Many-Body Fermion Density Matrices and Pattern-Forming Cellular Automata

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Why Numerical Methods?

- Ground-state properties (energy, correlations, $T = 0$ phase diagram) of $N \to \infty$ interacting QM degrees of freedom (spins, bosons, fermions) can be calculated from the ground-state wave function.

- Exact analytical many-body wave functions rare.

- Approximate analytical many-body wave functions
  - Perbutative: not valid over all Hamiltonian parameter(s); or
  - Variational: involve \textit{a priori} assumptions on structure of wave function.

- Numerical methods like
  - Exact Diagonalization (ED); and/or
  - Quantum Monte Carlo (QMC)

to obtain numerical wave functions or correlations of finite systems. Extrapolations then needed for $N \to \infty$. 
Why Density Matrices?

- Build up QM state of infinite system from QM states of finite subsystems.

- Pure state on infinite system $\iff$ mixed state on finite subsystem.
  
  (wave function $\Psi$) $\iff$ (density matrix $\rho$)

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Why Density Matrices?

- Calculation of correlations of products of local observables.

\[
\langle \Psi | c_1^\dagger c_2^\dagger c_3 c_4 | \Psi \rangle = \langle c_1^\dagger c_2^\dagger c_3 c_4 \rangle = \text{Tr} \rho_{AB} c_1^\dagger c_2^\dagger c_3 c_4.
\]
Quantum Renormalization Group (QRG)

- Repeated cycles of truncation and renormalization. [S. R. White, PRL 69, 2863 (1992); R. J. Bursill, PRB 60, 1643 (1999)]

- Truncation naturally guided by density matrix (DM).

- Understanding structure of DM may lead to algorithmic improvements (e.g. Transfer-Matrix Renormalization Group (TMRG)) and better ways to build symmetries of problem into RG.
Noninteracting Spinless Fermions in $d$ Dimensions

\[ H = -t \sum_{\langle r_i, r_j \rangle} \left[ c_i^\dagger c_j + c_j^\dagger c_i \right], \quad |\Psi_F\rangle = \text{Fermi sea ground state} \]
Exact Formula for Cluster DM

- For cluster of $N_C$ sites, DM found to have the structure [M.-C. Chung and I. Peschel, PRB 64, 064412 (2001)]

\[ \rho_C \propto \exp \left[ - \sum_{i=1}^{N_C} \varphi_i f_i^\dagger f_i \right], \quad \{f_i, f_i^\dagger\} = 1. \]

- Start from normalized grand-canonical DM of system

\[ \rho_0 = \mathcal{Z}^{-1} \exp \left[ -\beta (H - \mu F) \right] = \mathcal{Z}^{-1} \exp \left[ \sum_{i,j} \Gamma_{i,j} c_i^\dagger c_j \right] = \mathcal{Z}^{-1} \exp \left[ \sum_k \tilde{\Gamma}_{kk} \tilde{c}_k^\dagger \tilde{c}_k \right], \]

chemical potential $\mu$, inverse temperature $\beta$, fermion number operator $F = \sum_i c_i^\dagger c_i = \sum_k \tilde{c}_k^\dagger \tilde{c}_k$, grand-canonical partition function $\mathcal{Z}$, and coefficient matrices $\Gamma$ ($\tilde{\Gamma}$ in momentum space).

- Introduce fermionic coherent states

\[ |\xi \eta\rangle = |\xi_1 \cdots \xi_{N_C}; \eta_1 \cdots \eta_{N-N_C}\rangle = \exp \left( - \sum_{i=1}^{N_C} \xi_i c_i^\dagger - \sum_{j=1}^{N-N_C} \eta_j c_j^\dagger \right) |0\rangle. \]

$\xi_i$ and $\eta_j$ are anticommuting Grassman variables.
Exact Formula for Cluster DM

• Matrix elements of $\rho_0$ are

$$\langle \xi \eta | \rho_0 | \xi' \eta' \rangle = D^{-1} \exp \left[ (\xi^* \eta^*) e^\Gamma (\xi' \eta') \right].$$

• Coefficient matrices

$$1 + e^\Gamma = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad (1 + e^\Gamma)^{-1} = \begin{bmatrix} D & E \\ E^T & F \end{bmatrix},$$

$A$ and $D$ square $N_C \times N_C$ symmetric matrices, $B$ and $E$ nonsquare $N_C \times (N - N_C)$ matrices, $C$ and $F$ square $(N - N_C) \times (N - N_C)$ symmetric matrices.

