Exact Ground States and Correlation Functions of Interacting Spinless Fermions on a Two-Legged Ladder

SIEW-ANN CHEONG
Cornell Theory Center, Cornell University

School of Physical and Mathematical Sciences
Nanyang Technological University
29 March 2006
Overview of Talk

- **Bosons and Fermions**: Brief review of Jordan-Wigner transformation.

- **Exact Ground State**: Trio of analytical maps relating 1D nearest-neighbor excluded and nearest-neighbor included periodic chains.

- **Correlation Functions**: Corresponding observables and the intervening-particle expansion.

- **Three Limiting Cases**: Extended Hubbard ladder of spinless fermions, overview of results, and zeroth-order ground-state phase diagram.
  - Strong correlated hopping limit.
  - Weak inter-leg hopping limit.
  - Strong inter-leg hopping limit.

- **Conclusions**.
The Jordan-Wigner Transformation

- $P$ noninteracting spinless fermions on a 1D periodic chain of $L$ sites,

$$H_c = -t \sum_{j=1}^{L} \left( c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right).$$

- Ground state is a Fermi sea

$$|\Psi_F\rangle = \prod_{|k|<k_F} \tilde{c}_k^\dagger |0\rangle = \sum_{j_1<\cdots<j_P} \Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P) c_{j_1}^\dagger c_{j_2}^\dagger \cdots c_{j_P}^\dagger |0\rangle,$$

- Amplitude given by Slater determinant

$$\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P) = \frac{1}{L^{P/2}} \begin{vmatrix} e^{-ik_1j_1} & e^{-ik_1j_2} & \cdots & e^{-ik_1j_P} \\ e^{-ik_2j_1} & e^{-ik_2j_2} & \cdots & e^{-ik_2j_P} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-ik_Pj_1} & e^{-ik_Pj_2} & \cdots & e^{-ik_Pj_P} \end{vmatrix}.$$

- Two-point function decays as power law, $\langle \Psi_F | c_i^\dagger c_j | \Psi_F \rangle \sim |i - j|^{-1}$. 

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The Jordan-Wigner Transformation

- $P$ hard-core bosons on a 1D periodic chain of $L$ sites,

$$H_b = -t \sum_{j} \left( b_{j}^\dagger b_{j+1} + b_{j+1}^\dagger b_{j} \right) + U \sum_{j} n_j(1 - n_j), \quad U \to \infty.$$ 

- Map to noninteracting spinless fermion using Jordan-Wigner transformation [P. Jordan and E. Wigner, Z. Phys. 47, 631 (1928)],

$$b_i = \prod_{j<i} (1 - 2n_j) \quad c_i = \prod_{j<i} (-1)^{n_j} c_i.$$ 

- Non-local operator $\prod_{j<i}(1 - 2n_j)$ called Jordan-Wigner string.

- Hard-core boson ground state

$$|\Psi\rangle = \sum_{j_1} \cdots \sum_{j_P} |\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)| b_{j_1}^\dagger b_{j_2}^\dagger \cdots b_{j_P}^\dagger |0\rangle.$$

- Two-point function also decays as power law, $\langle \Psi|b_i^\dagger b_j|\Psi\rangle \sim |i - j|^{-1/2}$ [K. B. Efetov and A. I. Larkin, Sov. Phys. JETP 42, 390 (1976)].
Nearest-Neighbor Inclusion & Exclusion

- 1D chain of hard-core bosons or spinless fermions with infinite nearest-neighbor repulsion

\[
H_A = H_a + V \sum_j n_j n_{j+1}, \quad V \to \infty,
\]

where \(A = B\) (boson) or \(C\) (fermion), and \(a = b\) (boson) or \(c\) (fermion).

- \(H_a\) allows nearest-neighbor occupation: Hilbert space \(\mathcal{V}_a\) consists of nearest-neighbor included configurations.

