Chapter 4: Identical Particles

These our actors,
As I foretold you, were all spirits and
Are melted into air, into thin air:
And, like the baseless fabric of this vision,
The cloud-capp'd towers, the gorgeous palaces,
The solemn temples, the great globe itself,
Yea, all which it inherit, shall dissolve
And, like this insubstantial pageant faded,
Leave not a rack behind.

William Shakespeare, *The Tempest*

4.1. QUANTUM STATES OF IDENTICAL PARTICLES

In the previous chapter, we discussed how quantum theory applies to systems of multiple particles. That discussion omitted an important feature of multi-particle systems, namely the fact that particles of the same type have absolutely identical properties. It turns out that indistinguishability imposes a strong constraint on multi-particle quantum mechanics. Exploring this idea will lead us to a fundamental re-interpretation of what "particles" are.

4.1.1. Indistinguishability and exchange symmetry

Suppose we have two particles of the same type, e.g. two electrons. It is a fact of Nature that all electrons have identical physical properties: the same mass, same charge, same total spin, etc. As a consequence, the single-particle Hilbert spaces of the two electrons must be mathematically identical. If the single-electron Hilbert space is $\mathscr{H}^{(1)}$, the two-electron Hilbert space should be

$$\mathscr{H}^{(2)} = \mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)}. \tag{4.1}$$

Since the two electrons have identical properties, any Hamiltonian must treat them on the same footing. An example of such a Hamiltonian is

$$\hat{H} = \frac{1}{2m_e}\Big(|\hat{\mathbf{p}}_1|^2 + |\hat{\mathbf{p}}_2|^2\Big) + \frac{e^2}{4\pi\varepsilon_0|\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2|}, \tag{4.2}$$

consisting of the non-relativistic kinetic energies and the Coulomb potential energy. Operators $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{r}}_1$ act on electron 1, while $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{r}}_2$ act on electron 2. Swapping $\hat{\mathbf{p}}_1 \leftrightarrow \hat{\mathbf{p}}_2$ and $\hat{\mathbf{r}}_1 \leftrightarrow \hat{\mathbf{r}}_2$ leaves $\hat{H}$ unchanged.

From our present understanding of multi-particle quantum mechanics, the tensor product space (4.1) allows for the definition of observables and states that single out individual slots in the tensor product. For example, we can write down an observable

$$\hat{p}_{z,1} = \hat{p}_z \otimes \hat{I} \tag{4.3}$$

that singles out the momentum of the particle in slot 1. Likewise, we can define a state

$$|\mathbf{r}_1\rangle|\mathbf{r}_2\rangle, \tag{4.4}$$

which says that the particle in slot 1 is at position $\mathbf{r}_1$, and the particle in slot 2 is in a different position $\mathbf{r}_2$. If such observables and states are allowed, then the slot number has physical meaning. In principle, we can construct experiments that can identify a given particle as being in slot 1 or slot 2.

Now, let us take the point of view of fundamental physics, and imagine that the universe contains only these two particles. From this fundamental perspective, what we call slot 1 or slot 2 should just be a matter of convention, not a physically meaningful (i.e., observable) degree of freedom. Contrary to the previous paragraph, it seems reasonable to demand that the particles are **indistinguishable**, in the sense that there is no physically possible method to perform a measurement to ascertain whether a given particle occupies slot 1 or slot 2.

How might indistinguishability be imposed on the theory? To help figure this out, let us first introduce a symmetry called **exchange symmetry**, which refers to the exchange of the slots in the tensor product. The Hamiltonian (4.2) evidently possess this symmetry. To formulate this more rigorously, we define the **exchange operator**, $\hat{P}$, as follows. Let $\{|\mu\rangle\}$ be a basis for $\mathscr{H}^{(1)}$, so that tensor products of the form $|\mu\rangle|\nu\rangle$ (constructed from vectors both drawn from that basis) form a basis for $\mathscr{H}^{(2)}$. Then $\hat{P}$ is the linear operator satisfying

$$\hat{P}|\mu\rangle|\nu\rangle = |\nu\rangle|\mu\rangle \tag{4.5}$$

for all μ, ν. Thus, when applied to general vectors,

$$\begin{aligned}\hat{P}\Big(\sum_{\mu\nu}\psi_{\mu\nu}|\mu\rangle|\nu\rangle\Big) &= \sum_{\mu\nu}\psi_{\mu\nu}|\nu\rangle|\mu\rangle \\ &= \sum_{\mu\nu}\psi_{\nu\mu}|\nu\rangle|\mu\rangle \quad \text{(interchanging } \mu\leftrightarrow\nu \text{ in the double sum)}.\end{aligned} \tag{4.6}$$

The exchange operator has the following properties:

1. $\hat{P}^2 = \hat{I}$, where $\hat{I}$ is the identity operator.
2. $\hat{P}$ is both unitary and Hermitian (see Exercise 1).
3. The "slot-swapping" effect of $\hat{P}$ is independent of the basis of $\mathscr{H}^{(1)}$ (see Exercise 1).
4. $\hat{P}$ commutes with any 2-particle operator that treats the two particles on equal footing, like the Hamiltonian in Eq. (4.2) (see Exercise 2).

According to Noether's theorem, any symmetry implies a conservation law. The exchange operator $\hat{P}$ is both Hermitian *and* unitary, so we can take the conserved quantity to be its eigenvalue, p, which is called the **exchange parity**. Since $\hat{P}^2 = \hat{I}$, the possibilities are:

$$p = \begin{cases} +1 & (\text{"symmetric state"}), \text{ or} \\ -1 & (\text{"antisymmetric state"}). \end{cases} \tag{4.7}$$

If the 2-particle system is initially in an eigenstate of $\hat{P}$ with exchange parity p, and the Hamiltonian $\hat{H}$ commutes with $\hat{P}$ (as it should), then p is conserved for all time.

The exchange symmetry concept generalize to systems of more than two particles. For N particles, we can define a set of exchange operators $\hat{P}_{ij}$ such that

$$\hat{P}_{ij}\,|\mu\rangle\cdots\underbrace{|\nu\rangle}_{\text{slot } i}\cdots\underbrace{|\lambda\rangle}_{\text{slot } j}\cdots = |\mu\rangle\cdots\underbrace{|\lambda\rangle}_{\text{slot } i}\cdots\underbrace{|\nu\rangle}_{\text{slot } j}\cdots, \tag{4.8}$$

where $i, j \in \{1, 2, \dots, N\}$ and $i < j$. Note that the subscripts in $\hat{P}_{ij}$ refer to the slots being swapped, not matrix indices.

We now postulate that for any system of N identical particles,

1. Any physically valid quantum states must be an eigenstate of every $\hat{P}_{ij}$.
2. The exchange parities p_{ij} are all $+1$, or all -1. The value is determined by the particle type; for example, photons always have $p_{ij} = 1$.
3. Any physically valid observable, including the Hamiltonian, commutes with every $\hat{P}_{ij}$.

If these statements hold, then we will have achieved our previously-stated goal of making the particles indistinguishable. At any time, the N-particle state is an eigenstate of every $\hat{P}_{ij}$, so no slot in the tensor product is singled out. The Hamiltonian commutes with every $\hat{P}_{ij}$, so this property is conserved under time evolution. Moreover, no observable is allowed to treat the slots on unequal footing.

Let us henceforth assume that these postulates are true facts about Nature. Later, we will circle back to examine the condition of particle distinguishability, which motivated these developments, from a different point of view (see Section 4.3.4). In Section 4.1.4, we will discuss the circumstances under which particles can be treated as distinguishable, despite being indistinguishable at a fundamental level.

The second postulate, in the above list, stated that the particle type determines the exchange parity. There are two cases:

- Particles with exchange parity $+1$ are called **bosons**. They include the elementary particles that carry the fundamental forces: photons (carriers of the electromagnetic force), gluons (carriers of the strong nuclear force), and W and Z bosons (carriers of the weak nuclear force). Bosons that are composite particles (particles made of smaller particles) include alpha particles (helium-4 nuclei) and phonons (particles of sound).
- Particles with exchange parity -1 are called **fermions**. These include the elementary particles making up ordinary matter: electrons and quarks, as well as protons and neutrons, which are composite particles made of three quarks each. There are many other elementary fermions including positrons, muons, and neutrinos.

Whether a particle type is bosonic or fermionic is often referred to as the choice of "**particle statistics**". This is because the exchange parity affects the statistical behavior of large ensembles of particles: bosons obey the Bose-Einstein distribution, while fermions obey the Fermi-Dirac distribution. This has profound implications for statistical mechanics, falling outside the scope of this course.

For elementary particles, the **spin-statistics theorem** states that particles with integer spin (i.e., total spin equal to an integer multiple of $\hbar$) are bosons, whereas particles with half-integer spin are fermions. For instance, photons are spin-1 bosons, and electrons are spin-1/2 fermions. Though the spin-statistics theorem is simple to state, its proof is surprisingly complicated, seeming to require at minimum the combination of quantum mechanics with relativity. In non-relativistic settings, the spin-statistics theorem holds in the vast majority of cases (e.g., electronic quasiparticles in ordinary solid-state materials are spin-1/2 fermions, like elementary electrons), but it lacks the status of a rigorous theorem and is violable. The development of materials containing emergent particles that violate the spin-statistics theorem is an active area of research in condensed matter physics.

Incidentally, exchange operators are not the only way to formulate particle indistinguishability. Leinaas and Myrheim developed an interesting alternative description of indistinguishable particles based on a redefinition of the quantum wavefunction. Normally, a multi-particle wavefunction $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)$ is a function that maps a vector $(\mathbf{r}_1, \dots, \mathbf{r}_N)$ to a complex number. In Leinaas and Myrheim's view, for identical particles the input is not a vector, but a more complicated mathematical object with built-in invariance under slot exchange: $(\dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots) \equiv (\dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots)$. By analyzing the properties of such functions, bosonic or fermionic wavefunctions can be constructed. One startling outcome of this analysis is that the distinction between fermions and bosons is not absolute. In two spatial dimensions, there is the fascinating possibility of particles called **anyons**, which are intermediate between fermions and bosons. A proper discussion of anyons requires knowledge about magnetic vector potentials in quantum mechanics, which will be discussed in Chapter 5; after having gone through that chapter, you may refer to Appendix F, which gives a brief introduction to the theory of anyons.

4.1.2. Bosons

A state of N bosons must be symmetric under every possible exchange operator:

$$\hat{P}_{ij}\,|\psi\rangle = |\psi\rangle. \tag{4.9}$$

There is a standard way to construct multi-particle states obeying this symmetry condition. First, consider a two-boson system ($N = 2$). If both bosons occupy the same single-particle state, $|\mu\rangle \in \mathscr{H}^{(1)}$, Eq. (4.9) can be satisfied using the two-boson state

$$|\mu, \mu\rangle = |\mu\rangle|\mu\rangle. \tag{4.10}$$

Next, suppose the bosons occupy two single-particle states, $|\mu\rangle$ and $|\nu\rangle$, which are orthonormal. The two-boson state cannot be $|\mu\rangle|\nu\rangle$, as this fails to satisfy Eq. (4.9). Instead, let

$$|\mu, \nu\rangle = \frac{1}{\sqrt{2}}\Big(|\mu\rangle|\nu\rangle + |\nu\rangle|\mu\rangle\Big). \tag{4.11}$$

You can check that $1/\sqrt{2}$ is the right normalization factor. This state has the right exchange symmetry:

$$\hat{P}_{12}\,|\mu, \nu\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|\nu\rangle|\mu\rangle + |\mu\rangle|\nu\rangle\Big) \;=\; |\mu, \nu\rangle. \tag{4.12}$$

We can generalize the above construction to N bosons, using the N single-particle states

$$|\phi_1\rangle,\ |\phi_2\rangle,\ |\phi_3\rangle,\ \dots,|\phi_N\rangle. \tag{4.13}$$

Each $|\phi_j\rangle$ is drawn from an orthonormal basis $\{|\mu\rangle\}$ for $\mathscr{H}^{(1)}$. Duplicate draws are allowed, meaning that it is okay for two or more $|\phi_j\rangle$'s to be drawn from the same basis state. Then the N-boson state is

$$|\phi_1, \phi_2, \dots, \phi_N\rangle = \mathcal{N} \sum_p \Big(|\phi_{p(1)}\rangle|\phi_{p(2)}\rangle|\phi_{p(3)}\rangle \cdots |\phi_{p(N)}\rangle\Big). \tag{4.14}$$

The sum is taken over all $N!$ permutations of $(1, 2, \dots, N)$, with $p(j)$ denoting what j is permuted to. The normalization constant $\mathcal{N}$ will be discussed later.