• Partial trace over environment, gaussian integration and matrix block inversion gives matrix elements of cluster DM

$$\langle \xi | \rho_C | \xi' \rangle = \int d\eta^* d\eta e^{-\eta^* \eta} \langle \xi - \eta | \rho_0 | \xi' \eta \rangle$$

$$= \det D \exp \left\{ \xi^* \left[ D^{-1} - 1 \right] \xi' \right\}.$$
• Organize two-point functions $G_{ij} = \langle \Psi_F | c_i^\dagger c_j | \Psi_F \rangle$ into Green-function matrix $\mathcal{G}$ and cluster Green-function matrix $G_C$:

\[
\mathcal{G} = \begin{bmatrix}
G_{11} & G_{12} & \cdots & G_{1N_C} & G_{1N_C+1} & \cdots & G_{1N} \\
G_{21} & G_{22} & \cdots & G_{2N_C} & G_{2N_C+1} & \cdots & G_{2N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
G_{N_{C1}} & G_{N_{C2}} & \cdots & G_{N_{CN_C}} & G_{N_{CN_C}+1} & \cdots & G_{N_{CN}} \\
G_{N_{C+11}} & G_{N_{C+12}} & \cdots & G_{N_{C+1N_C}} & G_{N_{C+1N_C}+1} & \cdots & G_{N_{C+1N}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
G_{N_1} & G_{N_2} & \cdots & G_{NN_C} & G_{NN_C+1} & \cdots & G_{NN} \\
\end{bmatrix}
\]

$G_C$
Exact Formula for Cluster DM

- Momentum space matrix elements of $\tilde{G}$ and $\tilde{\Gamma}$,

$$\tilde{G}_{kk} = \langle \Psi_F | \tilde{c}_k^\dagger \tilde{c}_k | \Psi_F \rangle = \frac{1}{\exp \beta (\epsilon_k - \mu) + 1}, \quad \tilde{\Gamma}_{kk} = -\beta (\epsilon_k - \mu)$$

- Matrix relations

$$e^{\tilde{\Gamma}} = \tilde{G} (1 - \tilde{G})^{-1} \implies e^{\Gamma} = G (1 - G)^{-1}, \quad 1 + e^{\Gamma} = (1 - G)^{-1}.$$

- Cluster matrix relations

$$D = 1 - G_C, \quad D^{-1} = (1 - G_C)^{-1}, \quad D^{-1} - 1 = G_C (1 - G_C)^{-1}.$$

- Cluster DM matrix elements

$$\langle \xi | \rho_C | \xi' \rangle = \det (1 - G_C) \exp \left[ \xi^* G_C (1 - G_C)^{-1} \xi' \right].$$


$$\rho_C = \det (1 - G_C) \exp \left\{ \sum_{i,j} \left[ \log G_C (1 - G_C)^{-1} \right]_{ij} c_i^\dagger c_j \right\}$$
Many-Body Eigenstates and Eigenvalues of Cluster DM

- Eigenstates and eigenvalues of cluster Green-function matrix

\[ |\lambda_l\rangle = f_l^\dagger |0\rangle, \quad G_C |\lambda_l\rangle = \lambda_l |\lambda_l\rangle. \]

- \( |\lambda_l\rangle \) simultaneous 1-particle eigenstates of \( \rho_C \),

\[ \rho_C |\lambda_l\rangle = \det(1 - G_C) e^{-\varphi_l} |\lambda_l\rangle, \quad \varphi_l = -\ln \left[ \lambda_l(1 - \lambda_l)^{-1} \right]. \]

- \( P \)-particle eigenstate of \( \rho_C \) described by a set of numbers \( (n_1, \ldots, n_l, \ldots, n_{NC}) \), \( n_l = 0, 1, \)

\[ |w\rangle = f_1^\dagger f_2^\dagger \cdots f_P^\dagger |0\rangle, \quad n_l = \delta_{l,l_i}, \]

with eigenvalue (DM weight)

\[ w = \det(1 - G_C) \exp (-\Phi), \quad \Phi = \sum_{l=1}^{NC} n_l \varphi_l. \]
Statistical Mechanics Analogy

- [SAC and C. L. Henley, PRB 69, 075112 (2004)]