- \(H_A\) forbids nearest-neighbor occupation: Hilbert space \(\mathcal{V}_A\) consists of nearest-neighbor excluded configurations.
- **Right exclusion map**: nearest-neighbor excluded configuration to nearest-neighbor included configuration.

\[
|\alpha\rangle \quad \boxed{\begin{array}{cccccc}
\bullet & \times & \bullet & \times & \bullet & \times \\
\end{array}} \quad L = 11, P = 4
\]

\[
|\alpha'\rangle \quad \boxed{\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\end{array}} \quad L' = L - P = 7, P' = P = 4
\]

- Check that if \(|\alpha\rangle \leftrightarrow |\alpha'\rangle\) and \(|\beta\rangle \leftrightarrow |\beta'\rangle\), then \(\langle \alpha|H_A|\beta\rangle = \langle \alpha'|H_a|\beta'\rangle\).

- **Right exclusion map not one-to-one.**

- **Right inclusion map**: nearest-neighbor included configuration to nearest-neighbor excluded configuration,

\[
a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_P}^\dagger |0\rangle \leftrightarrow A_{j_1}^\dagger A_{j_2+1}^\dagger \cdots A_{j_{P+P-1}}^\dagger |0\rangle.
\]
• **Bloch-State-to-Bloch-State Map**

• Adopt **closed-shell boundary conditions**: $P$-fermion configuration incurs no sign change when translated across boundary. Treat bosons and fermions in same way.

• **Translational invariance**: define the Bloch states

$$ |\alpha; q\rangle = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{-i q_j T_j} |\alpha\rangle, $$

where $|\alpha\rangle$ is generating $P$-particle nearest-neighbor excluded configuration, and $T_j$ is translation operator.

• Eigenstates of $H_A$ have definite total linear momentum, and thus $H_A$ block-diagonal in basis of Bloch states. Each diagonal block $H_A(q)$ characterized by total momentum wave vector $q$.

• Number of Bloch states = number of translationally inequivalent configurations.
Example: $L = 6$, $P = 2$
Example: $L' = 4$, $P = 2$

- For each $q$, two nearest-neighbor excluded Bloch states $|\alpha'; q\rangle$ and $|\beta'; q\rangle$.
- See that $|\alpha\rangle \leftrightarrow |\alpha'; \rangle$ and $|\beta\rangle \leftrightarrow |\beta'; \rangle$ under right-exclusion map.
- For each $q'$, two nearest-neighbor included Bloch states $|\alpha'; q'\rangle$ and $|\beta'; q'\rangle$.
- Can we choose $q$ and $q'$ such that $\langle \alpha; q|H_A|\beta; q\rangle = \langle \alpha'; q'|H_A|\beta'; q'\rangle$?
Wave-Vector-To-Wave-Vector Map

- First note that nearest-neighbor excluded chain of length $L$ maps to nearest-neighbor included chain of length $L' = L - P$.

- Allowed total-momentum wave vectors are
  \[ q = \frac{2\pi n}{L}, \quad q' = \frac{2\pi n'}{L'}, \quad n, n' \in \mathbb{Z}. \]

- Find that $\langle \alpha; q | H_A | \beta; q \rangle = \langle \alpha'; q' | H_a | \beta'; q' \rangle$ for all $|\alpha\rangle \leftrightarrow |\alpha\rangle$ and $|\beta\rangle \leftrightarrow |\beta'\rangle$ if we have
  \[ q = \frac{2\pi n}{L} \leftrightarrow q' = \frac{2\pi n}{L'}, \quad n \in \mathbb{Z}. \]

- In case of $P = 1$, $n$ simply the number of nodes in wave function.
Corollary of Combined Map

- $H_A(q)$ and $H_a(q')$ are identical as matrices. Same eigenvalues and eigenvectors.

- All nearest-neighbor excluded chain eigenstates can be written in terms of nearest-neighbor included chain eigenstates, and vice versa.

- In particular, if we know a nearest-neighbor included eigenstate with energy eigenvalue $E'$ is

$$|\Psi'; q'\rangle = \sum_{j_1<\cdots<j_P} \Psi'(q'; j_1, \ldots, j_P) a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_P}^\dagger |0\rangle,$$

then nearest-neighbor excluded eigenstate with the same energy eigenvalue $E = E'$ is

$$|\Psi; q\rangle = \sum_{j_1<\cdots<j_P} \Psi'(q'; j_1, \ldots, j_P) A_{j_1}^\dagger A_{j_2+1}^\dagger \cdots A_{j_P+P-1}^\dagger |0\rangle,$$

- Exact solution of nearest-neighbor excluded chain in terms of nearest-neighbor included chain!
Corresponding Observables

- Since $|\Psi'; q'\rangle$ and $|\Psi; q\rangle$ share the same amplitudes, want to cast problem of calculating $\langle O \rangle = \langle \Psi; q|O|\Psi; q \rangle$ in nearest-neighbor excluded chain as problem of calculating $\langle O' \rangle = \langle \Psi'; q'|O'|\Psi'; q' \rangle$ in nearest-neighbor included chain.

