Classical and Quantum Entropy

Siew-Ann Cheong

July 22, 1999
Equilibrium Entropy: A Static Picture

In dimensionless form, with $k_B = 1$,

Microcanonical: $S = \log \Omega \quad (1)$

Canonical: $S = -\log Z - \beta \langle E \rangle = \langle \log p_i \rangle = \sum_i p_i \log p_i \quad (2)$

where $\beta = T^{-1}$, $Z$ is the canonical partition function and $\langle E \rangle$ the expectation value of the internal energy $E$.

In study of equilibrium at $t \to \infty$, dynamical details deliberately discarded. How to incorporate dynamical details into a “dynamical entropy”? 
Notes on Ensemble Picture

1. No equilibrium for each member of Gibbs ensemble, for example, ideal gas in cube of edge $L$.

2. Microscopic and macroscopic properties of each member fluctuates from time to time.

3. Macroscopic measurements = “coarse-graining” in time, i.e. time averaging.

4. Gibbs ensemble “equates” time average to phase space average, provided system ergodic.

5. Collisionless gas in a cube not ergodic. Need binary or higher-order collisions.
Motivating Dynamical Entropy

\[
\text{KEY: Entropy = Information Loss}
\]

Consider time evolution of (micro)state from \( t = 0 \) to \( t = T \),

\[
\{x_i, v_i\} \quad \overset{\text{\( t = 0 \)}}{\mapsto} \quad \{x_f, v_f\} \quad \overset{\text{\( t = T \)}}{\mapsto}
\]

(3)

If dynamics time-reversal invariant, then \( \{x_f, v_f\} \rightarrow \{x_i, v_i\} \) possible if **ALL** information known about final state.

What if we don’t? If mistake made in determining final state, how close can we get back to initial state?
The Home-Coming . . .

Attach $\epsilon$-ball to $\{x_i, v_i\}$, evolve it forward in time by $\Delta t = T$, make a small mistake $\epsilon'$, reverse time and evolve back:

\[ \text{overlap} \% \sim \frac{\text{overlap}}{\epsilon^D} \quad (4) \]

where $D$ is dimension of phase space, ignoring factors of $O(1)$. 
Entropy & Information

Intuitively expect that overlap% ↓ as $T \uparrow$. Thus should seek meaningful quantity in the limit

$$\lim_{\epsilon, \eta \to 0} \lim_{T \to \infty} \frac{\text{overlap} \%}{T} = \text{rate of information loss} \quad (5)$$

Truly dynamical because time evolution taken into account.

Also, $0 \leq \text{overlap} \% \leq 1$ interpret as some sort of probability $p_i$, and deduced the dynamical entropy as

$$H = - \sum_i p_i \log p_i \leq S \quad (6)$$

i.e. condition of equilibrium is condition of maximization of $H$ or state of maximum loss of information about system.
In totality, must consider entire phase space.

But \{\text{phase space}(t = 0)\} \equiv \{\text{phase space}(t = T)\}, must find way of introducing dynamics \implies \text{concept of a partition.}

\text{partition } \xi = \{\text{countable } \# \text{ of subsets } C_\alpha \text{ of } X \mid \bigcup_\alpha C_\alpha = \bigcup_\alpha C_\alpha \text{ of } X, C_\alpha \cap C_\beta = \emptyset \text{ if } \alpha \neq \beta\} \text{ up to sets of measure zero with respect to some measure } \mu, \text{ e.g. phase space volume.}
Dynamical Refinement of a Partition

Can define

\[ H = \sum_{\alpha} \mu(C_{\alpha}) \log \mu(C_{\alpha}) \]  

but not dynamical. Need concept of refinement of a partition.

This refinement can be brought about by a (measure-preserving) dynamical map: \( \varphi^T : X(t = 0) \rightarrow X(t = T) \), and denote by \( \xi^{\varphi^T} \) the dynamically refined partition of \( \xi \).
The Kolmogorov-Sinai Metric Entropy

An important proposition:

\[ h_\mu(\varphi, \xi) = \lim_{T \to \infty} \frac{H(\xi^{\varphi_T})}{T} = \lim_{T \to \infty} \sum_{\alpha} \frac{\mu(C_\alpha) \log \mu(C_\alpha)}{T}, \quad C_\alpha \in \xi^{\varphi_T} \]

exists for all measurable partition.

The \textit{Kolmogorov-Sinai metric entropy} of \( \varphi \) with respect to \( \mu \) is

\[ h_\mu(\varphi) = \sup \{ h_\mu(\varphi, \xi) \mid \text{all } \xi \text{ such that } H(\xi) < \infty \} \]

which is a topological quantity.

