Market Crashes as Critical Points

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Stock Market Crashes

- In the last century, we can identify a total of five large market crashes:
  - 1914 (out-break of World War I),
  - October 1929 (triggering the Great Depressions of the 1930s),
  - October 1987 (the Black Monday),
  - July 1997 (onset of the Asian Economic Crisis), and most recently,
  - the NASDAQ crash of April 2000.
Large Market Crashes are Extraordinary!

- By plotting the cumulative frequencies of daily loses in the stock market against the log-magnitude of the loss, Sornette et al found that:
  - “normal day-to-day” trading results in an exponential daily loss distribution.
  - the above large crashes are statistical outliers, very much out of the ordinary.
Market Crashes and Earthquakes
Precursors and Aftershocks

\[ \text{S&P 500} \]

\[ \sigma^2 (\text{S&P 500}) \]
Modeling Market Crashes — Macroscopic Considerations

- Rational market with incomplete information: not every aspect of the market is known, but whatever is known is reflected in the price of stocks and their fluctuations.

- Price of stock reflects not only the fundamental worth of whatever it represents, but also possible future gains, which comes at a risk (or hazard rate).

- The more risky the stock, the higher it is priced.
Modeling Market Crashes — the Microscopic Model of Sornette and Johansen

- System of $N$ traders in a trading network, in which each trader $i = 1, \ldots, N$ is connected to $N(i)$ nearest neighbors in its neighborhood $\mathcal{N}_i$ according to some graph.

- Traders interact only locally via such a network.

- In this highly simplified model, each trader $i$ can only be in one of two states $s_i = -1$ (BUY) or $s_i = +1$ (SELL) at any one time, reflecting the major pre-occupation of the trader at that time.
• Decision process of trader \( i \) only influenced by
  
  – opinions of the \( N(i) \) traders in its neighborhood;
  
  – an idiosyncratic signal received by trader \( i \) alone.

• Time evolution governed by a cellular automaton rule:

\[
  s_i(t + 1) = \text{sign} \left[ K \sum_{j \in \mathcal{N}_i} s_j(t) + \sigma \epsilon_i(t) \right], \quad i = 1, \ldots, N.
\]

• \( K = \) coupling strength — orders the system of traders, \( \sigma = \) strength of noise term — disorders the system of traders. Relative size of \( K \) and \( \sigma \) determines whether system is ordered or disordered.

• In this language, a market crash occurs whenever instantaneous correlations due to local fluctuations get magnified to \( O(N) \) proportions by the positive feedback intrinsic in the interactions.
Results from the Sornette-Johansen Model

- Existence of critical points $K_C$ on most networks.

- Susceptibility $\chi = \text{sensitivity of system to small perturbations} \text{ diverges as power law as } K \text{ approaches } K_C \text{ from below:}$

$$\chi \sim A(K_C - K)^{-\gamma},$$

where $\gamma > 0$ is the critical exponent of the susceptibility.

- Assume that the system is driven exogeneously slowly, such that $K$ approaches $K_C$ linearly as

$$K_C - K(t) \approx \alpha(t_C - t),$$

where the critical time $t_C = \text{most probable time for market crash.}$
Reasonable to assume that the average hazard rate $h(t)$ in the market should be positively correlated with $\chi(K(t))$, i.e.

$$h(t) \sim B(t_C - t)^{-\alpha},$$

for $0 < \alpha < 1$ (so that the stock price remains finite at the critical point).

Rational Expectation Model implies

$$p(t) = \int_{t_0}^{t} h(t') \, dt',$$

where $p(t) =$ price of stock, or stock index at time $t \implies$ power-law acceleration of price increase near $t_C$. 
Additional Results of Model on Hierarchical Lattice

- Power-law divergence of $\chi$ decorated by *log-periodic oscillations* due to discrete scale invariance in hierarchical lattice;

  $$\chi \sim A_0^l (K_C - K)^{-\gamma} + A_1^l (K_C - K)^{-\gamma} \cos [\omega \log (K_C - K) + \psi] + \cdots .$$

- Such log-periodic oscillations reflected in $p(t)$ too;

- Sornette *et al* and Feigenbaum *et al* fitted market data to extract $t_C$ — reasonable agreement. (include Feigenbaum’s graphs)
Examples of Small-World Networks

(a) = regular 1-dimensional clustered lattice with range of interaction $k$,
(b) = Watts-Strogatz small-world network — randomly rewiring $qkN$ bonds,
(c) = Newman-Watts small-world network — addition of $qkN$ random bonds.
Majority-Rule

- I chose states $s_i \in \{0, 1\}$ to use boolean variables. $0 = \text{BUY}, 1 = \text{SELL}$.

- Majority-rule to generate time evolution. Stochastic parameter $p = \text{probability that trader will take risk to change trading strategy when local trading network ambivalent.}$

$$s_i(t + 1) = \begin{cases} 
\text{MAJORITY} \left[ s_i(t); \{ s_j(t) \mid j \in \mathcal{N}_i \} \right], & \text{if } R_i(t) < 1 - p, \\
\text{MAJORITY} \left[ \text{NOT}[s_i(t)]; \{ s_j(t) \mid j \in \mathcal{N}_i \} \right], & \text{otherwise};
\end{cases}$$

where MAJORITY-function returns 0 if majority of traders buying and 1 if majority of traders selling. NOT is binary negation.
Preliminary Results

Random Network

Newman-Watts Small-World Network

- Equilibrium state when roughly half of the traders buying and half of the traders selling;

- Equilibrium state of random network much more sensitive to small perturbations than that for Newman-Watts small-world network.
Continuous Phase Transition for $k = 1$!

Asymptotic average state $\langle s_i(\infty) \rangle$ for various $p$.

Order parameter $m(p) = \langle s_i(\infty) \rangle - \frac{1}{2}$ as a function of $p$. 
• Signs of continuous phase transition;

• $k = 1$ corresponds to 1-dimensional Ising model. If $p \leftrightarrow T$, then $T_C = 0$ for 1-dimensional Ising model $\implies p_C = 0$. But not the case: $p_C > 0$!

• No such qualitative changes for $k > 2$. 
Further Investigations

- Critical exponent of $k = 1$ phase transition in Newman-Watts small-world network;

- Comparison between majority-rule and Sornette-Johansen sign-rule — can we have phase transitions for $k > 2$ Newman-Watts small-world networks?

- Sornette and Johansen used hierarchical lattices — real world traders organized into hierarchies. But hierarchical lattice exhibit no clustering — Newman-Watts random rewiring to give hybrid hierarchical small-world networks?

- Stock index as an endogeneous global influence term?
• Fundamental diagram for the stock market?

• Random update rules and effective update rules?
Comments, suggestions and collaborations welcomed!