- **Corresponding observables** $O$ and $O'$ defined by their matrix elements between Bloch states,

  $$\sqrt{l_\alpha l_\beta} \langle \alpha; q|O|\beta; q \rangle = \sqrt{l'_{\alpha'} l'_{\beta'}} \langle \alpha'; q'|O'|\beta'; q' \rangle,$$

  where $l_\alpha$ is period of $|\alpha\rangle$ and $l'_{\alpha'}$ is period of $|\alpha'\rangle$.

- Can check from right-exclusion map that $l'/l = \tilde{n'}/\tilde{n}$, where $\tilde{n}$ is filling fraction in nearest-neighbor excluded chain, and $\tilde{n'}$ is filling fraction in nearest-neighbor included chain.

- Expectation of corresponding observables related by

  $$\langle O \rangle = \frac{\tilde{n}}{\tilde{n'}} \langle O' \rangle.$$
The Intervening-Particle Expansion

- Defining condition of corresponding observables stringent, satisfied by few observables. For generic observables, need to use intervening-particle expansion.

- Example: The intervening-particle expansion for $\langle A_i^\dagger A_{i+r} \rangle$ is

$$\langle A_i^\dagger A_{i+r} \rangle = \langle A_i^\dagger (1 - N_{i+1}) \cdots (1 - N_{i+r-1}) A_{i+r} \rangle +$$

$$\langle A_i^\dagger N_{i+1} \cdots (1 - N_{i+r-1}) A_{i+r} \rangle + \cdots +$$

$$\langle A_i^\dagger (1 - N_{i+1}) \cdots N_{i+r-1} A_{i+r} \rangle +$$

$$\langle A_i^\dagger N_{i+1} N_{i+2} \cdots (1 - N_{i+r-1}) A_{i+r} \rangle + \cdots +$$

$$\langle A_i^\dagger (1 - N_{i+1}) \cdots N_{i+r-2} N_{i+r-1} A_{i+r} \rangle + \cdots +$$

$$\langle A_i^\dagger N_{i+1} N_{i+2} \cdots N_{i+r-1} A_{i+r} \rangle .$$

- Each term in expansion contains $p = 0, 1, \ldots, r$ intervening particles at fixed sites.

- Map each term $\langle A_i^\dagger O_p A_{i+r} \rangle$ to its corresponding expectation $\langle a_i^\dagger O_p' a_{i+r} \rangle$, and then sum over $(\tilde{n}/\tilde{n}') \langle a_i^\dagger O_p' a_{i+r} \rangle$ to get $\langle A_i^\dagger A_{i+r} \rangle$. 

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**Rules for Corresponding Intervening-Particle Observables**

- **Nearest-neighbor exclusion:** Drop terms $\langle A_i^{\dagger} O_p A_{i+r} \rangle$ in expansion if $A_j^{\dagger} A_{j+1}^{\dagger}, \; A_j A_{j+1}, \; A_j^{\dagger} N_{j+1}, \; N_j A_{j+1}$ appear.

- **Right-exclusion map:** In the surviving terms, making the replacements
  
  \[ A_j^{\dagger} (1 - N_{j+1}) \leftrightarrow a_j^{\dagger}, \quad A_j (1 - N_{j+1}) \leftrightarrow a_j, \quad N_j (1 - N_{j+1}) \leftrightarrow n_j. \]

- **Re-indexing:** Because right-exclusion map merges sites $j$ and $j + 1$, sites to right of $j + 1$ must be re-indexed. For example,
  
  \[ N_j (1 - N_{j+1}) N_{j+2} \leftrightarrow n_j n_{j+1}. \]

In general, site $j$ on nearest-neighbor excluded chain becomes site $j - p$ on nearest-neighbor included chain if there are $p$ particles between sites $i$ and $j$ (and including $i$).
Where We Are Right Now . . .

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- **Exact Ground State**: Trio of analytical maps relating 1D nearest-neighbor excluded and nearest-neighbor included periodic chains.

