LASSP Pizza Talk
June 13, 2002

Cellular Automata & Pattern Formation

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Prologue

• Not an advertisement for Wolfram…

• Not a Wolfite…

• Rather…
  – P681 Pattern Formation and Spatio-Temporal Chaos/Prof Eberhard Bodenschatz.
  – End-of-course poster…
  – …and adventures beyond.
Cellular Automata

- A collection of finite state machines. The state of the $i^{th}$ machine at time $t$ given by $s_i(t) \in \mathcal{A}$, where $\mathcal{A}$ is a finite set, also called the \textit{alphabet};

- A collection of neighborhoods. The neighborhood of the $i^{th}$ machine is denoted by $\mathcal{N}_i$;

- A dynamical rule $\varphi : \mathcal{N}_i \rightarrow \mathcal{A}$, such that $s_i(t+1) = \varphi(s_j(t) \mid j \in \mathcal{N}_i)$. 
Classification of CAs

- Elementary and compound CAs. Examples are Game of Life (GOL) and the Nagel-Schreckenberg model of traffic flow respectively.

- Wolfram classified all 256 1-D elementary CAs (ECAs) by their dynamical properties. Types I, II and III.

- Wolfram naming convention: if the ECA is

  \[
  \begin{array}{cccccccc}
  1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\
  \end{array}
  \]

  then Wolfram rule number is \( \sum_{j=0}^{7} \alpha_j 2^j \).

- No known attempts at classifying ECAs of higher dimensions.
From Pattern to ECA

• In P681, given PDE model, find what patterns form spontaneously. Can do the same for CA models.

• Ask the inverse question instead: given a pattern, what are all the possible CAs that spontaneously generate it?

• Two parts to this question:
  – what CA rules will have given pattern as fixed point; and
  – under which CA rules is the pattern stable?
Stripped Phase in 1-D

• Consider stripped phase in 1-D:

  \[ \begin{array}{cccccccccccccccc}
  \text{O} & \text{X} & \text{O} & \text{X} & \text{O} & \text{X} & \text{O} & \text{X} & \text{O} & \text{X} & \text{O} & \text{X} & \text{O} & \text{X} & \text{O} & \text{X} \\
  \end{array} \]

• Fixed point requirement implies the transition rules

  \[ \begin{array}{c}
  \text{O} \rightarrow \text{X} \\
  \text{O} \rightarrow \text{X} \\
  \end{array} \]

• Does not uniquely determine ECA rule, 6 more transition rules to specify.
Defects in Stripped Phase

- To analyze stability of stripped phase, need to investigate behaviour of departures from pattern, i.e. defects, under various ECA rules.

- Point defects:
  - vacancy
  - interstitial

- Domain walls:
  - $-1$ domain wall
  - $+1$ domain wall
Strips Stable in Presence of Point Defects

- Since ECA not completely specified, can choose remaining transition rules to stabilize stripped phase in presence of point defects.

- Demand that isolated vacancy ‘heals’: implies transition rules

  \[ \text{O O O O} \rightarrow \text{X \bullet \times}, \quad \text{\bullet O O O} \rightarrow \text{X O \times}, \quad \text{O O \bullet \bullet} \rightarrow \text{X O \times}. \]

- Demand that isolated interstitial ‘heals’: implies transition rules

  \[ \text{\bullet \bullet \bullet \bullet} \rightarrow \text{X \times \times}, \quad \text{O \bullet \bullet \bullet} \rightarrow \text{X \times \times}, \quad \text{\bullet \bullet \bullet \cdot} \rightarrow \text{X \times \times}. \]

- ECA completely specified by requirements that: (a) stripped phase is fixed point; (b) isolated vacancies ‘heal’; and (c) isolated interstitials ‘heal’.
ECA is Rule 77 in Wolfram’s classification scheme:

<table>
<thead>
<tr>
<th></th>
<th>$s_{j-1}(t)$</th>
<th>$s_j(t)$</th>
<th>$s_{j+1}(t)$</th>
<th>$s_j(t + 1)$</th>
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Further Considerations

• **Domain Wall Dynamics.** Both $\pm 1$ domain walls stationary under Rule 77, i.e. if start from random initial configuration, all domain walls initially present will be ‘frozen in’.