Example—Consider a system of $N = 3$ bosons, with two particles in single-particle state $|\mu\rangle$ and one in an orthogonal single-particle state $|\nu\rangle$. That is to say,

$$\begin{aligned} |\phi_1\rangle &= |\phi_2\rangle = |\mu\rangle, \\ |\phi_3\rangle &= |\nu\rangle. \end{aligned} \tag{4.15}$$

To get the 3-boson state, we plug this into Eq. (4.14). The sum has one term for each permutation of the sequence $(1, 2, 3)$, and there are $3! = 6$ such permutations:

$$\begin{matrix} (1,2,3), & (2,3,1), & (3,1,2) \\ (1,3,2), & (3,2,1), & (2,1,3). \end{matrix} \tag{4.16}$$

Hence we get (we'll discuss the normalization later):

$$\begin{aligned} |\phi_1, \phi_2, \phi_3\rangle = \frac{1}{\sqrt{12}}\Big(& |\phi_1\rangle|\phi_2\rangle|\phi_3\rangle + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle + |\phi_3\rangle|\phi_1\rangle|\phi_2\rangle \\ &+ |\phi_1\rangle|\phi_3\rangle|\phi_2\rangle + |\phi_3\rangle|\phi_2\rangle|\phi_1\rangle + |\phi_2\rangle|\phi_1\rangle|\phi_3\rangle\Big) \\ = \frac{1}{\sqrt{3}}\ &\Big(|\mu\rangle|\mu\rangle|\nu\rangle + |\mu\rangle|\nu\rangle|\mu\rangle + |\nu\rangle|\mu\rangle|\mu\rangle\Big). \end{aligned} \tag{4.17}$$

We can verify that this is an eigenstate of every exchange symmetry operator:

$$\hat{P}_{12}|\phi_1, \phi_2, \phi_3\rangle = \frac{1}{\sqrt{3}}\Big(|\mu\rangle|\mu\rangle|\nu\rangle + |\nu\rangle|\mu\rangle|\mu\rangle + |\mu\rangle|\nu\rangle|\mu\rangle\Big) = |\phi_1, \phi_2, \phi_3\rangle \tag{4.18}$$

$$\hat{P}_{23}|\phi_1, \phi_2, \phi_3\rangle = \frac{1}{\sqrt{3}}\Big(|\mu\rangle|\nu\rangle|\mu\rangle + |\mu\rangle|\mu\rangle|\nu\rangle + |\nu\rangle|\mu\rangle|\mu\rangle\Big) = |\phi_1, \phi_2, \phi_3\rangle \tag{4.19}$$

$$\hat{P}_{13}|\phi_1, \phi_2, \phi_3\rangle = \frac{1}{\sqrt{3}}\Big(|\nu\rangle|\mu\rangle|\mu\rangle + |\mu\rangle|\nu\rangle|\mu\rangle + |\mu\rangle|\mu\rangle|\nu\rangle\Big) = |\phi_1, \phi_2, \phi_3\rangle. \tag{4.20}$$

To see why the state $|\phi_1, \dots, \phi_N\rangle$ defined in Eq. (4.14) is symmetric under every exchange operator, consider a representative term in the sum, corresponding to a permutation p. When an exchange operator $\hat{P}_{ij}$ acts on it, the single-particle states in slots i and j swap places:

$$\hat{P}_{ij}\Big(\cdots \underbrace{|\phi_{p(i)}\rangle}_{\text{slot } i} \cdots \underbrace{|\phi_{p(j)}\rangle}_{\text{slot } j} \cdots\Big) = \Big(\cdots \underbrace{|\phi_{p(j)}\rangle}_{\text{slot } i} \cdots \underbrace{|\phi_{p(i)}\rangle}_{\text{slot } j} \cdots\Big). \tag{4.21}$$

The tensor product on the right represents a different permutation of $(1, \dots, N)$, which we denote by p'. The p' permutation occurs as another term in $|\phi_1, \dots, \phi_N\rangle$, and applying $\hat{P}_{ij}$ turns this term into the p term. Thus, $\hat{P}_{ij}$ merely swaps two terms in the sum, without affecting the overall sum. This holds for every p, so $\hat{P}_{ij}$ leaves $|\phi_1, \dots, \phi_N\rangle$ unchanged.

If the preceding paragraph is unclear, try tracing its logic using Eqs. (4.18)–(4.20).

As for the normalization constant in Eq. (4.14), it can be shown that

$$\mathcal{N} = \sqrt{\frac{1}{N!\,\prod_\mu n_\mu!}}, \tag{4.22}$$

where n_μ denotes the number of particles occupying the single-particle basis state $|\varphi_\mu\rangle$, such that $\sum_\mu n_\mu = N$. The proof of Eq. (4.22) is left as an exercise (Exercise 3).

In place of Eq. (4.14), N-boson states can alternatively be expressed as

$$|\phi_1, \phi_2, \dots, \phi_N\rangle = \mathcal{M} \sum_{p,\,\text{no dupes}} \Big(|\phi_{p(1)}\rangle|\phi_{p(2)}\rangle|\phi_{p(3)}\rangle \cdots |\phi_{p(N)}\rangle\Big), \tag{4.23}$$

where the sum omits duplicate permutations (i.e., those yielding the same tensor product). For instance, in the previous example of $|\phi_1\rangle = |\phi_2\rangle = |\mu\rangle$ and $|\phi_3\rangle = |\nu\rangle$, instead of having $|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle$ and $|\phi_2\rangle|\phi_1\rangle|\phi_3\rangle$ appear separately in the sum, we only write it once, as both permutations ultimately refer to $|\mu\rangle|\mu\rangle|\nu\rangle$. If we adopt this convention, the normalization condition (4.22) is replaced by

$$\mathcal{M} = \sqrt{\frac{\prod_\mu n_\mu!}{N!}}. \tag{4.24}$$

The N-boson states constructed using Eq. (4.14)—or, if you prefer, Eq. (4.23)—have an important property: if we build two such states from different sets of single-particle states, they are orthogonal. To prove this, take two sets of single-particle states, $\{\phi_1, \dots, \phi_N\}$ and $\{\phi_1, \dots, \phi'_N\}$. The two N-boson states, constructed by Eq. (4.14), have the inner product

$$\langle\phi_1, \dots, \phi_N|\phi'_1, \dots, \phi'_N\rangle = \mathcal{N}\mathcal{N}' \sum_{pp'} \langle\phi_{p(1)}|\phi'_{p'(1)}\rangle \dots \langle\phi_{p(N)}|\phi'_{p'(N)}\rangle. \tag{4.25}$$

Take each term in the double sum, involving permutations p and p'. If *every* bra $\langle\phi_{p(j)}|$ matches the corresponding ket $|\phi_{p'(j)}\rangle$, the term is unity; otherwise it is zero since at least one of the bra-kets in the product vanishes. There is some p, p' achieving this condition if and only if the two sets of single-particle states are equivalent: i.e., $(\phi_1, \dots, \phi_N) \equiv (\phi'_1, \dots, \phi'_N)$ up to a permutation, in which case $|\phi_1, \dots, \phi_N\rangle$ and $|\phi'_1, \dots, \phi'_N\rangle$ refer to the same state.

4.1.3. Fermions

A state of N fermions must be antisymmetric under every exchange operator $\hat{P}_{ij}$:

$$\hat{P}_{ij}\,|\psi\rangle = -|\psi\rangle. \tag{4.26}$$

Similar to the bosonic case, we can construct N-fermion states from a set of single-particle states. First, consider $N=2$. If the fermions occupy the single-particle states $|\mu\rangle$ and $|\nu\rangle$, which are orthonormal, then the appropriate 2-fermion state is

$$|\mu,\nu\rangle = \frac{1}{\sqrt{2}}\Big(|\mu\rangle|\nu\rangle - |\nu\rangle|\mu\rangle\Big). \tag{4.27}$$

We can easily check that this is antisymmetric:

$$\hat{P}_{12}|\mu,\nu\rangle = \frac{1}{\sqrt{2}}\Big(|\nu\rangle|\mu\rangle - |\mu\rangle|\nu\rangle\Big) = -|\mu,\nu\rangle. \tag{4.28}$$

If $|\mu\rangle$ and $|\nu\rangle$ are the same single-particle state, Eq. (4.27) doesn't work, since the two terms would cancel to give the zero vector, which is not a valid quantum state. This is a manifestation of the **Pauli exclusion principle**, which states that two fermions cannot occupy the same single-particle state.

To generalize this, let $|\phi_1\rangle$, $|\phi_2\rangle$, $\dots$, $|\phi_N\rangle$ be a set of N orthonormal single-particle states, which are occupied by identical fermions. Then the N-fermion state is

$$|\phi_1,\dots,\phi_N\rangle = \frac{1}{\sqrt{N!}}\sum_p s(p)\,|\phi_{p(1)}\rangle|\phi_{p(2)}\rangle\cdots|\phi_{p(N)}\rangle. \tag{4.29}$$

Here, the sum is taken over every permutation p of the sequence $\{1,2,\dots,N\}$, and each term in the sum has a coefficient $s(p)=\pm1$ denoting the **parity** of permutation p, defined below. The $1/\sqrt{N!}$ prefactor can be shown to normalize $|\phi_1,\dots,\phi_N\rangle$ to unity.

Any permutation p can be generated by starting from the trivial sequence $(1,2,\dots,N)$, and repeatedly applying transpositions (i.e., choosing two slots and swapping them). The parity $s(p)$ is defined as $+1$ if p is generated by an even number of transpositions, and -1 if it is generated by an odd number of transpositions. Although it is generally possible to generate p from different sequences of transpositions, a result called the "parity theorem" states that these all give the same $s(p)$. In other words, $s(p)$ is a property of p itself.

Let's look at a couple of examples.

Example—Consider $N=2$. The sequence $(1,2)$ has two permutations:

$$\begin{aligned} p_1 &: (1,2)\rightarrow(1,2), \quad s=+1 \\ p_2 &: (1,2)\rightarrow(2,1), \quad s=-1. \end{aligned} \tag{4.30}$$

Plugging these into Eq. (4.29) yields

$$|\phi_1,\phi_2\rangle = \frac{1}{\sqrt{2}}\Big(|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle\Big),$$

which is equivalent to the previously-discussed example (4.27).

Example—For $N = 3$, the sequence $(1, 2, 3)$ has $3! = 6$ permutations:

$$\begin{aligned} p_1 &: (1,2,3) \rightarrow (1,2,3), \quad s = +1 \\ p_2 &: (1,2,3) \rightarrow (2,1,3), \quad s = -1 \\ p_3 &: (1,2,3) \rightarrow (2,3,1), \quad s = +1 \\ p_4 &: (1,2,3) \rightarrow (3,2,1), \quad s = -1 \\ p_5 &: (1,2,3) \rightarrow (3,1,2), \quad s = +1 \\ p_6 &: (1,2,3) \rightarrow (1,3,2), \quad s = -1. \end{aligned} \tag{4.31}$$

In this list, we start from $(1, 2, 3)$ and apply one transposition at a time to generate the next line. Hence, the signs of $s(p)$ alternate. This sequence of transpositions is not unique; for example, we can generate p_4 by starting from $(1, 2, 3)$ and transposing 1 and 3. This involves one transposition, rather than three transpositions as in the above list. But both routes give the same parity, $s(p_4) = -1$.

Plugging these permutations into Eq. (4.29), we obtain the 3-fermion state

$$\begin{aligned} |\phi_1, \phi_2, \phi_3\rangle = \frac{1}{\sqrt{6}}\Big(&|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle - |\phi_2\rangle|\phi_1\rangle|\phi_3\rangle + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle \\ &- |\phi_3\rangle|\phi_2\rangle|\phi_1\rangle + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle - |\phi_1\rangle|\phi_3\rangle|\phi_2\rangle\Big). \end{aligned} \tag{4.32}$$

Let us examine the properties of the N-fermion states constructed from Eq. (4.29).

Firstly, the N-fermion state obeys the Pauli exclusion principle: it is valid only if the single-particle states used to construct it, $\{|\phi_1\rangle, \ldots, |\phi_N\rangle\}$, contain no duplicates. If there is a duplicate, then each term in the sum within Eq. (4.29) must contain, somewhere in the tensor product, two kets referring to the same single-particle state $|\mu\rangle$, say in slots i and j:

$$s(p)\big(\cdots \underbrace{|\phi_{p(i)}\rangle}_{=|\mu\rangle} \cdots \underbrace{|\phi_{p(j)}\rangle}_{=|\mu\rangle} \cdots\big).$$

But there is another permutation, p', which is the same as p except with the i and j slots transposed. That term appears with a parity of the opposite sign. This implies that the terms in the sum cancel pair-wise, so $|\phi_1, \ldots, \phi_N\rangle = 0$, which is not a valid quantum state.