- **Correlation Functions**: Corresponding observables and the intervening-particle expansion.

- **Three Limiting Cases**: Extended Hubbard ladder of spinless fermions, overview of results, and zeroth-order ground-state phase diagram.
  
  - Strong correlated hopping limit.
  
  - Weak inter-leg hopping limit.
  
  - Strong inter-leg hopping limit.

- **Conclusions**.
Extended Hubbard Ladder of Spinless Fermions

\[
H_{t \parallel t'V} = -t_\parallel \sum_i \sum_j \left( c_{i,j} \dagger c_{i,j+1} + c_{i,j+1} \dagger c_{i,j} \right) - t_\perp \sum_i \sum_j \left( c_{i,j} \dagger c_{i+1,j} + c_{i+1,j} \dagger c_{i,j} \right) \\
- t' \sum_i \sum_j \left( c_{i,j} \dagger n_{i+1,j+1} c_{i,j+2} + c_{i,j+2} \dagger n_{i+1,j+1} c_{i,j} \right) \\
- t' \sum_i \sum_j \left( c_{i+1,j} \dagger n_{i,j+1} c_{i+1,j+2} + c_{i+1,j+2} \dagger n_{i,j+1} c_{i+1,j} \right) \\
+ V \sum_i \sum_j n_{i,j} n_{i,j+1} + V \sum_i \sum_j n_{i,j} n_{i+1,j}, \quad V \to \infty.
\]
Overview of Three Limiting Cases

- Strong correlated-hopping limit, $t' \gg t_{\parallel}, t_{\perp}$:
  - universal SC power-law correlations dominate over non-universal hard-core-boson CDW power-law correlations at large distances.
  - FL correlations decay exponentially.

- Weak inter-leg hopping limit, $t_{\perp} \ll t_{\parallel}, t' = 0$:
  - universal CDW power-law correlations dominate over universal SC power-law correlations at large distances.
  - FL correlations decay exponentially.

- Strong inter-leg hopping limit, $t_{\perp} \gg t_{\parallel}, t' = 0$:
  - True long-range CDW when $\bar{n}_2 = \frac{1}{4}$.
  - Phase separation for $\bar{n}_2 > \frac{1}{4}$.
  - For $\bar{n}_2 < \frac{1}{4}$, universal SC power-law correlations dominate universal FL and CDW power-law correlations at large distances.
Zeroth-Order Phase Diagram

- strong inter-leg hopping limit
- weak inter-leg hopping limit
- \( t_{\perp}/t_{\parallel} \)
- \( t'/t_{\parallel} \)
- LR-CDW
- SC
- PL-CDW

strong correlated hopping limit

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Strong Correlated Hopping Limit

- When $t' \gg t_\parallel, t_\perp$, ladder spinless fermions form well-defined pairs: 1D problem of interacting hard-core bosons.

- Two flavors of interacting hard-core bosons. Call them even and odd, or red ($R$) and green ($G$). Flavor conserved as fermion pair correlated-hops.

- Bound-pair-to-hard-core boson map:

  \[
  B_j^\dagger = \begin{cases} 
  c_{1,j}^\dagger c_{2,j+1}^\dagger, & j \text{ even;} \\
  c_{1,j+1}^\dagger c_{2,j}^\dagger, & j \text{ odd,}
  \end{cases}
  \]\[
  B_j^\dagger = \begin{cases} 
  c_{1,j+1}^\dagger c_{2,j}^\dagger, & j \text{ even;} \\
  c_{1,j}^\dagger c_{2,j+1}^\dagger, & j \text{ odd.}
  \end{cases}
  \]
Strong Correlated Hopping Limit

- Hard-core boson of each flavor can come within two sites of another hard-core boson of the same flavor, but can only come within three sites of a hard-core boson of different flavor. Hard-core bosons cannot exchange positions.

- For $2P$ spinless fermions on ladder of length $L$, Hilbert space breaks up into sectors of immutable flavor sequences. Example: For $P = 4$, the distinct flavor sequences are $RRRR$, $RRRG$, $RRGG$, $RGRG$, $RGGR$, and $GGGG$. 
Kinetic Energy Argument

- Each hard-core boson confined to hop within interval of chain between the two hard-core bosons closest to it: particle-in-a-box problem!

