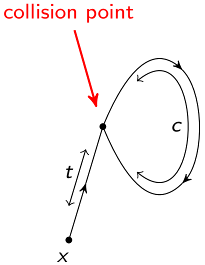

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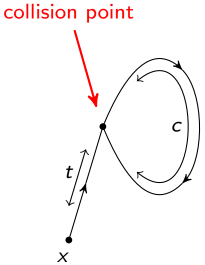

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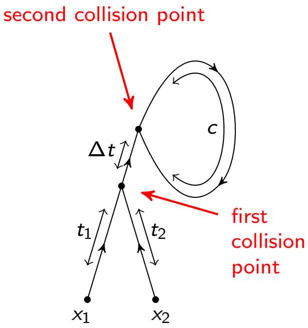

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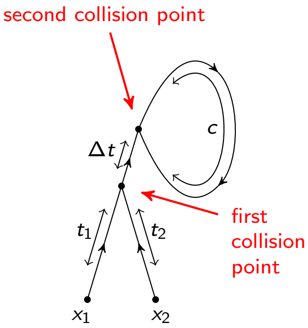

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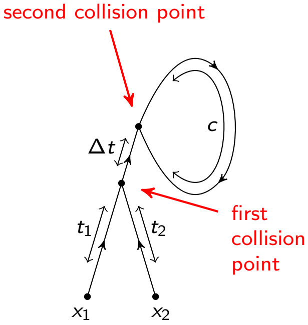

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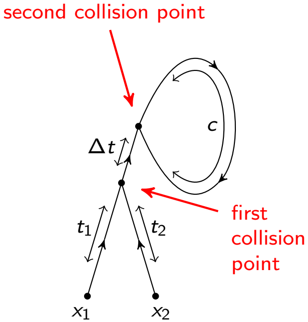

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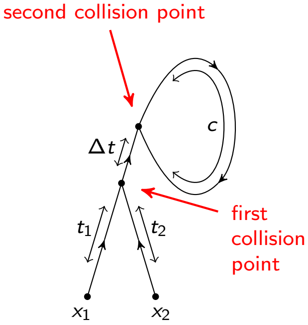

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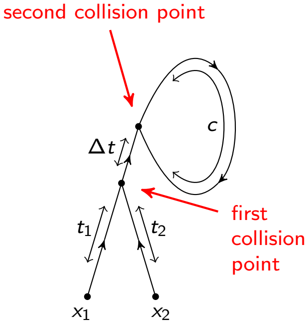

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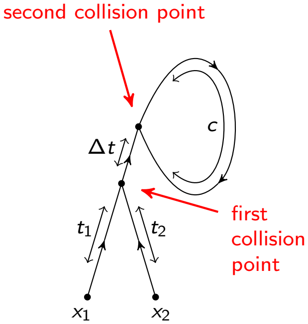

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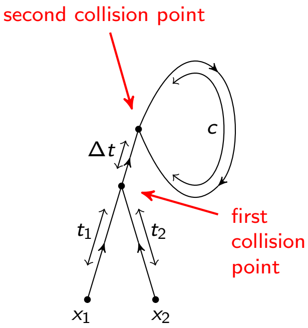

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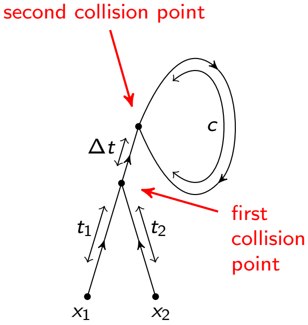

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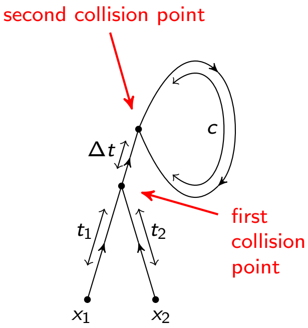

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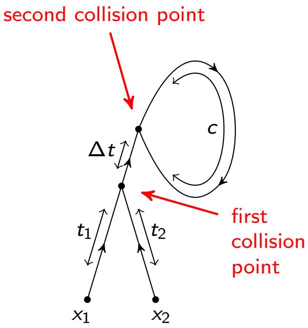

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- Security bound tight up to a factor of $(\log r)^3$

Lower Bound on Collision Probability

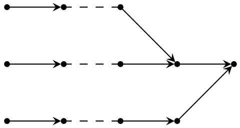
$x := (x_1, x_2, \ldots, x_q)$, x_i are distinct blocks from $\{0,1\}^n$.

Let $\mathbf{coll}_f(x_i; x_j)$ denote the event $f^{(\ell)}(x_i) = f^{(\ell)}(x_j)$ and $\mathbf{coll}_f(x) := \bigcup_{x_i, x_j \in x} \mathbf{coll}_f(x_i; x_j)$.