Secondly, we can verify that the N-fermion state is totally antisymmetric. Consider what happens when we apply an exchange operator $\hat{P}_{ij}$ to a term in the sum in Eq. (4.29):

$$\hat{P}_{ij}\Big[s(p)\big(\cdots \underbrace{|\phi_{p(i)}\rangle}_{\text{slot } i} \cdots \underbrace{|\phi_{p(j)}\rangle}_{\text{slot } j} \cdots\big)\Big] = s(p)\big(\cdots \underbrace{|\phi_{p(j)}\rangle}_{\text{slot } i} \cdots \underbrace{|\phi_{p(i)}\rangle}_{\text{slot } j} \cdots\big) \tag{4.33}$$

The single-particle states for $p(i)$ and $p(j)$ have exchanged places. This yields a new sequence of state labels, corresponding to a new permutation which we call p'. Since p' differs from p by a single transposition (of slots i and j), $s(p') = -s(p)$. Hence, (4.33) is the negative of another term in Eq. (4.29). By the same logic, applying $\hat{P}_{ij}$ to the p' term yields the negative of the p term. We thereby conclude that

$$\hat{P}_{ij}|\phi_1, \ldots, \phi_N\rangle = -|\phi_1, \ldots, \phi_N\rangle. \tag{4.34}$$

Thirdly, two N-fermion states constructed using Eq. (4.29) are orthogonal unless they consist of identical sets of single-particle states. In other words, $\langle\phi_1,\ldots,\phi_N|\phi'_1,\ldots,\phi'_N\rangle = 0$ unless $(\phi_1,\ldots,\phi_N) \equiv (\phi'_1,\ldots,\phi'_N)$ are equivalent up to a permutation. The proof follows the same logic as the N-boson case discussed earlier [see Eq. (4.25)].

4.1.4. Distinguishing particles

It might have struck you that the requirement of particle indistinguishbility, introduced in Section 4.1.1, could undermine what we said about partial measurements and entanglement from Chapter 3. For example, in the EPR thought experiment, we used the singlet state

$$|\psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\Big), \tag{4.35}$$

which looks like an antisymmetric state. Does this mean we can only prepare this state with fermions? More disturbingly, measuring S_z on one particle collapses the 2-particle state into $|\uparrow\rangle|\downarrow\rangle$ or $|\downarrow\rangle|\uparrow\rangle$, which are not eigenstates of $\hat{P}_{12}$. Is this now forbidden?

The answer to each question is no. The confusion arises because identical particles are only indistinguishable at a *fundamental* level, meaning when all degrees of freedom are accounted for. If we deal with a restricted set of degrees of freedom, like Eq. (4.35) which only involves the particles' spins, then particles *can* be distinguished from each other.

As an illustration, let us see how the EPR thought experiment is done with two identical bosons. Apart from the spin degree of freedom, the particles must also have a position degree of freedom; that's how we can assign one particle to each location (Alpha Centauri and Betelgeuse). This means the single-particle Hilbert space has the form

$$\mathscr{H}^{(1)} = \mathscr{H}_{\mathrm{spin}} \otimes \mathscr{H}_{\mathrm{position}}. \tag{4.36}$$

For simplicity, let us treat position as a twofold degree of freedom, treating $\mathscr{H}_{\mathrm{position}}$ as a 2D space spanned by the basis $\{|A\rangle, |B\rangle\}$.

Previously, we used $|\uparrow\rangle|\downarrow\rangle$ to denote a spin-up particle at A and a spin-down particle at B. This implicitly assumed that A refers to the left slot of the tensor product, and B refers to the right slot. Using the position degrees of freedom, we can instead write this as

$$|\uparrow, A;\, \downarrow, B\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle|A\rangle|\downarrow\rangle|B\rangle + |\downarrow\rangle|B\rangle|\uparrow\rangle|A\rangle\Big), \tag{4.37}$$

where the kets are written in the following order:

$$\Big[(\text{spin 1}) \otimes (\text{position 1})\Big] \otimes \Big[(\text{spin 2}) \otimes (\text{position 2})\Big]. \tag{4.38}$$

The exchange operator $\hat{P}_{12}$ swaps the two particles' Hilbert spaces, which includes both position and spin. Hence, Eq. (4.37) is explicitly symmetric:

$$\begin{aligned}\hat{P}_{12}\,|\uparrow, A\,;\,\downarrow, B\rangle &= \frac{1}{\sqrt{2}}\Big(|\downarrow\rangle|B\rangle|\uparrow\rangle|A\rangle + |\uparrow\rangle|A\rangle|\downarrow\rangle|B\rangle\Big)\\ &= |\uparrow, A\,;\,\downarrow, B\rangle.\end{aligned} \tag{4.39}$$

Likewise, if there is a spin-down particle at A and a spin-up particle at B, the bosonic 2-particle state is

$$|\downarrow, A\,;\uparrow, B\rangle = \frac{1}{\sqrt{2}}\Big(|\downarrow\rangle|A\rangle|\uparrow\rangle|B\rangle + |\uparrow\rangle|B\rangle|\downarrow\rangle|A\rangle\Big). \tag{4.40}$$

Using Eqs. (4.37) and (4.40), we can rewrite the EPR singlet state (4.35) as

$$\begin{aligned}|\psi_{\mathrm{EPR}}\rangle &= \frac{1}{\sqrt{2}}\Big(|\uparrow, A\,;\downarrow, B\rangle - |\downarrow, A\,;\uparrow, B\rangle\Big)\\ &= \frac{1}{2}\Big(|\uparrow\rangle|A\rangle|\downarrow\rangle|B\rangle + |\downarrow\rangle|B\rangle|\uparrow\rangle|A\rangle - |\downarrow\rangle|A\rangle|\uparrow\rangle|B\rangle - |\uparrow\rangle|B\rangle|\downarrow\rangle|A\rangle\Big).\end{aligned} \tag{4.41}$$

Eq. (4.41) can be further simplified by re-ordering the tensor product slots. In place of (4.38), let us order by spins and then positions:

$$\Big[(\text{spin }1)\otimes(\text{spin }2)\Big]\otimes\Big[(\text{position }1)\otimes(\text{position }2)\Big]. \tag{4.42}$$

With this convention, Eq. (4.41) becomes

$$|\psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\Big)\otimes\frac{1}{\sqrt{2}}\Big(|A\rangle|B\rangle - |B\rangle|A\rangle\Big). \tag{4.43}$$

The spin degrees of freedom indeed form an antisymmetric combination, as described by Eq. (4.35). But the position degrees of freedom, which we previously did not explicit monitor, are also antisymmetric. The overall state is therefore symmetric, consistent with the two particles being identical bosons.

Now suppose we measure S_z on the particle at A, and find that it is spin-up. This measurement outcome is associated with a projector, which in the ordering convention (4.38) can be written as

$$\hat{\Pi} = \Big(|\uparrow\rangle\langle\uparrow|\otimes|A\rangle\langle A|\Big)\otimes\Big(\hat{I}\otimes\hat{I}\Big) + \Big(\hat{I}\otimes\hat{I}\Big)\otimes\Big(|\uparrow\rangle\langle\uparrow|\otimes|A\rangle\langle A|\Big). \tag{4.44}$$

This operator treats the two particles on equal footing, and commutes with $\hat{P}_{12}$. Applying it to the EPR state, and renormalizing, yields the post-collapse state

$$|\psi'\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle|A\rangle|\downarrow\rangle|B\rangle + |\downarrow\rangle|B\rangle|\uparrow\rangle|A\rangle\Big). \tag{4.45}$$

This is precisely the bosonic state $|\uparrow, A;\downarrow, B\rangle$ defined in Eq. (4.37), which is equivalent to the singlet state (4.41).

From this discussion, we conclude that exchange symmetry is compatible with all our previous discussions about partial measurements, entanglement, etc.

4.2. SECOND QUANTIZATION

In the usual tensor product notation, symmetric and antisymmetric states become quite cumbersome to deal with when the number of particles is large. We will now introduce a formalism called **second quantization**, which greatly simplifies manipulations of such multi-particle states. (The reason for the name "second quantization" will not be apparent until later; it is a bad name, but one we are stuck with for historical reasons.)

We start by introducing a convenient way to specify multi-particle states, called the **occupation number representation**. Suppose $\{|1\rangle, |2\rangle, |3\rangle, \cdots\}$ is an orthonormal basis for the single-particle Hilbert space $\mathscr{H}^{(1)}$. From Sections 4.1.2 and 4.1.3, we know how to use the single-particle states to construct N-particle bosonic or fermionic states. Letting n_j denote the occupancy number of state $|j\rangle$, we define

$$|n_1, n_2, n_3, \ldots\rangle$$

as the multi-particle state constructed by Eq. (4.14) for bosons, or Eq. (4.29) for fermions.

Example—The 2-particle state $|0, 2, 0, 0, \ldots\rangle$ has both particles in the single-particle state $|2\rangle$ (which is only possible if the particles are bosons):

$$|0, 2, 0, 0, \ldots\rangle \equiv |2\rangle|2\rangle. \tag{4.46}$$

Example—The 3-particle state $|1, 1, 1, 0, 0, \ldots\rangle$ has one particle each occupying $|1\rangle$, $|2\rangle$, and $|3\rangle$. If the particles are bosons, this corresponds to the symmetric state

$$\begin{aligned}|1, 1, 1, 0, 0, \ldots\rangle \equiv \frac{1}{\sqrt{6}}\Big(&|1\rangle|2\rangle|3\rangle + |3\rangle|1\rangle|2\rangle + |2\rangle|3\rangle|1\rangle \\ &+ |1\rangle|3\rangle|2\rangle + |2\rangle|1\rangle|3\rangle + |3\rangle|2\rangle|1\rangle\Big).\end{aligned} \tag{4.47}$$

And if the particles are fermions,

$$\begin{aligned}|1, 1, 1, 0, 0, \ldots\rangle \equiv \frac{1}{\sqrt{6}}\Big(&|1\rangle|2\rangle|3\rangle + |3\rangle|1\rangle|2\rangle + |2\rangle|3\rangle|1\rangle \\ &- |1\rangle|3\rangle|2\rangle - |2\rangle|1\rangle|3\rangle - |3\rangle|2\rangle|1\rangle\Big).\end{aligned} \tag{4.48}$$

4.2.1. Fock space

There is a subtle point we have glossed over: what Hilbert space do these state vectors reside in? For instance, the state $|0, 2, 0, 0, \ldots\rangle$ is a bosonic 2-particle state, which is a vector in the 2-particle Hilbert space:

$$|0, 2, 0, 0, \ldots\rangle \in \mathscr{H}^{(2)} = \mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)}. \tag{4.49}$$

However, this statement is over-broad, since $\mathscr{H}^{(2)}$ also contains 2-particle states that do not satisfy the bosonic exchange symmetry. To be more precise, we can shrink the Hilbert space to the space of 2-particle state vectors with exchange parity +1, denoted by $\mathscr{H}_+^{(2)}$:

$$|0,2,0,0,\dots\rangle \in \mathscr{H}_+^{(2)}. \tag{4.50}$$

In a similar fashion, for the 3-particle state $|1,1,1,0,\dots\rangle$, we can write

$$|1,1,1,0,\dots\rangle \in \begin{cases} \mathscr{H}_+^{(3)} & \text{(for bosons)} \\ \mathscr{H}_-^{(3)} & \text{(for fermions)}, \end{cases} \tag{4.51}$$

where $\mathscr{H}_\pm^{(3)}$ is the space of 3-particle states with exchange parity +1 or -1, respectively.