- At given filling fraction $\bar{n}$,
  - $L_{\text{eff}}$ larger if $R$ particle bounded by $R$ particles, and $G$ particle bounded by $G$ particles.
  - $L_{\text{eff}}$ smaller if $R$ particle bounded by $G$ particles, or $G$ particle bound by $R$ particles.
  - Kinetic energy of bound particle lowest if bound by particles of the same flavor.

- Two-fold-degenerate ground state for $2P$ spinless fermions: $P R$ bound pairs or $P G$ bound pairs. Ground-state wave functions of each can be mapped to ground-state wave function of $P$ noninteracting spinless fermions.
Ground-State Wave Functions

- Start with ground-state wave function of $P$ noninteracting spinless fermions on periodic chain of length $L' = L - P$, 
  \[ |\Psi_F\rangle = \sum_{j_1<\ldots<j_P} \Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P) c_{j_1}^\dagger c_{j_2}^\dagger \cdots c_{j_P}^\dagger |0\rangle, \]
  where $k_1, \ldots, k_P$ are the $P$ occupied single-particle wave vectors.

- Use Jordan-Wigner map to get ground-state wave function of $P$ nearest-neighbor included hard-core bosons on periodic chain of length $L' = L - P$, 
  \[ |\Psi_b\rangle = \sum_{j_1<\ldots<j_P} |\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)| b_{j_1}^\dagger b_{j_2}^\dagger \cdots b_{j_P}^\dagger |0\rangle, \]

- Use right-inclusion map to get ground-state wave function of $P$ nearest-neighbor excluded hard-core bosons on periodic chain of length $L$, 
  \[ |\Psi_B\rangle = \sum_{j_1<\ldots<j_P} |\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)| B_{j_1}^\dagger B_{j_2+1}^\dagger \cdots B_{j_P+P-1}^\dagger |0\rangle, \]

- Use bound-pair-to-hard-core-boson map to get ground-state wave function of $P$ ($R$ or $G$) bound pairs on ladder of length $L$. 

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Correlation Functions

- Only simple to calculate correlation functions which can be written in terms of $B_j$ and $B_j^\dagger$.
  - SC correlations $\langle B_i^\dagger B_{i+r} \rangle$.
  - CDW-$\pi$ correlations $\langle B_i^\dagger B_i B_{i+r}^\dagger B_{i+r} \rangle$.

- Correlation functions not readily expressible in terms of $B_j$ and $B_j^\dagger$ difficult to calculate.
  - FL correlation $\langle c_{i,j}^\dagger c_{i',j+r} \rangle$, understood using semi-quantitative arguments.
  - CDW-$\sigma$ correlations $\langle c_{i,j}^\dagger c_{i,j'}^\dagger c_{i',j+r} c_{i',j+r} \rangle$.

- Numerically, summing the intervening-particle expansion for correlation functions involve summing over various minors of an $r \times r$ matrix. Without acceleration schemes, only feasible up to separations of $r \approx 20$.

- Correlation exponents, wave vectors, amplitudes and phase shifts obtained through nonlinear curve fitting.
CDW-$\pi$ Correlations

\[ \langle N_j N_{j+r} \rangle - \langle N_j \rangle \langle N_{j+r} \rangle \]

- $\bar{N}_1 = 0.20$
- $\bar{N}_1 = 0.25$
- $\bar{N}_1 = 0.30$

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FL Correlations

- Configurations containing unpaired spinless fermions cannot occur in ground state.

- FL correlations $\langle c_{i,j}^\dagger c_{i',j+r} \rangle$ nonzero only when $r$ even.

- For $r = 2p$, only compact $p$-bound-pair configurations with one end at $j$ and the other end at $j + r$ contribute to $\langle c_{i,j}^\dagger c_{i',j+r} \rangle$.

- $\langle c_{i,j}^\dagger c_{i',j+r} \rangle$ proportional to probability of finding compact $p$-bound-pair cluster in ground state.