• **Robustness of Stripped Phase.** By modifying some transition rules in Rule 77, can test genericity of stripped pattern. Found that:
  
  – Stripped phase *most* stable under Rule 77, but also stable under 6 other ECA rules derived from Rule 77, in which a single transition rule is modified.
  
  – Stripped phase *marginally* stable under 4 ECA rules derived from Rule 77, in which one or two transition rules are modified.
  
  – Stripped phase unstable once more than two transition rules are modified from Rule 77. Oscillatory phase nucleates.
2-D ECAs

• In 1-D, neighborhood simple, unless one wants to go to next nearest neighbor.

• In 2-D, greater variety of neighborhoods. Simplest neighborhood for 2-D CA is von Neumann (VN) neighborhood:

\[
\begin{array}{ccc}
N & & C \\
W & C & E \\
S & & E \\
\end{array}
\]

• With VN neighborhood, total of \(2^5 = 32\) possible local configurations \(\Rightarrow\) total of \(2^{32} = 4,294,967,296\) 2-D ECAs.
\[ \lambda = 4, \nu = +1 \text{ Traveling Wave Phase} \]

- A traveling wave phase with \( \lambda = 4 \) and \( \nu = +1 \) looks like

- The traveling wave transition rules are

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Multiple Defect Analysis & Transition Rule Conflict

• Unlike in 1-D, point defect analysis alone cannot fully specify ECA. Need to do multiple defect analysis.

• Four types of point defect:

  \[ V_L \]  \[ V_R \]  \[ I_L \]  \[ I_R \]

• In this chosen pattern, transition rules implied by \( V_L \) conflicts with that implied by \( V_R \), and transition rules implied by \( I_L \) conflicts with that implied by \( I_R \).

• Generic problem.
Protocol for Conflict Resolution

• When transition rule implied by two configurations in conflict, give precedence to configuration with lower number of defects.

• When transition rule implied by leading edge configuration conflicts with that implied by trailing edge configuration, give precedence to trailing edge configuration.

• Can show that some multi-defect configurations whose implied transition rules are forfeited will still be ‘healed’.

• Compromise necessary because traveling wave breaks left-right symmetry.

• Completed CA rule is Rule 2,383,284,874.
Simulating Rule 2,383,284,874
Compound CAs

- Some patterns cannot be achieved using ECAs because conflict resolution protocol used cannot ensure stability of desired pattern.

- What to do?
  - Use larger neighborhoods — equivalent to a restricted class of compound ECAs.
  - Use larger state space, say $s_i(t) = 0, \frac{1}{2}, 1$.

- The main idea is to increase the number of transition rules available for pattern matching.

- Another way is to compound together ECAs.
How to Compound ECAs

- Enumerate all defect configurations that can be ‘healed’ in a few time steps.
- For each defect configuration, find the ECA that ‘heals’, while acting as identity map on other configurations, other than the desired patterned configurations.
\[ \lambda = 4, \nu = +1 \] Traveling Wave Phase in 1-D

<table>
<thead>
<tr>
<th>config</th>
<th>(V_L + I_L)</th>
<th>(V_L + I_R)</th>
<th>(V_R + I_L)</th>
<th>(V_R + I_R)</th>
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**rule** | 139 | 43 | 142 | 46
Does It Work?

- Rules 43 and 142 by themselves most readily generate the desired pattern for initial density $\rho = \frac{1}{2}$. Not good away from half-filling.
- Rules 46 and 139 less readily generate desired pattern.
- Compounding 46 + 139 or 43 + 142 does not make desired pattern any more stable.
- Reason: competing fixed points. Back to square one — need to find fixed points or limit cycles of given ECA.
References