Thus far, if two state have different total particle number $N = n_1 + n_2 + \cdots$, they are regarded as lying in different spaces. We can consolidate them by defining the space of all multi-particle states regardless of N, called the **Fock space**. In the formal language of linear algebra, this extended space is

$$\mathscr{H}_\pm^F = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)} \oplus \mathscr{H}_\pm^{(2)} \oplus \mathscr{H}_\pm^{(3)} \oplus \mathscr{H}_\pm^{(4)} \oplus \cdots, \tag{4.52}$$

where the $\pm$ subscripts specify whether we are dealing with bosons (+) or fermions (−). The $\oplus$ symbol denotes the **direct sum** operation, which combines vector spaces by grouping their basis vectors into a larger basis set. For example,

$$\begin{array}{l} \mathscr{H}_1 \text{ has basis } \big\{|a\rangle, |b\rangle\big\}, \\ \mathscr{H}_2 \text{ has basis } \big\{|c\rangle, |d\rangle, |e\rangle\big\} \end{array} \quad \Rightarrow \quad \mathscr{H}_1 \oplus \mathscr{H}_2 \text{ has basis } \big\{|a\rangle, |b\rangle, |c\rangle, |d\rangle, |e\rangle\big\}. \tag{4.53}$$

The direct sum should not be confused with the tensor product introduced in Chapter 3. In particular, we can see from Eq. (4.53) that

$$\dim\left[\mathscr{H}_1 \oplus \mathscr{H}_2\right] = \dim\left[\mathscr{H}_1\right] + \dim\left[\mathscr{H}_2\right]. \tag{4.54}$$

By contrast, for the tensor product,

$$\dim\left[\mathscr{H}_1 \otimes \mathscr{H}_2\right] = \dim\left[\mathscr{H}_1\right] \dim\left[\mathscr{H}_2\right]. \tag{4.55}$$

In Eq. (4.52), the first term of the direct sum, $\mathscr{H}^{(0)}$, is the Hilbert space of 0 particles. This Hilbert space contains only one distinct state vector, denoted by

$$|\varnothing\rangle \equiv |0,0,0,0,\dots\rangle. \tag{4.56}$$

This called the **vacuum state**. Note that $|\varnothing\rangle$ is *not* the same thing as a zero vector; it has the standard normalization

$$\langle\varnothing|\varnothing\rangle = 1. \tag{4.57}$$

The concept of "a state of zero particles" may seem silly, but we will see that there are very good reasons to include it in the formalism.

Now that we have defined the Fock space, any state written using the occupation number representation, $|n_1, n_2, n_3, \dots\rangle$, can be regarded as a vector in $\mathscr{H}_\pm^F$, regardless of its total particle number. Such states moreover form a complete basis for $\mathscr{H}_\pm^F$.

A subtle consequence of the Fock space concept is that we can now define quantum states that do not have a definite number of particles. For example,

$$\frac{1}{\sqrt{2}}\Big(|1,0,0,0,0,\dots\rangle + |1,1,1,0,0,\dots\rangle\Big) \in \mathscr{H}_{\pm}^{F}$$

is neither a 1-particle state nor a 3-particle state, but a superposition of the two. We will discuss the significance of states with indefinite particle number later.

4.2.2. Second quantization for bosons

After this lengthy prelude, we are ready to introduce the formalism of second quantization. Let us concentrate on bosons first. We define a **boson creation operator**, denoted by $\hat{a}_\mu^\dagger$ and acting in the following way:

$$\hat{a}_\mu^\dagger \big|n_1, n_2, \dots, n_\mu, \dots\big\rangle = \sqrt{n_\mu + 1}\,\big|n_1, n_2, \dots, n_\mu + 1, \dots\big\rangle. \tag{4.58}$$

In this definition, there is one particle creation operator for each state in the single-particle basis $\{|\varphi_1\rangle, |\varphi_2\rangle, \dots\}$. Each creation operator is defined as an operator acting on state vectors in the Fock space $\mathscr{H}_S^F$, and has the effect of incrementing the occupation number of its single-particle state by one. The prefactor of $\sqrt{n_\mu + 1}$ is defined for later convenience.

Applying a creation operator to the vacuum state yields a single-particle state:

$$\hat{a}_\mu^\dagger|\varnothing\rangle \;=\; |0,\dots,0,\underset{\hookleftarrow\,\mu}{1},0,0,\dots\rangle. \tag{4.59}$$

The creation operator's Hermitian conjugate, $\hat{a}_\mu$, is the **boson annihilation operator**. To characterize it, first take the Hermitian conjugate of Eq. (4.58):

$$\big\langle n_1, n_2, \dots\big|\hat{a}_\mu = \sqrt{n_\mu + 1}\,\big\langle n_1, n_2, \dots, n_\mu + 1, \dots\big|. \tag{4.60}$$

Right-multiplying by another occupation number state $|n_1', n_2', \dots\rangle$ results in

$$\begin{aligned}\big\langle n_1, n_2, \dots\big|\hat{a}_\mu\big|n_1', n_2', \dots\big\rangle &= \sqrt{n_\mu + 1}\,\big\langle \dots, n_\mu + 1, \dots\big|\dots, n_\mu', \dots\big\rangle \\ &= \sqrt{n_\mu + 1}\;\delta^{n_1}_{n_1'}\;\delta^{n_2}_{n_2'}\cdots\delta^{n_\mu+1}_{n_\mu'}\cdots \\ &= \sqrt{n_\mu'}\;\delta^{n_1}_{n_1'}\;\delta^{n_2}_{n_2'}\cdots\delta^{n_\mu+1}_{n_\mu'}\cdots\end{aligned} \tag{4.61}$$

From this, we can deduce that

$$\hat{a}_\mu\big|n_1', n_2', \dots, n_\mu', \dots\big\rangle = \begin{cases}\sqrt{n_\mu'}\;\big|n_1', n_2', \dots, n_\mu' - 1, \dots\big\rangle, & \text{if } n_\mu' > 0\\ 0, & \text{if } n_\mu' = 0.\end{cases} \tag{4.62}$$

In other words, the annihilation operator decrements the occupation number of a specific single-particle state by one (hence its name). As a special exception, if the given single-particle state is unoccupied ($n_\mu = 0$), applying $\hat{a}_\mu$ results in a zero vector (note that this is *not* the same thing as the vacuum state $|\varnothing\rangle$).

The boson creation/annihilation operators obey the following commutation relations:

$$[\hat{a}_\mu, \hat{a}_\nu] = [\hat{a}_\mu^\dagger, \hat{a}_\nu^\dagger] = 0, \qquad [\hat{a}_\mu, \hat{a}_\nu^\dagger] = \delta_{\mu\nu}. \tag{4.63}$$

These can be derived by taking the matrix elements with respect to the occupation number basis. We will go through the derivation of the last commutation relation; the others are left as an exercise (Exercise 5).

To prove that $[\hat{a}_\mu, \hat{a}_\nu^\dagger] = \delta_{\mu\nu}$, first consider the case where the creation/annihilation operators act on the same single-particle state:

$$\begin{aligned}
\big\langle n_1, n_2, \dots \big| \hat{a}_\mu \hat{a}_\mu^\dagger \big| n_1', n_2' \dots \big\rangle &= \sqrt{(n_\mu + 1)(n_\mu' + 1)}\, \big\langle \dots, n_\mu + 1, \dots \big| \dots, n_\mu' + 1, \dots \big\rangle \\
&= \sqrt{(n_\mu + 1)(n_\mu' + 1)}\delta_{n_1'}^{n_1}\, \delta_{n_2'}^{n_2} \cdots \delta_{n_\mu'+1}^{n_\mu+1} \cdots \\
&= (n_\mu + 1)\delta_{n_1'}^{n_1}\, \delta_{n_2'}^{n_2} \cdots \delta_{n_\mu'}^{n_\mu} \cdots \\
\big\langle n_1, n_2, \dots \big| \hat{a}_\mu^\dagger \hat{a}_\mu \big| n_1', n_2' \dots \big\rangle &= \sqrt{n_\mu n_\mu'}\, \big\langle \dots, n_\mu - 1, \dots \big| \dots, n_\mu' - 1, \dots \big\rangle \\
&= \sqrt{n_\mu n_\mu'}\delta_{n_1'}^{n_1}\, \delta_{n_2'}^{n_2} \cdots \delta_{n_\mu'-1}^{n_\mu-1} \cdots \\
&= n_\mu \delta_{n_1'}^{n_1}\, \delta_{n_2'}^{n_2} \cdots \delta_{n_\mu'}^{n_\mu} \cdots
\end{aligned} \tag{4.64}$$

In the second equation, we were a bit sloppy in handling the $n_\mu = 0$ and $n_\mu' = 0$ cases, but you can check for yourself that the result on the last line remains correct. Upon taking the difference of the two equations, we get

$$\big\langle n_1, n_2, \dots \big| \left(\hat{a}_\mu \hat{a}_\mu^\dagger - \hat{a}_\mu^\dagger \hat{a}_\mu \right) \big| n_1', n_2' \dots \big\rangle = \delta_{n_1'}^{n_1}\, \delta_{n_2'}^{n_2} \cdots \delta_{n_\mu'}^{n_\mu} \cdots = \big\langle n_1, n_2, \dots \big| n_1', n_2' \dots \big\rangle. \tag{4.65}$$

Since the occupation number states form a basis for $\mathscr{H}_S^F$, we conclude that

$$\hat{a}_\mu \hat{a}_\mu^\dagger - \hat{a}_\mu^\dagger \hat{a}_\mu = \hat{I}. \tag{4.66}$$

Next, consider the case where $\mu \neq \nu$:

$$\begin{aligned}
\big\langle n_1, \dots \big| \hat{a}_\mu \hat{a}_\nu^\dagger \big| n_1', \dots \big\rangle &= \sqrt{(n_\mu + 1)(n_\nu' + 1)}\, \langle \dots, n_\mu + 1, \dots, n_\nu, \dots | \dots, n_\mu', \dots, n_\nu' + 1, \dots \rangle \\
&= \sqrt{n_\mu' n_\nu}\; \delta_{n_1'}^{n_1} \cdots \delta_{n_\mu'}^{n_\mu+1} \cdots \delta_{n_\nu'+1}^{n_\nu} \cdots \\
\big\langle n_1, \dots \big| \hat{a}_\nu^\dagger \hat{a}_\mu \big| n_1', \dots \big\rangle &= \sqrt{n_\mu' n_\nu}\, \langle \dots, n_\mu, \dots, n_\nu - 1, \dots | \dots, n_\mu' - 1, \dots, n_\nu' \dots \rangle \\
&= \sqrt{n_\mu' n_\nu}\; \delta_{n_1'}^{n_1} \cdots \delta_{n_\mu'-1}^{n_\mu} \cdots \delta_{n_\nu'}^{n_\nu-1} \cdots \\
&= \sqrt{n_\mu' n_\nu}\; \delta_{n_1'}^{n_1} \cdots \delta_{n_\mu'}^{n_\mu+1} \cdots \delta_{n_\nu'+1}^{n_\nu} \cdots
\end{aligned}$$

Hence,

$$\hat{a}_\mu \hat{a}_\nu^\dagger - \hat{a}_\nu^\dagger \hat{a}_\mu = 0 \quad \text{for} \quad \mu \neq \nu. \tag{4.67}$$

Combining these two results gives the desired commutation relation, $[\hat{a}_\mu, \hat{a}_\nu^\dagger] = \delta_{\mu\nu}$.

Another useful result which emerges from the first part of this proof is that

$$\big\langle n_1, n_2, \dots \big| \hat{a}_\mu^\dagger \hat{a}_\mu \big| n_1', n_2' \dots \big\rangle = n_\mu \big\langle n_1, n_2, \dots \big| n_1', n_2' \dots \big\rangle. \tag{4.68}$$

Hence, we can define an observable corresponding to the occupation number of single-particle state μ:

$$\hat{n}_\mu \equiv \hat{a}_\mu^\dagger \hat{a}_\mu. \tag{4.69}$$

If you are familiar with the method of ladder operators for solving the quantum harmonic oscillator, you will have noticed the striking similarity to the creation and annihilation operators for bosons. This is no mere coincidence. We will examine the relationship between harmonic oscillators and bosons in Section 4.3.2.

4.2.3. Second quantization for fermions

For fermions, the multi-particle states are antisymmetric. The fermion creation operator can be defined as follows:

$$\begin{aligned}\hat{c}_\mu^\dagger |n_1, n_2, \dots, n_\mu, \dots\rangle &= \begin{cases}(-1)^{n_1+n_2+\cdots+n_{\mu-1}} |n_1, n_2, \dots, n_{\mu-1}, 1, \dots\rangle & \text{if } n_\mu = 0 \\ 0 & \text{if } n_\mu = 1.\end{cases} \\ &= (-1)^{n_1+n_2+\cdots+n_{\mu-1}} \, \delta_0^{n_\mu} \, \big|n_1, n_2, \dots, n_{\mu-1}, 1, \dots\big\rangle.\end{aligned} \tag{4.70}$$

In other words, if state μ is unoccupied, then $\hat{c}_\mu^\dagger$ increments the occupation number to 1 and introduces a factor of $(-1)^{n_1+n_2+\cdots+n_{\mu-1}}$ (i.e, $+1$ if there is an even number of occupied states preceding μ, and -1 if there is an odd number). The role of this factor will be apparent later. Note that this definition requires the single-particle states to be ordered in some way; otherwise, it would not make sense to speak of the states "preceding" μ. It does not matter which ordering we choose, so long as we make *some* choice and stick to it consistently.