- Compact $p$-bound-pair cluster $\leftrightarrow$ compact $p$-hard-core-boson cluster $\leftrightarrow$ compact $p$-noninteracting-spinless-fermion cluster.
From SAC and C. L. Henley, Phys. Rev. B 69, 075112 (2004), know that probability of fully-occupied $p$-site cluster in 1D Fermi sea is

$$\det G_C(p) = \prod_{l=1}^{p} \lambda_l = \prod_{l=1}^{p} \frac{1}{e^{\varphi_l} + 1},$$

where $\lambda_l$ are eigenvalues of the cluster Green-function matrix $G_C(p)$, and $\varphi_l$ are the single-particle pseudo-energies of the cluster density matrix $\rho_C$.

For $p \gg 1$, know that

$$\det G_C(p) \approx \exp \left( -p \int_{0}^{1-\tilde{n}'} f(\tilde{n}', x) \, dx \right),$$

i.e. FL correlations decay exponentially for large $r$, with $\tilde{n}$-dependent correlation length ($\tilde{n}'$ is filling fraction of nearest-neighbor included chain).
# Summary of Correlation Exponents

<table>
<thead>
<tr>
<th>limit</th>
<th>correlation function</th>
<th>correlation exponent</th>
<th>wave vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t' \gg t_{\parallel}, t_{\perp}$</td>
<td>CDW-$\pi$</td>
<td>$\frac{1}{2} + \frac{5}{2} \left( \frac{1}{2} - \bar{N}_1 \right)$</td>
<td>$2k_F$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>SC</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{3}{2} \rightarrow \frac{1}{2}$</td>
<td>$2k_F$</td>
</tr>
</tbody>
</table>
Weak Inter-Leg Hopping Limit

- When $t_\perp \to 0$ and $t' = 0$, the two legs of ladder coupled only by infinite nearest-neighbor repulsion.

- Each spinless fermion carries permanent leg index $i$.

- Spinless fermion cannot move past each other, even if they are on different legs (because of infinite nearest-neighbor repulsion).

- For $P$ spinless fermions on ladder of length $L$, Hilbert space breaks up into sectors of immutable leg indices. Example: For $P = 4$, the distinct leg-index sequences are 1111, 1112, 1122, 1212, 1222, and 2222.

- Again use kinetic energy argument to determine structure of ground state:
  - Compare locally the sequences $\{\cdots 111222 \cdots\}$ and $\{\cdots 112122 \cdots\}$, find that third and fourth particles in $\{\cdots 112122 \cdots\}$ have longer intervals to hop around, compared to their counterparts in $\{\cdots 111222 \cdots\}$. 
Kinetic energies of particles forming leg-index domain wall lower.

Overall ground state must therefore have as many domain walls as possible, i.e. sequence must be \(\cdots 121212 \cdots\) or \(\cdots 212121 \cdots\).

Two-fold-degenerate staggered ground state.
Ground-State Wave Functions

- Again, start with ground-state wave function of $P$ noninteracting spinless fermions on periodic chain of length $L' = L$,

$$|\Psi_F\rangle = \sum_{j_1<\ldots<j_P} \Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P) c_{j_1}^{\dagger} c_{j_2}^{\dagger} \cdots c_{j_P}^{\dagger} |0\rangle,$$

where $k_1, \ldots, k_P$ are the $P$ occupied single-particle wave vectors. Infinite nearest-neighbor repulsion between different legs do not result in need to exclude sites.

- Without loss of generality, assume $P$ even. Then two-fold-degenerate staggered ground-state wave functions are

$$|\Psi_{\pm}\rangle = \sum_{j_1<\ldots<j_P} \Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P) \times$$

$$\frac{1}{\sqrt{2}} \left( c_{1,j_1}^{\dagger} c_{2,j_2}^{\dagger} \cdots c_{1,j_{P-1}}^{\dagger} c_{2,j_P}^{\dagger} \pm c_{2,j_1}^{\dagger} c_{1,j_2}^{\dagger} \cdots c_{2,j_{P-1}}^{\dagger} c_{1,j_P}^{\dagger} \right) |0\rangle.$$

$|\Psi_{+}\rangle$ symmetric with respect to reflection about ladder axis, while $|\Psi_{-}\rangle$ anti-symmetric with respect to reflection about ladder axis.