<table>
<thead>
<tr>
<th>free spinless fermion</th>
<th>(\rho_C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamiltonian</td>
<td>(\hat{H} = \sum_l \varphi_l f_l^\dagger f_l)</td>
</tr>
<tr>
<td>1-particle energy</td>
<td>(\varphi_l)</td>
</tr>
<tr>
<td>(\epsilon_k)</td>
<td>(f_l)</td>
</tr>
<tr>
<td>(\tilde{c}_k)</td>
<td>(n_l)</td>
</tr>
<tr>
<td>occupation number</td>
<td>(n_l)</td>
</tr>
<tr>
<td>(n_k)</td>
<td>(\Phi = \sum_l n_l \varphi_l)</td>
</tr>
<tr>
<td>total energy</td>
<td>total pseudo-energy</td>
</tr>
<tr>
<td>(E = \sum_l n_k \epsilon_k)</td>
<td>(\varphi_F)</td>
</tr>
<tr>
<td>Fermi level</td>
<td>pseudo-Fermi level</td>
</tr>
<tr>
<td>(\epsilon_F)</td>
<td></td>
</tr>
</tbody>
</table>

- Based on analogy, average pseudo-occupation is

\[
\langle n_l \rangle = \lambda_l = \frac{1}{\exp \varphi_l + 1}.
\]

- Most probable eigenstate of \(\rho_C\) has structure of Fermi sea: \(\varphi_l \leq \varphi_F\) occupied, \(\varphi_l > \varphi_F\) empty.

- Other eigenstates look like ‘excitations’ about Fermi sea.
Operator-Based DM Truncation Scheme

- DM eigenstates with largest weights always have $\varphi_l \ll \varphi_F$ occupied and $\varphi_l \gg \varphi_F$ empty. These differ in $n_l$ for $\varphi_l \approx \varphi_F$;
- Keep only $f_l^\dagger$ with $\varphi_l \approx \varphi_F$:

- Compare with weight-ranked truncation (used for e.g., in the DMRG):
  - eigenstates with largest weights all kept;
  - some eigenstates with intermediate weights not kept, but replaced with eigenstates with slightly smaller weights;
  - eigenstates with small weights not kept.
Results: 1D Noninteracting Spinless Fermions

\[ \varepsilon(k) \]

\[
\begin{align*}
\Delta E &= \varepsilon_F \\
N_C &= 10^{-18} 10^{-16} 10^{-14} 10^{-12} 10^{-10} 10^{-8} 10^{-6} 10^{-4} 10^{-2} 10^0
\end{align*}
\]

\[ \gamma = 0.25, 0.33, 0.50, 0.67, 0.75, 0.80 \]

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Cluster DM on 2D Square Lattice

- Definition of system:

- 5-site cluster, various system sizes $N = |R_1 \times R_2|$.

- Computation of cluster DM $\rho_C$:
  - obtain ground state $|\Psi\rangle$ (exact diagonalization or otherwise);
  - $\rho_0 = |\Psi\rangle \langle \Psi| \xrightarrow{\text{partial trace}} \rho_C$ (care with fermion sign!);
  - translational invariance;
  - degeneracy, orientation and twist boundary conditions averaging.
2D Cluster DM Weights

- nearest neighbor hopping (noninteracting) and nearest neighbor hopping + infinite nearest neighbor repulsion (strongly interacting);
- 0-particle weight not interesting — monotonic decreasing with filling $\bar{n}$, very similar for noninteracting and strongly interacting systems;
- 5 1-particle weights, characterized by “angular momentum” quantum numbers $s_1$, $p_x$, $p_y$, $d$, $s_2$.
- Infinite system limit for noninteracting system, $\sim 200$ sites for a squarish finite system without twist boundary conditions averaging;
- Small finite systems (noninteracting & interacting) of $\sim 20$ sites, strong influence from finite size effects (most severe for $d$ state, least severe for $s_1$ state) $\Rightarrow$ require twist boundary conditions averaging.
1-Particle Weights (Noninteracting)
1-Particle Weights (Strongly Interacting)

- $w_s^1$
- $w_p$
- $w_s^2$
- $w_d$

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Correlation DM

- Entanglement entropy $S = - \text{Tr} \rho_C \log \rho_C$ as gross diagnostic of correlations. [Vidal et al, PRL 90, 227902 (2003)].