If μ is occupied, applying $\hat{c}_\mu^\dagger$ gives the zero vector. The occupation numbers are therefore forbidden from being larger than 1, consistent with the Pauli exclusion principle.

The conjugate operator, $\hat{c}_\mu$, is the fermion annihilation operator. To see what it does, take the Hermitian conjugate of the definition of the creation operator:

$$\langle n_1, n_2, \dots, n_\mu, \dots | \hat{c}_\mu = (-1)^{n_1+n_2+\cdots+n_{\mu-1}} \, \delta_0^{n_\mu} \, \big\langle n_1, n_2, \dots, n_{\mu-1}, 1, \dots \big|. \tag{4.71}$$

Right-multiplying this by $|n_1', n_2', \dots\rangle$ gives

$$\langle n_1, n_2, \dots, n_\mu, \dots | \hat{c}_\mu | n_1', n_2', \dots\rangle = (-1)^{n_1+\cdots+n_{\mu-1}} \, \delta_{n_1'}^{n_1} \cdots \, \delta_{n_{\mu-1}'}^{n_{\mu-1}} \left(\delta_0^{n_\mu} \delta_{n_\mu'}^1\right) \delta_{n_{\mu+1}'}^{n_{\mu+1}} \cdots \tag{4.72}$$

Hence, we deduce that

$$\begin{aligned}\hat{c}_\mu |n_1', \dots, n_\mu', \dots\rangle &= \begin{cases}0 & \text{if } n_\mu' = 0 \\ (-1)^{n_1'+\cdots+n_{\mu-1}'} |n_1', \dots, n_{\mu-1}', 0, \dots\rangle & \text{if } n_\mu' = 1.\end{cases} \\ &= (-1)^{n_1'+\cdots+n_{\mu-1}'} \, \delta_{n_\mu'}^1 \, \big|n_1', \dots, n_{\mu-1}', 0, \dots\big\rangle.\end{aligned} \tag{4.73}$$

In other words, if state μ is unoccupied, then applying $\hat{c}_\mu$ gives the zero vector; if state μ is occupied, applying $\hat{c}_\mu$ decrements the occupation number to 0, and multiplies the state by the aforementioned factor of ± 1.

With these definitions, the fermion creation/annihilation operators can be shown to obey the following *anticommutation* relations:

$$\left\{\hat{c}_\mu, \hat{c}_\nu\right\} = \left\{\hat{c}_\mu^\dagger, \hat{c}_\nu^\dagger\right\} = 0, \qquad \left\{\hat{c}_\mu, \hat{c}_\nu^\dagger\right\} = \delta_{\mu\nu}. \tag{4.74}$$

Here, $\{\cdot, \cdot\}$ denotes an anticommutator, which is defined by

$$\left\{\hat{A}, \hat{B}\right\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}. \tag{4.75}$$

Similar to the bosonic commutation relations (4.63), the anticommutation relations (4.74) can be derived by taking matrix elements with occupation number states. We will only go over the last one, $\left\{\hat{c}_\mu, \hat{c}_\nu^\dagger\right\} = \delta_{\mu\nu}$; the others are left for the reader to verify.

First, consider creation/annihilation operators acting on the same single-particle state μ:

$$\begin{aligned}\left\langle \dots, n_\mu, \dots \middle| \hat{c}_\mu \hat{c}_\mu^\dagger \middle| \dots, n'_\mu, \dots \right\rangle &= (-1)^{n_1+\cdots+n_{\mu-1}} (-1)^{n'_1+\cdots+n'_{\mu-1}} \, \delta_0^{n_\mu} \delta_{n'_\mu}^0 \\ &\quad \times \left\langle n_1, \dots, n_{\mu-1}, 1, \dots \middle| n'_1, \dots, n'_{\mu-1}, 1, \dots \right\rangle \\ &= \delta_{n'_\mu}^0 \;\cdot\; \delta_{n'_1}^{n_1} \delta_{n'_2}^{n_2} \cdots \delta_{n'_\mu}^{n_\mu} \cdots \end{aligned} \tag{4.76}$$

By a similar calculation,

$$\left\langle \dots, n_\mu, \dots \middle| \hat{c}_\mu^\dagger \hat{c}_\mu \middle| \dots, n'_\mu, \dots \right\rangle = \delta_{n'_\mu}^1 \;\cdot\; \delta_{n'_1}^{n_1} \delta_{n'_2}^{n_2} \cdots \delta_{n'_\mu}^{n_\mu} \cdots \tag{4.77}$$

By adding these two equations, and using the fact that $\delta_{n'_\mu}^0 + \delta_{n'_\mu}^1 = 1$, we get

$$\left\langle \dots, n_\mu, \dots \middle| \left\{\hat{c}_\mu, \hat{c}_\mu^\dagger\right\} \middle| \dots, n'_\mu, \dots \right\rangle = \left\langle \dots, n_\mu, \dots \middle| \dots, n'_\mu, \dots \right\rangle \tag{4.78}$$

And hence,

$$\left\{\hat{c}_\mu, \hat{c}_\mu^\dagger\right\} = \hat{I}. \tag{4.79}$$

Next, we must prove that $\left\{\hat{c}_\mu, \hat{c}_\nu^\dagger\right\} = 0$ for $\mu \neq \nu$. We will show this for $\mu < \nu$ (the $\mu > \nu$ case follows by Hermitian conjugation). This is, once again, by taking matrix elements:

$$\begin{aligned}\left\langle \dots, n_\mu, \dots, n_\nu, \dots \middle| \hat{c}_\mu \hat{c}_\nu^\dagger \middle| \dots, n'_\mu, \dots, n'_\nu, \dots \right\rangle &= (-1)^{n_1+\cdots+n_{\mu-1}} (-1)^{n'_1+\cdots+n'_{\nu-1}} \, \delta_0^{n_\mu} \delta_{n'_\nu}^0 \\ &\quad \times \left\langle \dots, 1, \dots, n_\nu, \dots \middle| \dots, n'_\mu, \dots, 1, \dots \right\rangle \\ &= (-1)^{n'_\mu+\cdots+n'_{\nu-1}} \, \delta_{n'_1}^{n_1} \delta_{n'_2}^{n_2} \cdots \left(\delta_0^{n_\mu} \delta_{n'_\mu}^1\right) \cdots \left(\delta_1^{n_\nu} \delta_{n'_\nu}^0\right) \cdots \\ &= (-1)^{1+n_{\mu+1}+\cdots+n_{\nu-1}} \, \delta_{n'_1}^{n_1} \delta_{n'_2}^{n_2} \cdots \left(\delta_{n_\mu}^0 \delta_{n'_\mu}^1\right) \cdots \left(\delta_{n'_\nu}^0 \delta_{n_\nu}^1\right) \cdots \\ \left\langle \dots, n_\mu, \dots, n_\nu, \dots \middle| \hat{c}_\nu^\dagger \hat{c}_\mu \middle| \dots, n'_\mu, \dots, n'_\nu, \dots \right\rangle &= (-1)^{n_1+\cdots+n_{\nu-1}} (-1)^{n'_1+\cdots+n'_{\mu-1}} \, \delta_1^{n_\nu} \delta_{n'_\mu}^1 \\ &\quad \times \left\langle \dots, n_\mu, \dots, 0, \dots \middle| \dots, 0, \dots, n'_\nu, \dots \right\rangle \\ &= (-1)^{n_\mu+\cdots+n_{\nu-1}} \, \delta_{n'_1}^{n_1} \delta_{n'_2}^{n_2} \cdots \left(\delta_0^{n_\mu} \delta_{n'_\mu}^1\right) \cdots \left(\delta_1^{n_\nu} \delta_{n'_\nu}^0\right) \cdots \\ &= (-1)^{0+n_{\mu+1}+\cdots+n_{\nu-1}} \, \delta_{n'_1}^{n_1} \delta_{n'_2}^{n_2} \cdots \left(\delta_0^{n_\mu} \delta_{n'_\mu}^1\right) \cdots \left(\delta_1^{n_\nu} \delta_{n'_\nu}^0\right) \cdots \end{aligned}$$

The two equations differ by a factor of -1, so adding them gives zero. Putting everything together, we conclude that $\left\{c_\mu, c_\nu^\dagger\right\} = \delta_{\mu\nu}$, as stated in (4.74).

As you can see, the derivation of the fermionic anticommutation relations is quite hairy, in large part due to the $(-1)^{(\cdots)}$ factors in the definitions of the creation and annihilation operators. But once these relations have been derived, we can deal entirely with the creation and annihilation operators, without worrying about the underlying occupation number representation and its $(-1)^{(\cdots)}$ factors. By the way, if we had chosen to omit the $(-1)^{(\cdots)}$ factors in the definitions, the creation and annihilation operators would still satisfy the anticommutation relation $\{\hat{c}_\mu, \hat{c}_\nu^\dagger\} = \delta_{\mu\nu}$, but two creation operators or two annihilation operators would *commute* rather than *anticommute*. During subsequent calculations, the algebra of creation and annihilation operators ends up being much harder to deal with.

4.2.4. Hamiltonians for non-interacting particles

Second quantization provides a powerful way to express quantum operators, such as the Hamiltonian, for multi-particle systems. The basic idea is to use creation and annihilation operators as the basic building blocks for constructing all other operators.

Suppose the Hamiltonian for a single-particle ($N = 1$) system is $\hat{H}^{(1)}$. This is a Hermitian operator acting on the single-particle Hilbert space $\mathscr{H}^{(1)}$. For general N, the Hamiltonian $\hat{H}$ should be a Hermitian operator acting on the Fock space $\mathscr{H}^F$. What is this operator, and how is it related to $\hat{H}^{(1)}$?

Let us assume, for specificity, that the particles are bosons. We assert that if the particles are *non-interacting* (the precise meaning of which will be discussed shortly), the multi-particle Hamiltonian is

$$\hat{H} = \sum_{\mu\nu} \hat{a}_\mu^\dagger H_{\mu\nu} \hat{a}_\nu, \tag{4.80}$$

where $\hat{a}_\mu$ and $\hat{a}_\mu^\dagger$ are the boson creation and annihilation operators, defined using any orthonormal basis for $\mathscr{H}^{(1)}$, and

$$H_{\mu\nu} = \langle\mu|\hat{H}^{(1)}|\nu\rangle \tag{4.81}$$

are the single-particle Hamiltonian's matrix elements in that basis.

To justify Eq. (4.80), consider how it acts on various states in $\mathscr{H}^F$. For the vacuum state,

$$\hat{H}|\varnothing\rangle = 0, \tag{4.82}$$

so $|\varnothing\rangle$ is an energy eigenstate with zero energy, as expected. For single-particle states,

$$\hat{H}|n_\nu = 1\rangle = \sum_{\mu\nu'} \hat{a}_\mu^\dagger H_{\mu\nu'} \hat{a}_{\nu'}\, \hat{a}_\nu^\dagger |\varnothing\rangle \tag{4.83}$$

$$= \sum_{\mu} H_{\mu\nu} \hat{a}_\mu^\dagger |\varnothing\rangle. \tag{4.84}$$

In going from Eq. (4.83) to (4.84), we used the commutation relation $\hat{a}_{\nu'}\hat{a}_\nu^\dagger = \delta_{\nu\nu'} + \hat{a}_\nu^\dagger \hat{a}_{\nu'}$, and the fact that $\hat{a}_{\nu'}|\varnothing\rangle = 0$. We see that Eq. (4.84) exactly matches what we get from the single-particle Hamiltonian defined in Eq. (4.81):

$$\hat{H}^{(1)} = \sum_{\mu\nu} |\mu\rangle H_{\mu\nu} \langle\nu| \quad \Rightarrow \quad \hat{H}^{(1)}|\nu\rangle = \sum_{\mu} H_{\mu\nu}|\mu\rangle. \tag{4.85}$$

As a final justification, suppose the single-particle basis states are eigenstates of $\hat{H}^{(1)}$:

$$\hat{H}^{(1)}|\mu\rangle = E_\mu|\mu\rangle \quad \Rightarrow \quad \hat{H} = \sum_\mu E_\mu\, \hat{a}_\mu^\dagger \hat{a}_\mu = \sum_\mu E_\mu \hat{n}_\mu, \tag{4.86}$$

where $\hat{n}_\mu$ is the number operator defined in Eq. (4.69). Thus, the total energy of the system is the sum of the energies of the individual particles, as we expect for particles not interacting with one another.