- Note that ladder with filling fraction $\tilde{n}_2$ maps onto chain of filling fraction $\tilde{n}_1 = 2\tilde{n}_2$. 
Correlation Functions

- CDW+ correlations
  \[ \langle n_1,j n_1,j+r \rangle + \langle n_1,j n_2,j+r \rangle, \]
  \[ \langle n_2,j n_1,j+r \rangle + \langle n_2,j n_2,j+r \rangle \]
  both equal \( \frac{1}{2} \langle \Psi_F | n_j n_{j+r} | \Psi_F \rangle \), the CDW correlation in 1D Fermi sea.

- SC+ correlations
  \[ \langle c_2,j+1^\dagger c_1,j c_1,j+r c_2,j+r+1 \rangle + \langle c_2,j+1^\dagger c_1,j c_2,j+r c_1,j+r+1 \rangle, \]
  \[ \langle c_1,j+1^\dagger c_2,j c_1,j+r c_2,j+r+1 \rangle + \langle c_1,j+1^\dagger c_2,j c_2,j+r c_1,j+r+1 \rangle \]
  both equal \( \frac{1}{2} \langle c_{j+1}^\dagger c_j^\dagger c_{j+r} c_{j+r+1} \rangle \), the SC correlation in 1D Fermi sea.

- CDW− and SC− correlations need to calculate numerically.

- Staggered FL correlations \( \langle c_1,j c_2,j+r \rangle = 0 = \langle c_2,j c_1,j+r \rangle \) vanish identically.

- FL correlations \( \langle c_1,j c_1,j+r \rangle \) and \( \langle c_2,j c_2,j+r \rangle \) decay exponentially with \( r \), understood using semi-quantitative arguments.
CDW – Correlations

\[ \langle n_{-j} n_{+j+r} \rangle - \langle n_{-j} \rangle \langle n_{+j+r} \rangle \]

\( n_2 = 0.20 \)
\( n_2 = 0.25 \)
\( n_2 = 0.30 \)
SC– Correlations

\[ r^2 \langle \Delta_{z,j} \Delta_{z,j+r} \rangle \]

- \( n_2 = 0.20 \)
- \( n_2 = 0.25 \)
- \( n_2 = 0.30 \)
To contribute to $\langle c_{i,j}^\dagger c_{i,j+r} \rangle$, there must be no spinless fermions (on either legs) between rung $j$, where spinless fermion will be created, and rung $j+r$, where spinless fermion will be annihilated.

Configurations satisfying this condition are those in which rung $j+r$ sits in a gap of length $s \geq r$. 
FL Correlations

- Can write FL correlation as

\[ \langle c_{i,j}^\dagger c_{i,j+r} \rangle = \sum_s P(s) \sum_{s_i'} \psi^*(s'_f) \psi(s'_i), \]

where \( P(s) \) is probability of finding a gap of length \( s \) in ground state, and \( \psi(s') \) is ‘amplitude’ of single spinless fermion at site \( s' \) within gap.

- \( \sum_{s_i',s_f'} \psi^*(s'_f) \psi(s'_i) \) is \( O(1) \) number, so \( \langle c_{i,j}^\dagger c_{i,j+r} \rangle \sim \sum_s P(s) \).

- Gap of \( s \) rungs on ladder \( \leftrightarrow \) gap of \( s \) sites on chain.

- From SAC and C. L. Henley, Phys. Rev. B 69, 075112 (2004), know that probability of a gap of \( s \) sites in 1D Fermi sea is

\[ P(s) = \det(\mathbb{1} - G_C(s)), \]

where \( G_C(s) \) is cluster Green-function matrix.
FL Correlations

- For $s \gg 1$, know that

$$P(s) \approx \exp \left\{ -s \int_0^{\bar{n}_1} f(1 - \bar{n}_1, x) \, dx \right\},$$

and thus

$$\langle c_{i,j}^\dagger c_{i,j+r} \rangle \sim \frac{\exp \left\{ -r \int_0^{\bar{n}_1} f(1 - \bar{n}_1, x) \, dx \right\}}{1 - \exp \left\{ - \int_0^{\bar{n}_1} f(1 - \bar{n}_1, x) \, dx \right\}},$$

i.e. FL correlation decays exponentially with separation $r$. 
### Summary of Correlation Exponents

<table>
<thead>
<tr>
<th>limit</th>
<th>correlation function</th>
<th>correlation exponent</th>
<th>wave vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_\perp \ll t_\parallel, t' = 0$</td>
<td>CDW+</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>$2k_F$</td>
</tr>
<tr>
<td></td>
<td>CDW−</td>
<td>$\frac{1}{2}$</td>
<td>$2k_F$</td>
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<td>2</td>
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</tr>
<tr>
<td></td>
<td>SC+</td>
<td>2</td>
<td>0</td>
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<td>$2k_F$</td>
</tr>
<tr>
<td></td>
<td>SC−</td>
<td>$\frac{5}{2}$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
When $t_\perp \gg t_\parallel, t' = 0$, spinless fermions very nearly localized onto rungs of ladder, hopping to adjacent rungs only very rarely.