Quantum Static Entropy

Shannon-Rényi Entropy: \[ H(\psi) = \sum_i |\langle i | \psi | i \rangle|^2 \log |\langle i | \psi | i \rangle|^2 \]

von Neumann Entropy: \[ S(\varrho) = - \text{tr} \varrho \log \varrho \]

Quantum Relative Entropy: \[ S(\psi, \chi) = \text{tr} \varrho_\psi (\log \varrho_\psi - \log \varrho_\chi) \]

Effective State Entropy: \[ S(\varrho) = - \int dE \text{ tr} \varrho_E \log \varrho_E \]

Quantum Dynamical Entropy

1. Connes-Størmer-Narnhofer-Thirring entropy ✓
2. Alicki-Fannes entropy
3. Coherent State entropy ✓
Connes-Størmer-Narnhoffer-Thirring
Algebraic Entropy: The Essentials

1. C*-algebra $L^\infty(M)$ of infinitely integrable complex-valued functions on compact phase space $M$ with Borel probability measure $\mu$, equipped with faithful normal trace

$$\tau(f) = \int_M f \, d\mu < \infty, \quad f \in L^\infty(M) \quad (10)$$

take the place of minimal $\sigma$-algebra on $M$.

2. Finite-dimensional subalgebras $\mathcal{N}$ generated by complete set of minimal projection operators $\{p_i\}$ such that $p_i \cdot p_j = \delta_{ij}$ takes the place of partition $\xi$. 
3. Define

\[ H(\mathcal{N}) = \sum_i \tau (p_i \log p_i) = H(\xi) \]  

(11)

4. \( \mu \)-preserving dynamical map \( \varphi \) on \( M \) induces \( \tau \)-preserving dynamical map \( \Phi \) on \( L^\infty(M) \). Use \( \Phi \) to generate dynamical refinement.

5. Define Connes-Størmer-Narnhoffer-Thirring entropy as

\[ h(\mathcal{N}) = \sup_{\Phi} h(\mathcal{N}, \Phi) \]

\[ = \sup_{\Phi} \lim_{n \to \infty} n^{-1} H(\mathcal{N} \vee \Phi(\mathcal{N}) \vee \cdots \vee \Phi^{n-1}(\mathcal{N})) \]  

(12)

6. C*-algebra \( L^\infty(M) \) commutative, can be shown to equal Kolmogorov-Sinai metric entropy.
Connes-Størmer-Narnhoffer-Thirring
Quantum Dynamical Entropy

von Neumann algebra of projection operators used in quantum mechanics is C*-algebra, i.e. basic recipe same as classical mechanics.

In defining procedure of Connes-Størmer-Narnhoffer-Thirring entropy, as in Kolmogorov-Sinai entropy, partition must remain finite. However, quantum algebra not commutative $\Rightarrow$ dynamical refinement of $\mathcal{N}$ problematic. Quantum dynamically refined partition infinite (R. Alicki and M. Fannes, Lett. Math. Phys. 32, 75 (1994)).

Trick: Perform abelianized refinement of $\mathcal{N}$, i.e. redundant physical information of incompatible observables removed.
Coherent State Entropy

Follows same cookbook recipe as Kolmogorov-Sinai and Connes-Størmer-Narnhofer-Thirring entropies, but measure-theoretic rather than algebraic.

Define measures on quantum-mechanical phase space whose density functions are Husimi functions, defined as

\[ \Phi(q, p, t) = \frac{1}{\pi \hbar} \int dQ dP \, e^{-\left(\frac{(q-Q)^2}{\hbar w^2} + w^2 \frac{(p-P)^2}{\hbar}\right)} \Psi(Q, P, t) \]  \hspace{1cm} (13)

where

\[ \Psi(q, p, t) = \frac{1}{(2\pi \hbar)^N} \int dQ \, \psi(q - Q, t) \psi^*(q + Q, t) e^{-2ipQ/\hbar} \] \hspace{1cm} (14)

are the Wigner functions.
Coherent States and Measurement

*Coherent states* are *a posteriori* states associated with measurement, i.e. states we *want* to see.

Second Postulate of QM – after exact measurement, state collapses to one of eigenspaces of observable, but in experiment with uncertainty $w$, state collapses to group of eigenspaces centered around dominant eigenspace $\implies$ wavepackets of width $w$.

Gaussians are instantaneously minimum uncertainty wavepackets, i.e. what an experimenter want to see.

Only such states projected out of quantum state $\psi(q,t)$ to incorporate into Husimi functions.