It is interesting to interpret Eq. (4.80) in light of the Hamiltonian's role as the generator of time evolution. During an infinitesimal time step, the evolution operator is

$$\hat{U}(\delta t) \approx \hat{I} - \frac{i\delta t}{\hbar}\hat{H} \tag{4.87}$$

$$= \hat{I} - \frac{i\delta t}{\hbar}\sum_{\mu\nu} \hat{a}_\mu^\dagger H_{\mu\nu}\hat{a}_\nu. \tag{4.88}$$

The evolution includes processes whereby a particle is annihilated in state ν, and then immediately re-created in state μ, which we can interpret as transferring a particle from ν to μ. These processes, as well as the omitted higher-order terms, leave the total particle number unchanged. In fact, we can formally show that the total particle number is conserved:

$$[\hat{H}, \hat{N}] = 0, \quad \text{where} \;\; \hat{N} \equiv \sum_\mu \hat{a}_\mu^\dagger a_\mu. \tag{4.89}$$

This proof is left as an exercise (see Exercise 6).

Other multi-particle observables can be expressed similarly. Given any single-particle observable $\hat{A}^{(1)}$, we can define an observable for the "total A" of a multi-particle system:

$$\hat{A} = \sum_{\mu\nu} \hat{a}_\mu^\dagger A_{\mu\nu}\hat{a}_\nu, \quad \text{where} \;\; A_{\mu\nu} = \langle\mu|\hat{A}^{(1)}|\nu\rangle. \tag{4.90}$$

Similar to Eq. (4.89), the $\hat{A}$ operator commutes with the total particle number operator $\hat{N}$. It is also worth noting that the form of Eq. (4.90) is independent of the choice of single-particle basis. Suppose we replace the $\{|\mu\rangle, |\nu\rangle, \dots\}$ basis with a new basis $\{|p\rangle, |q\rangle, \dots\}$, related to the old basis states by some linear transformation

$$|p\rangle = \sum_\mu U_{p\mu}|\mu\rangle, \tag{4.91}$$

where U is unitary: $U_{p\mu}^* = (U^{-1})_{\mu p}$. We can write Eq. (4.91) in second quantized form as

$$\hat{a}_p^\dagger|\varnothing\rangle = \sum_\mu U_{p\mu}\hat{a}_\mu^\dagger|\varnothing\rangle, \tag{4.92}$$

where $\hat{a}_p^\dagger$ is the creation operator defined in the new basis. We thereby deduce that

$$\hat{a}_p^\dagger = \sum_\mu U_{p\mu}\hat{a}_\mu^\dagger \tag{4.93}$$

$$\hat{a}_p = \sum_\mu U_{p\mu}^*\hat{a}_\mu. \tag{4.94}$$

Using the unitarity of $U_{\alpha\mu}$, we can verify that the new $\hat{a}_p$ and $\hat{a}_p^\dagger$ operators satisfy the bosonic commutation relations. We can also invert Eq. (4.93)–(4.94) to obtain

$$\hat{a}_\mu^\dagger = \sum_p U_{p\mu}^* \hat{a}_p^\dagger \tag{4.95}$$

$$\hat{a}_\mu = \sum_p U_{p\mu} \hat{a}_p. \tag{4.96}$$

Plugging these into the multi-particle observable (4.90) yields

$$\hat{A} = \sum_{\mu\nu pq} \hat{a}_p^\dagger U_{p\mu}^* A_{\mu\nu} U_{q\nu} \hat{a}_q \tag{4.97}$$

$$= \sum_{pq} \hat{a}_p^\dagger A_{pq} \hat{a}_q, \tag{4.98}$$

where

$$A_{pq} = U_{p\mu}^* A_{\mu\nu} U_{q\nu} = \langle p|\hat{A}^{(1)}|q\rangle. \tag{4.99}$$

In other words, adopting a different single-particle basis simply alters the single-particle state labels in Eq. (4.90).

For fermions, all of the above holds true with the a operators replaced by c operators. For example, the Hamiltonian for a system of non-interacting fermions is

$$\hat{H} = \sum_{\mu\nu} \hat{c}_\mu^\dagger H_{\mu\nu} \hat{c}_\nu, \tag{4.100}$$

where the $\hat{c}_\nu$ and $\hat{c}_\mu^\dagger$ operators obey the anticommutation relations (4.74).

4.2.5. Particle interactions

There are other ways to combine creation and annihilation operators into multi-particle observables. For example, a pairwise potential for a system of bosons has the form

$$\hat{V} = \frac{1}{2} \sum_{\mu\nu\lambda\sigma} \hat{a}_\mu^\dagger \hat{a}_\nu^\dagger \, V_{\mu\nu\lambda\sigma} \, \hat{a}_\sigma \hat{a}_\lambda. \tag{4.101}$$

The prefactor of 1/2 is conventional. If such an operator is placed into the Hamiltonian, then during each infinitesimal time step it induces the annihilation and immediate re-creation of a pair of particles, which can be interpreted as a two-particle interaction. The total particle number is conserved during this process.

The coefficients $\{V_{\mu\nu\lambda\sigma}\}$ in Eq. (4.101) can be determined from the 2-particle form of the potential operator. First, note that we want $\hat{V}$ to be Hermitian, so

$$\hat{V}^\dagger = \frac{1}{2} \sum_{\mu\nu\lambda\sigma} \hat{a}_\lambda^\dagger \hat{a}_\sigma^\dagger \, V_{\mu\nu\lambda\sigma}^* \, \hat{a}_\nu \hat{a}_\mu = \hat{V}. \tag{4.102}$$

Comparing this to Eq. (4.101), we see that Hermiticity can be achieved by requiring that

$$V_{\lambda\sigma\mu\nu}^* = V_{\mu\nu\lambda\sigma}. \tag{4.103}$$

Now, let $\hat{V}^{(2)}$ be the potential operator in the 2-particle Hilbert space $\mathscr{H}^{(2)}$. We can compare the matrix elements for $\hat{V}$ and $\hat{V}^{(2)}$. For example, consider the 2-boson states

$$\begin{aligned} |n_\mu = 1, n_\nu = 1\rangle &= \frac{1}{\sqrt{2}}\Big(|\mu\rangle|\nu\rangle + |\nu\rangle|\mu\rangle\Big), \\ |n_\lambda = 1, n_\sigma = 1\rangle &= \frac{1}{\sqrt{2}}\Big(|\lambda\rangle|\sigma\rangle + |\sigma\rangle|\lambda\rangle\Big), \end{aligned} \tag{4.104}$$

where $\mu \neq \nu$ and $\lambda \neq \sigma$. The matrix elements of $\hat{V}^{(2)}$ are

$$\begin{aligned} &\langle n_\mu = 1, n_\nu = 1|\hat{V}^{(2)}|n_\lambda = 1, n_\sigma = 1\rangle \\ &\quad = \frac{1}{2}\Big(\langle\mu|\langle\nu|\hat{V}^{(2)}|\lambda\rangle|\sigma\rangle + \langle\nu|\langle\mu|\hat{V}^{(2)}|\lambda\rangle|\sigma\rangle + \langle\mu|\langle\nu|\hat{V}^{(2)}|\sigma\rangle|\lambda\rangle + \langle\nu|\langle\mu|\hat{V}^{(2)}|\sigma\rangle|\lambda\rangle\Big). \end{aligned} \tag{4.105}$$

On the other hand, the matrix elements of $\hat{V}$ are

$$\langle n_\mu = 1, n_\nu = 1|\hat{V}|n_\lambda = 1, n_\sigma = 1\rangle = \sum_{\mu'\nu'\lambda'\sigma'} V_{\mu'\nu'\lambda'\sigma'}\langle\varnothing|\hat{a}_\nu\hat{a}_\mu\hat{a}_{\mu'}^\dagger\hat{a}_{\nu'}^\dagger\hat{a}_{\sigma'}\hat{a}_{\lambda'}\hat{a}_\lambda^\dagger\hat{a}_\sigma^\dagger|\varnothing\rangle \tag{4.106}$$

$$= \frac{1}{2}\left(V_{\mu\nu\lambda\sigma} + V_{\mu\nu\sigma\lambda} + V_{\nu\mu\lambda\sigma} + V_{\nu\mu\sigma\lambda}\right). \tag{4.107}$$

In going from Eq. (4.106) to (4.107), we use the bosonic commutation relations repeatedly to "push" the annihilation operators to the right (so that they can act upon $|\varnothing\rangle$), and to push the creation operators to the left (so that they can act upon $\langle\varnothing|$). Comparing Eq. (4.105) to Eq. (4.107), we see that the matrix elements match if we simply take

$$V_{\mu\nu\lambda\sigma} = \langle\mu|\langle\nu|\hat{V}^{(2)}|\lambda\rangle|\sigma\rangle. \tag{4.108}$$

For instance, if the bosons have a position representation, we would have something like

$$V_{\mu\nu\lambda\sigma} = \int d^d r_1 \int d^d r_2\; \varphi_\mu^*(r_1)\,\varphi_\nu^*(r_2)\,V(r_1,r_2)\,\varphi_\lambda(r_1)\,\varphi_\sigma(r_2). \tag{4.109}$$

The appropriate coefficients for $\mu = \nu$ and/or $\lambda = \sigma$, and for the fermionic case, are left for the reader to work out.

4.2.6. Particle non-conserving observables

Another way to build an observable from creation and annihilation operators is

$$\hat{A} = \sum_\mu \left(\alpha_\mu \hat{a}_\mu^\dagger + \alpha_\mu^* \hat{a}_\mu\right). \tag{4.110}$$

If such a term is added to a Hamiltonian, it breaks the conservation of total particle number N. Each infinitesimal time step will involve processes that decrement N (due to $\hat{a}_\mu$), and processes that increment N (due to $\hat{a}_\mu^\dagger$). Thus, even if the initial state has definite N, it subsequently evolves into a state of indefinite N, consisting of a superposition of state vectors with different N.

Such operators are physically relevant. For example, we will see that in quantum electrodynamics (Chapter 5), this sort of number non-conserving operator is responsible for the emission and absorption of photons.

4.3. QUANTUM FIELD THEORY

4.3.1. Field operators

So far, we have been agnostic about the choice of single-particle basis states used in defining creation and annihilation operators. Let us now consider the common situation where the single-particle Hilbert space $\mathscr{H}^{(1)}$ has a position basis.

Assume, as before, that $\{|\mu\rangle\}$ is some discrete orthonormal basis for $\mathscr{H}^{(1)}$, from which are defined the creation and annihilation operators $\hat{a}_\mu^\dagger$ and $\hat{a}_\mu$ (we will adopt the notation for bosons, but most of this discussion applies without modification to fermions). Now, let $\mathscr{H}^{(1)}$ also have a position basis $\{|\mathbf{r}\rangle\}$, where the position eigenstates are indexed by d-dimensional position vectors $\mathbf{r}$. For each $|\mu\rangle$, we have a wavefunction defined by

$$\varphi_\mu(\mathbf{r}) = \langle \mathbf{r}|\mu\rangle. \tag{4.111}$$

Since the $|\mu\rangle$'s are orthonormal, the wavefunctions have the following completeness and orthonormality properties:

$$\int d^d r\; \varphi_\mu^*(\mathbf{r})\, \varphi_\nu(\mathbf{r}) = \delta_{\mu\nu}, \tag{4.112}$$

$$\sum_\mu \varphi_\mu^*(\mathbf{r})\varphi_\mu(\mathbf{r}') = \delta^d(\mathbf{r}-\mathbf{r}'). \tag{4.113}$$

Now consider the following "**field operators**":

$$\hat{\psi}(\mathbf{r}) = \sum_\mu \varphi_\mu(\mathbf{r})\, \hat{a}_\mu \tag{4.114}$$

$$\hat{\psi}^\dagger(\mathbf{r}) = \sum_\mu \varphi_\mu^*(\mathbf{r})\, \hat{a}_\mu^\dagger. \tag{4.115}$$

These are Fock space operators, and they depend parametrically on $\mathbf{r}$, just as $\hat{a}_\mu$ and $\hat{a}_\mu^\dagger$ depend on the choice of μ.

In classical physics, a field is something that maps the position $\mathbf{r}$ to one or more numbers, like the components of the electric field vector $\mathbf{E}(\mathbf{r})$. Here we have $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$, which map $\mathbf{r}$ to operators, rather than numbers. This can therefore be regarded as a quantum generalization of the classical notion of a field. A quantum theory involving such position-to-operator mappings is called a **quantum field theory**.