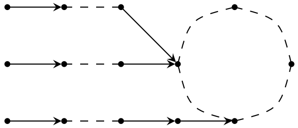
Lower Bound on Collision Probability

$$
\begin{aligned}
\Pr_f\left[\mathbf{coll}_f(x)\right] \geq{} & \sum_{i<j} \overbrace{\Pr_f[\mathbf{coll}_f(x_i; x_j)]}^{\mathbf{coll}_{i,j}} \\
& - 3 \sum_{i<j<k} \overbrace{\Pr_f[\mathbf{coll}_f(x_i; x_j) \cap \mathbf{coll}_f(x_j; x_k)]}^{\mathbf{coll}_{i,j,k}} \\
& - \frac{1}{2} \sum_{\substack{i<j, k<m \\ \{i,j\}\cap\{k,m\}=\emptyset}} \overbrace{\Pr_f[\mathbf{coll}_f(x_i; x_j) \cap \mathbf{coll}_f(x_k; x_m)]}^{\mathbf{coll}_{i,j,k,m}}
\end{aligned}
$$

Upper Bound on $\mathbf{coll}_{i,j,k}$



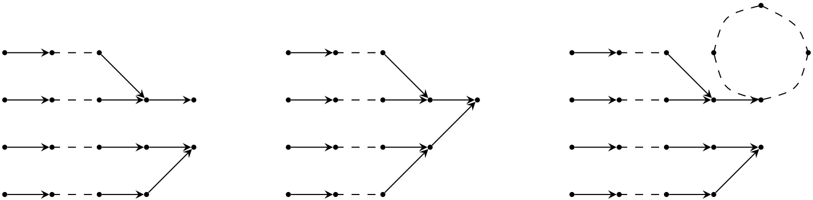
$\Pr[\text{Case 1}] \leq \frac{2\ell^2}{N^2}$



$\Pr[\text{Case 2}] \leq \frac{6\ell^6}{N^3}$

$$\boxed{\mathbf{coll}_{i,j,k} \leq \frac{2\ell^2}{N^2} + \frac{6\ell^6}{N^3}.}$$

Upper Bound on $\mathbf{coll}_{i,j,k,m}$



$\Pr[\texttt{Case 1}] \leq \frac{\ell^2}{N^2}$

$\Pr[\texttt{Case 2}] \leq \frac{6\ell^3}{N^3}$

$\Pr[\texttt{Case 3}] \leq \frac{2\ell^5}{N^3}$

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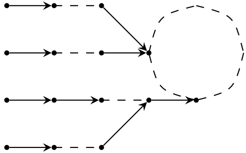

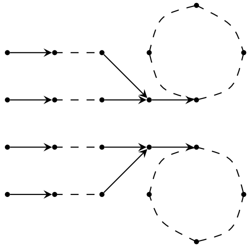

$$\Pr[\text{Case } 4] \leq \frac{24\ell^8}{N^4} \qquad \Pr[\text{Case } 5] \leq \frac{4\ell^8}{N^4}.$$

$$\boxed{\mathbf{coll}_{i,j,k,m} \leq \frac{\ell^2}{N^2} + \frac{6\ell^3 + 2\ell^5}{N^3} + \frac{28\ell^8}{N^4}.}$$

Lower Bound on $\mathbf{coll}_{i,j}$

Let cycle be the event that at least one of the walks (corresponding to x_i and x_j) has a cycle.

$$\mathbf{coll}_{i,j|\neg\texttt{cycle}} = \frac{\ell}{N} \qquad \Pr[\texttt{cycle}] \leq \frac{2\ell^2}{N}.$$

$$\boxed{\mathbf{coll}_{i,j} \geq \frac{\ell}{N}\left(1 - \frac{2\ell^2}{N}\right).}$$

Main Result on Lower Bound

Lower Bound Theorem

Let $x := (x_1, \ldots, x_q) \in (\{0,1\}^n)^q$ be a q tuple of distinct inputs. For $\ell, q \geq 3$, $\frac{q^2\ell}{N} < 1$ and $\ell < \min(\frac{N}{5184}, \frac{N^{\frac{1}{2}}}{4\sqrt{3}}, \frac{N^{\frac{1}{3}}}{\sqrt[3]{36}})$, we have

$$\Pr[\mathbf{coll}_f(x)] \geq \frac{q^2\ell}{12N}.$$

Example

Collision for $N = 2^{64}$. Hence taking $q = \sqrt{20} \cdot 2^{\frac{64}{3}}$, $\ell = 0.1 \times 2^{\frac{64}{3}}$, we get $\delta = 0.499$. $\square$

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THANK YOU

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