- Systematic extraction of order parameters from cluster DM:
  -Disconnected clusters $A$ at $r$ and $B$ at $r'$;
  -Cluster DMs $\rho_A$ and $\rho_B$, supercluster DM $\rho_{AB}$;
  -Define correlation DM, $\rho^c = \rho_{AB} - \rho_A \otimes \rho_B$;

- Correlation DM contains all correlations between $A$ and $B$ — want to attribute these correlations to various order parameters.
Singular Value Decomposition of Correlation DM

- Start from operator basis of referencing operators
  \[ K_n = \prod_i \left[ n_i c_i + (1 - n_i) c_i^\dagger \right], \quad K_n |n'\rangle = \delta_{nn'} |0\rangle. \]

- Write \( \rho^c = \sum_{n,n'} (-1)^{f_{nn'}} \langle n | \rho_{AB} | n' \rangle - \langle l | \rho_A | l' \rangle \langle m | \rho_B | m' \rangle \] \( K_l^\dagger K_{l'} K_m^\dagger K_{m'} \), where \( |n\rangle = |l\rangle |m\rangle \), and \( K_n = K_l K_m \).

- Product of referencing operators orthonormal with respect to Frobenius norm
  \[
  \text{Tr} \, X_{ll'} X_{l'l''} = \delta_{ll'}, \quad \text{Tr} \, Y_{mm'} Y_{m'm''} = \delta_{mm'}, \quad \text{Tr} \, X_{ll'} X_{l'l''} Y_{mm'} Y_{m'm''} = \delta_{ll'}, \delta_{mm'}, \delta_{ll''}, \delta_{mm''}. \]

- Numerical singular value decomposition (SVD) of coefficient matrix of \( \rho^c \) gives
  \[
  \rho^c = \sum_\alpha \sigma_\alpha X_\alpha Y_\alpha^\dagger. \]

- \( X_\alpha Y_\alpha^\dagger \) and \( X_\beta Y_\beta^\dagger \) represent independent quantum fluctuations on clusters A and B, i.e. can treat \( X_\alpha \) and \( Y_\alpha \) as order parameters.
Extended Hubbard Ladder of Spinless Fermions

\[ H = -t \sum_{a} \sum_{j} \left( c_{j,a}^{\dagger} c_{j+1,a} + c_{j+1,a}^{\dagger} c_{j,a} \right) - t \sum_{j} \left( c_{j,1}^{\dagger} c_{j,2} + c_{j,2}^{\dagger} c_{j,1} \right) \]

\[ - t' \sum_{j} \left( c_{j,1}^{\dagger} n_{j+1,2} c_{j+2,1} + c_{j+2,1}^{\dagger} n_{j+1,2} c_{j,1} \right) \]

\[ - t' \sum_{j} \left( c_{j,2}^{\dagger} n_{j+1,1} c_{j+2,2} + c_{j+2,2}^{\dagger} n_{j+1,1} c_{j,2} \right) \]

\[ + V \sum_{a} \sum_{j} n_{j,a} n_{j+1,a} + V \sum_{j} n_{j,1} n_{j,2} \]
Expected Order Parameters

- $V \to \infty$, no nearest-neighbor occupation, smaller Hilbert space for exact diagonalization.

- Basic physics that of spinless Luttinger liquid:
  - Power-law decay of charge density wave (CDW) and superconducting (SC) correlations;
  - CDW dominate at long distances if $K_\rho < 1$, SC dominate at long distances if $K_\rho > 1$, Fermi liquid (FL) if $K_\rho = 1$;
  - Insulator at half-filling.

- Tunable parameters in model:
  - Filling fraction $\bar{n}$: fermion fluid for $\bar{n} \gtrsim 0$, hole fluid for $\bar{n} \lesssim \frac{1}{2}$;
  - Correlated hop $t'$ favors pairing and hence SC correlations.
Results From SVD of Correlation DM

\[ \frac{t'}{t} = 0 \]
\[ \frac{t'}{t} = 1 \]
\[ \frac{t'}{t} = 10 \]
\[ \frac{t'}{t} = \infty \]

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Conclusions for Part I

- Learning from noninteracting spinless fermions:
  - Exact formula for cluster DM;
  - Statistical mechanics analogy;
  - Operator-Based DM Truncation Scheme;
  - When 2D infinite-system limit reached numerically;
  - Effectiveness of averaging apparatus.