To understand the meaning of these field operators, let us use Eqs. (4.114)–(4.115), along with Eq. (4.113), to derive the commutation relations

$$\begin{aligned}\left[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\right] &= \sum_{\mu\nu} \varphi_\mu(\mathbf{r})\varphi_\nu^*(\mathbf{r}')\,\left[\hat{a}_\mu, \hat{a}_\nu^\dagger\right] && (4.116)\\ &= \sum_\mu \varphi_\mu(\mathbf{r})\varphi_\mu^*(\mathbf{r}') && (4.117)\\ &= \delta^d(\mathbf{r}-\mathbf{r}'). && (4.118)\end{aligned}$$

Similarly, we can show that

$$\left[\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')\right] = \left[\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\right] = 0. \tag{4.119}$$

These are just like the original bosonic commutation relations (4.63), except that the state labels replaced by positions, and the Kronecker delta is replaced by a delta function. We can therefore interpret $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ as spatially localized creation and annihhilation operators: $\hat{\psi}^\dagger(\mathbf{r})$ creates a particle at $\mathbf{r}$, while $\hat{\psi}(\mathbf{r})$ annihilates a particle at $\mathbf{r}$. For fermions, the above commutators are simply replaced by anticommutators.

These field operators supply the motivation for the term "second quantization", introduced in Section 4.2 to refer to the use of creation and annihilation operators for multi-particle quantum systems. The idea is that single-particle quantum mechanics is derived by quantizing classical mechanics, which involves converting numbers like $\mathbf{r}$ and $\mathbf{p}$ into operators like $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$. The subsequent extension of single-particle quantum mechanics to multi-particle quantum mechanics involves creation and annihilation operators like $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$, whose definitions in Eqs. (4.114)–(4.115) are based on the single-particle wavefunctions. This can be viewed as a second quantization step, whereby the single-particle wavefunction $\varphi_\mu(\mathbf{r})$ is converted into a quantum field. Still, "second quantization" is not entirely satisfactory terminology, because quantum field theories are not necessarily derived using this two-step procedure. In Section 4.3.3, we will see an example of deriving field operators directly from a classical field theory.

Physically relevant observables can be expressed using the field operators $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$. As discussed in Section 4.2.4, given a single-particle observable $\hat{A}^{(1)}$ we can define a Fock space observable $\hat{A}$ corresponding to the "total A" of a multi-particle system:

$$\hat{A} = \sum_{\mu\nu} \hat{a}_\mu^\dagger \, \langle \mu|\hat{A}^{(1)}|\nu\rangle \, \hat{a}_\nu. \tag{4.120}$$

We can re-express such an operator using the field operators. First, using Eqs. (4.112)–(4.113), we derive the inverse relations

$$\hat{a}_\mu = \int d^d r \; \varphi_\mu^*(\mathbf{r}) \, \hat{\psi}(\mathbf{r}) \tag{4.121}$$

$$\hat{a}_\mu^\dagger = \int d^d r \; \varphi_\mu(\mathbf{r}) \, \hat{\psi}^\dagger(\mathbf{r}). \tag{4.122}$$

Plugging these into Eq. (4.120) yields

$$\hat{A} = \int d^d r \, d^d r' \; \hat{\psi}^\dagger(\mathbf{r}) \; \langle \mathbf{r}|\hat{A}^{(1)}|\mathbf{r}'\rangle \; \hat{\psi}(\mathbf{r}'). \tag{4.123}$$

For example, the single-particle Hamiltonian for a particle in a potential V is

$$\hat{H}^{(1)} = \frac{|\hat{\mathbf{p}}|^2}{2m} + V(\hat{\mathbf{r}}), \tag{4.124}$$

where $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are the single-particle position and momentum operators. We can convert the kinetic energy and potential energy terms to second quantized form using Eq. (4.123). The expression for the kinetic energy operator can be further simplified by integrating by parts twice, which yields

$$\hat{T} = \frac{\hbar^2}{2m} \int d^d r \; \nabla \hat{\psi}^\dagger(\mathbf{r}) \, \cdot \, \nabla \hat{\psi}(\mathbf{r}). \tag{4.125}$$

The potential energy operator, meanwhile, simplifies to

$$\hat{V} = \int d^d r \; \hat{\psi}^\dagger(\mathbf{r}) \, V(\mathbf{r}) \, \hat{\psi}(\mathbf{r}). \tag{4.126}$$

Note that the $\hat{\mathbf{r}}$ in Eq. (4.124) is an operator, whereas the the $\mathbf{r}$ in Eqs. (4.125) and (4.126) are vectors (i.e., arrays of numbers). In a quantum field theory, the position serves as an index for the various field operators, and is not itself an observable.

Eqs. (4.125) and (4.126) are strongly reminiscent of the expectation values for the kinetic and potential energy in single-particle quantum mechanics:

$$\langle T \rangle = \frac{\hbar^2}{2m} \int d^d r \, |\nabla \psi(\mathbf{r})|^2 \,, \quad \langle V \rangle = \int d^d r \; V(\mathbf{r}) \; |\psi(\mathbf{r})|^2, \tag{4.127}$$

where $\psi(\mathbf{r})$ is the single-particle wavefunction. This accords with the aforementioned hand-waving interpretation of $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ as being derived by "quantizing" the wavefunction.

As mentioned in Section 4.2.6, we can also use creation and annihilation operators to construct observables that do not commute with the total particle number. One example is

$$\psi(\mathbf{r}) + \psi(\mathbf{r})^\dagger.$$

More generally, we can come up with observables of the form

$$F(\mathbf{r}) = \int d^d r' \Big(f(\mathbf{r}, \mathbf{r}') \hat{\psi}(\mathbf{r}) + f^*(\mathbf{r}, \mathbf{r}') \hat{\psi}^\dagger(\mathbf{r}') \Big), \tag{4.128}$$

where $f(\mathbf{r}, \mathbf{r}')$ is some complex function. We will shortly see that observables of this sort are used in the quantum generalization of classical field variables, like the electric field $\mathbf{E}(\mathbf{r})$ or the magnetic field $\mathbf{B}(\mathbf{r})$.

In the next two sections, we will try to get a better understanding of the relationship between classical fields and bosonic quantum fields. (The situation for fermionic quantum fields is more complicated; these cannot be related to classical fields of the sort we are familiar with, for reasons that lie outside the scope of this course.)

4.3.2. Revisiting the harmonic oscillator

Before delving into the links between classical fields and bosonic quantum fields, let us first revisit the harmonic oscillator. Our objective is to see how the concept of a **normal mode** carries over from classical to quantum mechanics.

A classical 1D harmonic oscillator is described by the Hamiltonian

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2, \tag{4.129}$$

where x is the oscillator's position variable, p is the corresponding momentum variable, m is the mass, and ω is the natural frequency of oscillation. We know that the classical equations of motion have solutions of the form

$$x(t) = \mathcal{A} \, e^{-i\omega t} + \mathcal{A}^* \, e^{i\omega t}. \tag{4.130}$$

Such a solution describes an oscillation of frequency ω, and is parameterized by the **mode amplitude** $\mathcal{A}$, a complex number determining the magnitude and phase of the oscillation.

For the quantum harmonic oscillator, x and p are replaced by the observables $\hat{x}$ and $\hat{p}$. From these, we can define the **ladder operators** $\hat{a}$ and $\hat{a}^\dagger$:

$$\begin{cases} \hat{a} = \sqrt{\dfrac{m\omega}{2\hbar}} \left(\hat{x} + \dfrac{i\hat{p}}{m\omega}\right), \\ \hat{a}^\dagger = \sqrt{\dfrac{m\omega}{2\hbar}} \left(\hat{x} - \dfrac{i\hat{p}}{m\omega}\right). \end{cases} \quad \Leftrightarrow \quad \begin{cases} \hat{x} = \sqrt{\dfrac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger\right) \\ \hat{p} = -i\sqrt{\dfrac{m\omega\hbar}{2}} \left(\hat{a} - \hat{a}^\dagger\right). \end{cases} \tag{4.131}$$

It can be shown that

$$\left[\, \hat{a}\, ,\, \hat{a}^\dagger \,\right] = 1, \quad \hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right). \tag{4.132}$$

Furthermore, we can use the ladder operators to derive the energy spectrum. The details should have been covered in an earlier course.

How do the ladder operators relate to the dynamics of the quantum harmonic oscillator? Let us go to the Heisenberg picture, with $t = 0$ as the reference time. The Heisenberg picture position operator is

$$\hat{x}(t) = \hat{U}^\dagger(t)\, \hat{x}\, \hat{U}(t), \tag{4.133}$$

where

$$\hat{U}(t) \equiv \exp\left(-\frac{i}{\hbar}\hat{H}t\right). \tag{4.134}$$

We will adopt the convention that all operators written with an explicit t-dependence refer to Heisenberg picture operators, while those written without a t-dependence are Schrödinger picture operators. For example, $\hat{x} \equiv \hat{x}(0)$. Now, referring to Eq. (4.131), we see that $\hat{x}(t)$ is related to the Heisenberg picture ladder operators by

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}} \Big(\hat{a}(t) + \hat{a}^\dagger(t)\Big). \tag{4.135}$$

The Heisenberg equation for the annihilation operator is

$$\begin{aligned} \frac{d\hat{a}(t)}{dt} &= \frac{i}{\hbar}\big[\hat{H}, \hat{a}(t)\big] \\ &= \frac{i}{\hbar}\,\hat{U}^\dagger(t)\big[\hat{H}, \hat{a}\big]\hat{U}(t) \\ &= \frac{i}{\hbar}\,\hat{U}^\dagger(t)\Big(-\hbar\omega\hat{a}\Big)\hat{U}(t) \\ &= -i\omega\hat{a}(t). \end{aligned} \tag{4.136}$$

Hence,

$$\hat{a}(t) = \hat{a}\, e^{-i\omega t}. \tag{4.137}$$

We can apply this, along with its Hermitian conjugate, to Eq. (4.135), obtaining

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}} \Big(\hat{a}\, e^{-i\omega t} + \hat{a}^\dagger\, e^{i\omega t}\Big). \tag{4.138}$$

This has exactly the same form as the classical oscillatory solution (4.130)! Comparing the two, we see that $\hat{a}$ times the scale factor $\sqrt{\hbar/2m\omega}$ plays the role of the mode amplitude $\mathcal{A}$.

Now, suppose we come at things from the opposite direction. We start with a pair of ladder operators that satisfy Eq. (4.132), from which Eqs. (4.136)–(4.137) follow. We then want to construct an observable corresponding to a classical oscillator variable. A natural Hermitian ansatz is

$$\hat{x}(t) = \mathcal{C}\left(\hat{a}\, e^{-i\omega t} + \hat{a}^\dagger\, e^{i\omega t}\right), \tag{4.139}$$

where $\mathcal{C}$ is a constant that is conventionally taken to be real.

How might $\mathcal{C}$ be chosen? To figure this out, we can study the behavior *in the classical limit*. The classical limit of a quantum harmonic oscillator is described by a **coherent state**. The details of how this state is defined need not concern us for now (see Appendix E). The most important things to know are that (i) it can be denoted by $|\alpha\rangle$ where $\alpha \in \mathbb{C}$, (ii) it is an eigenstate of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{4.140}$$

Also, (iii) its energy expectation value is

$$\langle E\rangle = \langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right) \overset{|\alpha|^2\to\infty}{\longrightarrow} \hbar\omega|\alpha|^2. \tag{4.141}$$

When the system is in a coherent state, we can effectively substitute the $\hat{a}$ and $\hat{a}^\dagger$ operators in Eq. (4.139) with the complex numbers α and α^*, which gives a classical trajectory

$$x_{\text{classical}}(t) = \mathcal{C}\left(\alpha\, e^{-i\omega t} + \alpha^*\, e^{i\omega t}\right). \tag{4.142}$$

This has amplitude $2\mathcal{C}|\alpha|$. At maximum displacement, the classical momentum is zero, so the total energy of the classical oscillator must be

$$E_{\text{classical}} = \frac{1}{2}m\omega^2\Big(2\mathcal{C}|\alpha|\Big)^2 = 2m\omega^2\mathcal{C}^2|\alpha|^2. \tag{4.143}$$

Equating the classical energy (4.143) to the coherent state energy (4.141) gives

$$2m\omega^2\mathcal{C}^2|\alpha|^2 = \hbar\omega|\alpha|^2 \tag{4.144}$$

$$\Rightarrow \qquad \mathcal{C} = \sqrt{\frac{\hbar}{2m\omega}}. \tag{4.145}$$

This exactly matches the factor in Eq. (4.138).

4.3.3. A scalar boson field

We now have the basic ingredients for understanding the connection between a classical field and its quantum counterpart. Consider a classical scalar field variable $f(x,t)$, defined in 1D space, whose classical equation of motion is the wave equation:

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2}. \tag{4.146}$$

The constant c is a wave speed. Eq. (4.146) arises in many physical contexts, including the propagation of sound through air, in which case c is the speed of sound.