Each spinless fermion very nearly in rung ground state

$$|+, j\rangle = \frac{1}{\sqrt{2}} \left( c_{1,j}^\dagger + c_{2,j}^\dagger \right) |0\rangle = C_j^\dagger |0\rangle .$$

Call spinless fermion in rung ground state rung fermion.

Essentially problem of 1D rung fermions with infinite nearest-neighbor repulsion.

Use trio of maps to write rung-fermion ground state

$$|\Psi\rangle = \sum_{j_1<\ldots<j_P} \Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P) C_{j_1}^\dagger C_{j_{2+1}}^\dagger \cdots C_{j_{P+P-1}}^\dagger |0\rangle$$

in terms of 1D Fermi sea.
Long-Range Order and Phase Separation

- At $\tilde{n}_2 = \frac{1}{4} \equiv \tilde{n}_1 = \frac{1}{2}$, every other rung occupied. Spinless fermions can continue to hop back and forth along rung, but cannot hop to adjacent rungs (infinite nearest-neighbor repulsion). **Dynamic solid phase with long-range CDW order.**

- For $\tilde{n}_2 > \frac{1}{4}$, a fraction of spinless fermions become immobile (inert solid phase, $\tilde{n}_2 = \frac{1}{2} \equiv \tilde{n}_1 = 1$), while the rest remain in dynamic solid phase, $\tilde{n}_2 = \frac{1}{4}$.

- For given $\tilde{n}_2$, ground-state composition of dynamic and inert solid phases determined by having as many spinless fermions in dynamic solid phase as possible.

\begin{center}
\includegraphics[width=\textwidth]{diagram.png}
\end{center}

\textbf{inert solid phase} \hspace{1cm} \textbf{dynamic solid phase}
CDW Correlations

\[ \langle N_j, N_{j+r} \rangle - \langle N_j \rangle \langle N_{j+r} \rangle \]

- $n_2 = 0.10$
- $n_2 = 0.15$
- $n_2 = 0.20$

Nanyang Technological University, 29 March 2006
SC Correlations

\[- r^2 \langle \Delta_j \Delta_{j+r} \rangle \]

\[ N_1 = 0.20 \]
\[ N_1 = 0.30 \]
\[ N_1 = 0.40 \]
### Summary of Correlation Exponents

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<th>Wave Vector</th>
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<tbody>
<tr>
<td>$t_\perp \gg t_{\parallel}, t' = 0$</td>
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<tr>
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<tr>
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<td>$\frac{1}{4}$</td>
<td>$2k_F$</td>
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<td></td>
<td></td>
<td>2</td>
<td>$2k_F$</td>
</tr>
</tbody>
</table>
Conclusions

- Exact solution via
  
  (i) right-exclusion configuration-to-configuration map;
  
  (ii) Bloch-state-to-Bloch-state map; and
  
  (iii) wave-vector-to-wave-vector map

relating nearest-neighbor excluded chain and nearest-neighbor included chain.

- Corresponding observables and intervening-particle expansion allows some correlation functions to be calculated, either analytically or numerically.

- Study three limiting cases of the extended Hubbard ladder of spinless fermions:
  
  (i) strong correlated hopping;
  
  (ii) weak inter-leg hopping; and
  
  (iii) strong inter-leg hopping.
Conclusions

- Wrote down exact ground states, calculated various correlation functions, and perform nonlinear curve fitting to get correlation exponents.

- Many unexpected universal correlation exponents not found in existing literature on Luttinger liquids.

- Hard-core boson two-point function maps to nonlocal string observable in 1D Fermi sea. Correlation exponent $\beta = \frac{1}{2}$ calculated by Efetov and Larkin an example of string correlation exponent.

- Numerical results hints at rich physics of nonlocal string observables.