- Applying to strongly interacting spinless fermions:
  - Adaptation and extension of Operator-Based DM Truncation Scheme;
  - Signatures of quantum phase transitions.

- SVD of correlation DM
  - Systematic extraction of order parameters;
  - Signatures of quantum phase transitions.
Part II

Pattern-Forming Cellular Automata
Cellular Automata

- A collection of finite state machines. The state of the $i^{th}$ machine at time $t$ given by $s_i(t) \in \mathcal{A}$, where $\mathcal{A}$ is a finite set, also called the alphabet;

- A collection of neighborhoods. The neighborhood of the $i^{th}$ machine is denoted by $\mathcal{N}_i$;

- A dynamical rule $\varphi : \mathcal{N}_i \to \mathcal{A}$, such that $s_i(t + 1) = \varphi(s_j(t) \mid j \in \mathcal{N}_i)$. 
Classification of CAs

- Elementary and compound CAs. Examples are Game of Life (GOL) and the Nagel-Schreckenberg model of traffic flow respectively.

- Wolfram classified all 256 1D elementary CAs (ECAs) by their dynamical properties. Types I, II and III.

- Wolfram naming convention: if the ECA is

\[
\begin{array}{cccccccccc}
111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\
\end{array}
\]

then Wolfram rule number is \( \sum_{j=0}^{7} \alpha_j 2^j \).

- No known attempts at classifying ECAs of higher dimensions.
From Pattern to ECA

- In P681 Pattern Formation and Spatio-Temporal Chaos/Prof Eberhard Bodenschatz, given PDE model, find what patterns form spontaneously. Can do the same for CA models.

- Ask the inverse question instead: given a pattern, what are all the possible CAs that spontaneously generate it?

- Two parts to this question:
  - what CA rules will have given pattern as fixed point; and
  - under which CA rules is the pattern stable?
Consider striped phase in 1D:

\[ \begin{array}{cccccccccccc}
O & O & O & O & O & O & O & O & O & O & O & O \\
\end{array} \]

Fixed point requirement implies the transition rules

\[ \begin{array}{c}
\text{\textbullet\textbullet\textbullet} \rightarrow \times \bigcirc \times \\
\text{\bigcirc\textbullet\textbullet} \rightarrow \times \textbullet \times
\end{array} \]

Does not uniquely determine ECA rule, 6 more transition rules to specify.
Defects in Striped Phase

- To analyze stability of striped phase, need to investigate behaviour of departures from pattern, i.e. defects, under various ECA rules.

- Point defects:
  - vacancy
  - interstitial

- Domain walls:
  - $-1$ domain wall
  - $+1$ domain wall
Strips Stable in Presence of Point Defects

- Since ECA not completely specified, can choose remaining transition rules to stabilize striped phase in presence of point defects.

- Demand that isolated vacancy ‘heals’: implies transition rules

\[
\text{〇〇〇} \rightarrow \times \text{●} \times, \quad \text{●〇〇} \rightarrow \times \text{〇} \times, \quad \text{〇〇●} \rightarrow \times \text{〇} \times.
\]

- Demand that isolated interstitial ‘heals’: implies transition rules

\[
\text{●●●●} \rightarrow \times \text{〇} \times, \quad \text{〇●●●} \rightarrow \times \text{●} \times, \quad \text{●●〇} \rightarrow \times \text{●} \times.
\]

- ECA completely specified by requirements that: (a) striped phase is fixed point; (b) isolated vacancies ‘heal’; and (c) isolated interstitials ‘heal’.
Completed ECA Rule

ECA is Rule 77 in Wolfram’s classification scheme:

<table>
<thead>
<tr>
<th>$s_{j-1}(t)$</th>
<th>$s_j(t)$</th>
<th>$s_{j+1}(t)$</th>
<th>$s_j(t + 1)$</th>
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<td>0</td>
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</table>
Further Considerations

- **Domain Wall Dynamics.** Both $\pm 1$ domain walls stationary under Rule 77, i.e. if start from random initial configuration, all domain walls initially present will be ‘frozen in’.