Let us first consider the case where the field is defined in a finite interval of length L, with the periodic boundary conditions $f(x,t) \equiv f(x+L,t)$. Eq. (4.146) possesses solutions of the form

$$f(x,t) = \sum_n \left(\mathcal{A}_n\, \varphi_n(x)\, e^{-i\omega_n t} + \mathcal{A}_n^*\, \varphi_n^*(x)\, e^{i\omega_n t}\right), \tag{4.147}$$

which corresponds to a superposition of **normal modes**. Each normal mode (labelled n) varies harmonically in time with a mode frequency ω_n, and varies in space according to a complex mode profile $\varphi_n(x)$; its overall magnitude and phase is specified by the mode amplitude $\mathcal{A}_n$. The mode profiles are normalized according to some fixed convention, e.g.

$$\int_0^L dx\, |\varphi_n(x)|^2 = 1. \tag{4.148}$$

Substituting Eq. (4.147) into Eq. (4.146), and using the periodic boundary conditions, gives

$$\varphi_n(x) = \frac{1}{\sqrt{L}} \exp\left(ik_n x\right), \quad \omega_n = ck_n = \frac{2\pi c n}{L}, \quad n \in \mathbb{Z}. \tag{4.149}$$

These mode profiles are orthonormal:

$$\int_0^L dx\, \varphi_m^*(x)\, \varphi_n(x) = \delta_{mn}. \tag{4.150}$$

Each normal mode carries energy. By analogy with the classical harmonic oscillator—see Eqs. (4.142)–(4.143)—we assume that the energy per unit length is proportional to the square of the field variable. Let it have the form

$$U(x) = 2\rho \sum_n |\mathcal{A}_n|^2\, |\varphi_n(x)|^2, \tag{4.151}$$

where ρ is some parameter that has to be derived from the underlying physical context. For example, for acoustic modes, ρ is the mass density of the underlying acoustic medium. In the next chapter, we will see another example involving the energy density of an electromagnetic mode. From Eq. (4.151), we can express the total energy of the system as

$$E = \int_0^L dx\, U(x) = 2\rho \sum_n |\mathcal{A}_n|^2. \tag{4.152}$$

To quantize the classical field, we treat each normal mode as an independent oscillator, with creation and annihilation operators $\hat{a}_n^\dagger$ and $\hat{a}_n$ satisfying

$$\left[\,\hat{a}_m\,,\,\hat{a}_n^\dagger\right] = \delta_{mn}, \quad \left[\,\hat{a}_m\,,\,\hat{a}_n\right] = \left[\,\hat{a}_m^\dagger\,,\,\hat{a}_n^\dagger\right] = 0. \tag{4.153}$$

We then take the Hamiltonian to be that of a set of independent harmonic oscillators:

$$\hat{H} = \sum_n \hbar\omega_n \hat{a}_n^\dagger \hat{a}_n + E_0, \tag{4.154}$$

where E_0 is the ground-state energy. Just as in Section 4.3.2, the Heisenberg picture annihilation operator has the following solution:

$$\hat{a}_n(t) = \hat{a}_n e^{-i\omega_n t}. \tag{4.155}$$

As previously noted, we can use the creation and annihilation operators to construct observables. In the Schrödinger picture, consider an observable of the form

$$\hat{f}(x) = \sum_n \mathcal{C}_n \Big(\hat{a}_n \varphi_n(x) + \hat{a}_n^\dagger \varphi_n^*(x) \Big), \tag{4.156}$$

involving some real constant $\mathcal{C}_n$ for each normal mode n. In the Heisenberg picture,

$$\hat{f}(x,t) = \sum_n \mathcal{C}_n \Big(\hat{a}_n \varphi_n(x)\, e^{-i\omega_n t} + \hat{a}_n^\dagger \varphi_n^*(x)\, e^{i\omega_n t} \Big). \tag{4.157}$$

This is evidently the quantum version of the classical solution (4.147).

To determine the $\mathcal{C}_n$ factors, we consider the classical limit and follow the same procedure as Section 4.3.2. Let $|\alpha\rangle$ be a coherent state for all the normal modes; i.e., for each n,

$$\hat{a}_n |\alpha\rangle = \alpha_n |\alpha\rangle \tag{4.158}$$

for some $\alpha_n \in \mathbb{C}$. Then the expectation value for the total energy is

$$\langle E \rangle = \sum_n \hbar\omega_n |\alpha_n|^2. \tag{4.159}$$

In the coherent state, the $\hat{a}_n$ and $\hat{a}_n^\dagger$ operators in Eq. (4.157) can be replaced with α_n and α_n^* respectively. Hence, we identify $\mathcal{C}_n \alpha_n$ as the classical mode amplitude $\mathcal{A}_n$ in Eq. (4.147). In order for the classical energy (4.152) to match the coherent state energy (4.159),

$$2\rho|\mathcal{A}_n|^2 = 2\rho|\mathcal{C}_n\alpha_n|^2 = \hbar\omega_n|\alpha_n|^2 \tag{4.160}$$

$$\Rightarrow \qquad \mathcal{C}_n = \sqrt{\frac{\hbar\omega_n}{2\rho}}. \tag{4.161}$$

We thus conclude that the appropriate field operator is

$$\hat{f}(x,t) = \sum_n \sqrt{\frac{\hbar\omega_n}{2\rho}} \Big(\hat{a}_n \varphi_n(x)\, e^{-i\omega_n t} + \hat{a}_n^\dagger \varphi_n^*(x)\, e^{i\omega_n t} \Big). \tag{4.162}$$

Returning to the Schrödinger picture, and using the explicit mode profiles from Eq. (4.149),

$$\hat{f}(x) = \sum_n \sqrt{\frac{\hbar\omega_n}{2\rho L}} \Big(\hat{a}_n e^{ik_n x} + \hat{a}_n^\dagger e^{-ik_n x} \Big). \tag{4.163}$$

Finally, if we are interested in the infinite-L limit, we can convert the sum over n into an integral. The result is

$$\hat{f}(x) = \int dk\, \sqrt{\frac{\hbar\omega(k)}{4\pi\rho}} \Big(\hat{a}(k)\, e^{ikx} + \hat{a}^\dagger(k)\, e^{-ikx} \Big), \tag{4.164}$$

where $\hat{a}(k)$ denotes a rescaled annihilation operator defined by $\hat{a}_n \to \sqrt{2\pi/L}\; \hat{a}(k)$, satisfying

$$\big[\, \hat{a}(k)\,,\, \hat{a}^\dagger(k')\,\big] = \delta(k-k'). \tag{4.165}$$

4.3.4. Looking ahead

In the next chapter, we will see how quantum field theory can be used to formulate a quantum theory of electrodynamics. The electromagnetic field will be described by a boson field, whose $\hat{a}^\dagger$ and $\hat{a}$ operators create and annihilate photons, or particles of light. We will see how to define observables that correspond to the classical electromagnetic field variables, and how the quantum dynamics of the electromagnetic field reduces to Maxwell's equations in the classical limit. The electromagnetic field interacts with electrons, which are in turn described using a fermion field.

Quantum electrodynamics is only one example of the importance of quantum field theory. In fundamental physics, all currently-known elementary particles can be described using a quantum field theory called the Standard Model, which incorporates numerous boson fields (e.g., photons and gluons) and fermion fields (e.g., electrons and quarks) that interact with one another. Quantum field theory is also widely used in condensed matter physics, since it is the natural framework for dealing with systems of many particles.

Wilczek has pointed out that the modern picture of fundamental physics bears a striking resemblence to the old idea of the "luminiferous ether": a medium that fills all of space and time, whose vibrations are physically-observable light waves. The key difference is that this ether is not a classical medium, but one obeying the rules of quantum mechanics. (Another difference is that is that the fundamental quantum fields obey relativistic symmetries, so that their behavior is invariant under a change of reference frame.)

It is interesting to contemplate the idea of fields, not individual particles, as the fundamental objects in the universe. This would explain, in a sense, why all particles of the same type have identical properties—why, for example, all electrons have exactly the same mass and charge. Individual particles have no distinguishing properties since they are not themselves fundamental, but merely excitations of underlying entities—quantum fields!

EXERCISES

1. Consider a system of two identical particles. Each single-particle Hilbert space $\mathscr{H}^{(1)}$ is spanned by a basis $\{|\mu_i\rangle\}$. The exchange operator is defined on $\mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)}$ by

$$P\Big(\sum_{ij} \psi_{ij}|\mu_i\rangle|\mu_j\rangle\Big) \;\equiv\; \sum_{ij} \psi_{ij}|\mu_j\rangle|\mu_i\rangle. \tag{4.166}$$

   Prove that $\hat{P}$ is linear, unitary, and Hermitian. Moreover, prove that the operation is basis-independent: i.e., given any other basis $\{\nu_j\}$ that spans $\mathscr{H}^{(1)}$,

$$P\Big(\sum_{ij} \varphi_{ij}|\nu_i\rangle|\nu_j\rangle\Big) \;=\; \sum_{ij} \varphi_{ij}|\nu_j\rangle|\nu_i\rangle. \tag{4.167}$$

2. Prove that the exchange operator commutes with the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m_e}\Big(\nabla_1^2 + \nabla_2^2\Big) + \frac{e^2}{4\pi\varepsilon_0|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{4.168}$$

3. An N-boson state can be written as

$$|\phi_1, \phi_2, \dots, \phi_N\rangle = \mathcal{N} \sum_p \Big(|\phi_{p(1)}\rangle |\phi_{p(2)}\rangle |\phi_{p(3)}\rangle \cdots |\phi_{p(N)}\rangle \Big). \tag{4.169}$$

   Prove that the normalization constant is

$$\mathcal{N} = \sqrt{\frac{1}{N! \prod_\mu n_\mu!}}, \tag{4.170}$$

   where n_μ denotes the number of particles occupying the single-particle state μ.

4. $\mathscr{H}_+^{(N)}$ and $\mathscr{H}_-^{(N)}$ denote the Hilbert spaces of N-particle states that are totally symmetric and totally antisymmetric under exchange, respectively. Prove that

$$\dim\left(\mathscr{H}_+^{(N)}\right) = \frac{(d+N-1)!}{N!(d-1)!}, \tag{4.171}$$

$$\dim\left(\mathscr{H}_-^{(N)}\right) = \frac{d!}{N!(d-N)!}. \tag{4.172}$$

5. Prove that for boson creation and annihilation operators, $[\hat{a}_\mu, \hat{a}_\nu] = [\hat{a}_\mu^\dagger, \hat{a}_\nu^\dagger] = 0$.

6. Let $\hat{A}_1$ be an observable (Hermitian operator) for single-particle states. Given a single-particle basis $\{|\varphi_1\rangle, |\varphi_2\rangle, \dots\}$, define the bosonic multi-particle observable

$$\hat{A} = \sum_{\mu\nu} a_\mu^\dagger \, \langle \varphi_\mu | \hat{A}_1 | \varphi_\nu \rangle \, a_\nu, \tag{4.173}$$

   where $a_\mu^\dagger$ and a_μ are creation and annihilation operators satisfying the usual bosonic commutation relations, $[a_\mu, a_\nu] = 0$ and $[a_\mu, a_\nu^\dagger] = \delta_{\mu\nu}$. Prove that $\hat{A}$ commutes with the total number operator:

$$\left[\hat{A}, \sum_\mu a_\mu^\dagger a_\mu\right] = 0. \tag{4.174}$$

   Next, repeat the proof for a fermionic multi-particle observable

$$\hat{A} = \sum_{\mu\nu} c_\mu^\dagger \, \langle \varphi_\mu | \hat{A}_1 | \varphi_\nu \rangle \, c_\nu, \tag{4.175}$$

   where $c_\mu^\dagger$ and c_μ are creation and annihilation operators satisfying the fermionic anti-commutation relations, $\{c_\mu, c_\nu\} = 0$ and $\{c_\mu, c_\nu^\dagger\} = \delta_{\mu\nu}$. In this case, prove that

$$\left[\hat{A}, \sum_\mu c_\mu^\dagger c_\mu\right] = 0. \tag{4.176}$$

FURTHER READING

[1] Bransden & Joachain, §10.1–10.5

[2] Sakurai, §6

[3] I. Duck and E. C. G. Sundarshan, *Pauli and the Spin-Statistics Theorem*, World Scientific (1998).

[4] J. M. Leinaas and J. Myrheim, *On the Theory of Identical Particles*, Nuovo Cimento B **37**, 1 (1977).

[5] F. Wilczek, *The Persistence of Ether*, Physics Today **52**, 11 (1999). [link]