- **Robustness of Striped Phase.** By modifying some transition rules in Rule 77, can test genericity of striped pattern. Found that:
  - Striped phase *most* stable under Rule 77, but also stable under 6 other ECA rules derived from Rule 77, in which a single transition rule is modified.
  - Striped phase *marginally* stable under 4 ECA rules derived from Rule 77, in which one or two transition rules are modified.
  - Striped phase unstable once more than two transition rules are modified from Rule 77. Oscillatory phase nucleates.
2D ECAs

- In 1D, neighborhood simple, unless one wants to go to next nearest neighbor.
- In 2D, greater variety of neighborhoods. Simplest neighborhood for 2D CA is von Neumann (VN) neighborhood:

  ![VN Neighborhood Diagram]

  - With VN neighborhood, total of $2^5 = 32$ possible local configurations $\implies$ total of $2^{32} = 4,294,967,296$ 2D ECAs.
A traveling wave phase with $\lambda = 4$ and $\nu = +1$ looks like

The traveling wave transition rules are

$$\begin{align*}
\begin{array}{cccc}
\x & \rightarrow & \x \\
\x & \rightarrow & \x \\
\x & \rightarrow & \x \\
\x & \rightarrow & \x \\
\end{array}
\end{align*}$$
• Unlike in 1D, point defect analysis alone cannot fully specify ECA. Need to do multiple defect analysis.

• Four types of point defect:

\[ \begin{array}{c}
V_L \\
\hline
V_R \\
\hline
I_L \\
\hline
I_R \\
\end{array} \]

• In this chosen pattern, transition rules implied by \( V_L \) conflicts with that implied by \( V_R \), and transition rules implied by \( I_L \) conflicts with that implied by \( I_R \).

• Generic problem.
Protocol for Conflict Resolution

- When transition rule implied by two configurations in conflict, give precedence to configuration with lower number of defects.

- When transition rule implied by leading edge configuration conflicts with that implied by trailing edge configuration, give precedence to trailing edge configuration.

- Can show that some multi-defect configurations whose implied transition rules are forfeited will still be ‘healed’.

- Compromise necessary because traveling wave breaks left-right symmetry.

- Completed CA rule is Rule 2,383,284,874.
Simulating Rule 2,383,284,874
Compounds CAs

• Some patterns cannot be achieved using ECAs because conflict resolution protocol used cannot ensure stability of desired pattern.

• What to do?
  – Use larger neighborhoods — equivalent to a restricted class of compound ECAs.
  – Use larger state space, say $s_i(t) = 0, \frac{1}{2}, 1$.

• The main idea is to increase the number of transition rules available for pattern matching.

• Another way is to compound together ECAs.
  – Enumerate all defect configurations that can be ‘healed’ in a few time steps.
  – For each defect configuration, find the ECA that ‘heals’, while acting as identity map on other configurations, other than the desired patterned configurations.
\( \lambda = 4, \nu = +1 \) Traveling Wave Phase in 1-D

<table>
<thead>
<tr>
<th>config</th>
<th>( V_L + I_L )</th>
<th>( V_L + I_R )</th>
<th>( V_R + I_L )</th>
<th>( V_R + I_R )</th>
</tr>
</thead>
<tbody>
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<td>0 0 0 0</td>
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<td>1</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0 0 1 0</td>
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</tbody>
</table>

| rule  | 139 | 43  | 142 | 46  |

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Does It Work?

- Rules 43 and 142 by themselves most readily generate the desired pattern for initial density $\rho = \frac{1}{2}$. Not good away from half-filling.

- Rules 46 and 139 less readily generate desired pattern.

- Compounding 46 + 139 or 43 + 142 does not make desired pattern any more stable.

- Reason: competing fixed points. Back to square one — need to find fixed points or limit cycles of given ECA.
Conclusions for Part II

- Standard approach to pattern formation: given model, what patterns?
- Inverse approach to pattern formation: given pattern, what models? Studied in the context of ECAs.
- Requiring that pattern be fixed point of dynamics and stable with respect to point defects completely specify 1D ECA. Notion of genericity for pattern under ‘perturbations’ to 1D ECA.
- For patterns in higher-dimensional ECAs, multiple defect analysis necessary.
- Generic problem of transition rule conflict, leading to reduced stability of pattern.
- Compound CAs, more transition rules for pattern matching to avoid conflicts. More transition rules \(\Rightarrow\) more fixed points and limit cycles \(\Rightarrow\) competition between fixed points.
- Try compound CAs with transition rules designed to make all but desired fixed point unstable.