Wireless Networks : New Models and Results

Radhika Gowaikar

California Institute of Technology

Nanyang Technological University, Singapore

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A Quick History

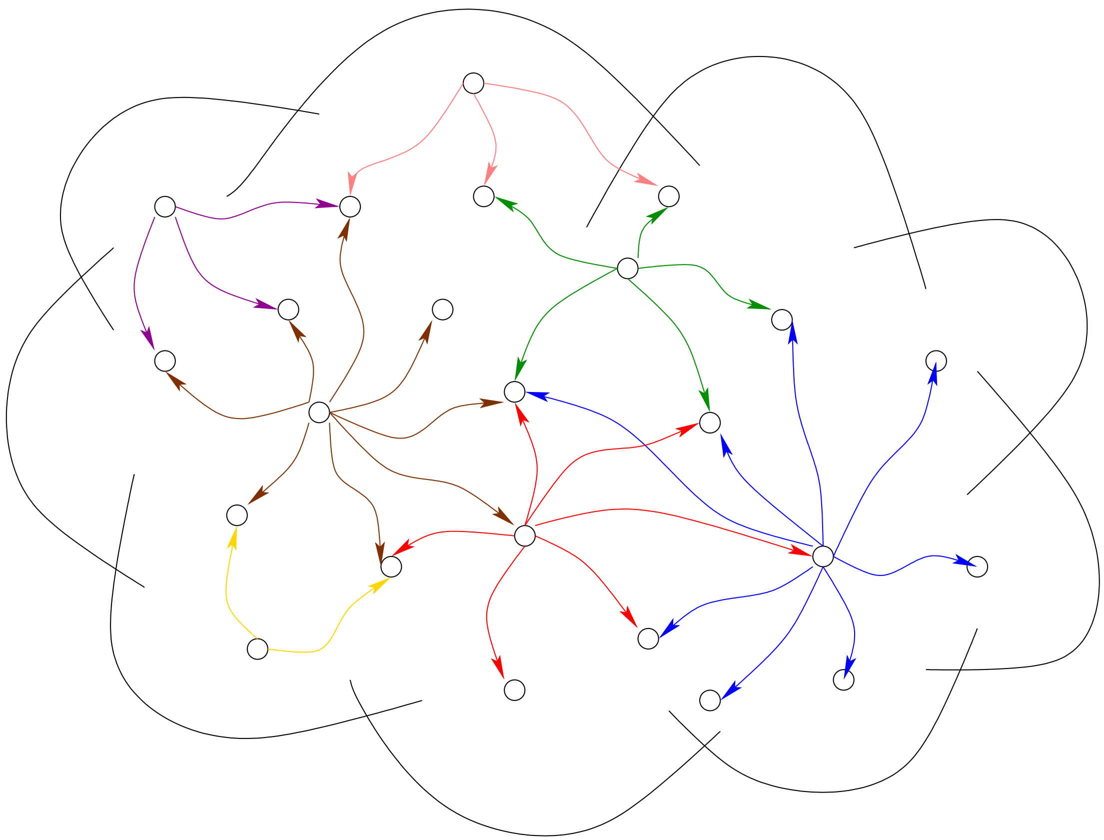


- Pre-1800s – fires, drums, mirrors, pigeons
- 1800s – telephone and telegraph cables
- 1900s – wireless communication
- Future – multimedia over wireless

Outline

- Introduction
  - Wireless connection
  - Wireless networks
- Networks with Random Connections
  - Outline, Model, Result
- Wireless Erasure Networks
  - Outline, Model, Result
- Further directions

The Wireless Connection



Shared medium, Broadcast, Interference, Randomness due to fading

Advantages of the Wireless Connection

- Randomness leads to diversity
- Less investment in infrastructure
- Mobile systems can be connected temporarily
- Can operate very different systems on the same platform

Wireless Networks

- Types and functions
  - Ad hoc, sensor, cellular, Local Area (LAN), Wide Area (WAN), hybrid
- Models
  - Geometric models, fading models, combinations
- Performance measures
  - Capacity, Throughput, Rate-Distortion, Delay, Robustness
- Constraints
  - Decentralized operation, real-time operation

General multi-terminal network problem is very much open.

Simplifications and Techniques

- Simplifications
  - Simple models – wireline, interference-free
  - Asymptotic regimes of parameters – low power, large networks
  - Scaling laws rather than exact analysis
- Techniques
  - Routing, treat information like “flow”
  - Co-operation, relaying
  - Combine and code different data streams – network coding
  - Game theoretic, random graphs

In this talk : Two network problems

- Network with Random Connections
  - Ad hoc network with fading connections
  - Throughput scaling laws
  - Relaying information
  - Random graphs
- Wireless Erasure Network
  - Erasure links
  - Capacity region
  - Network coding

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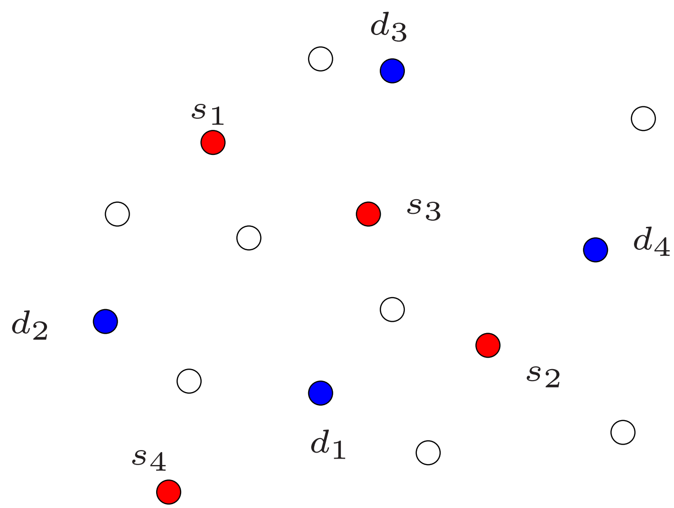
Achievable Throughputs for Random Networks

1. R. Gowaikar, B. Hochwald, B. Hassibi, "Communication over a Wireless Network with Random Connections," *IEEE Transactions on Information Theory, July 2006.*
2. R. Gowaikar, B. Hochwald, B. Hassibi, "An Achievability Result for Random Networks," *Proc. IEEE ISIT 2005.*
3. R. Gowaikar, B. Hassibi, "Achievable Throughput in Two-Scale Wireless Networks," *IEEE Journal on Selected Areas in Communications, September 2009.*
4. R. Gowaikar, B. Hassibi, "On the Achievable Throughput in Two-Scale Wireless Networks," *Proc. IEEE ISIT 2006.*

Outline for this section

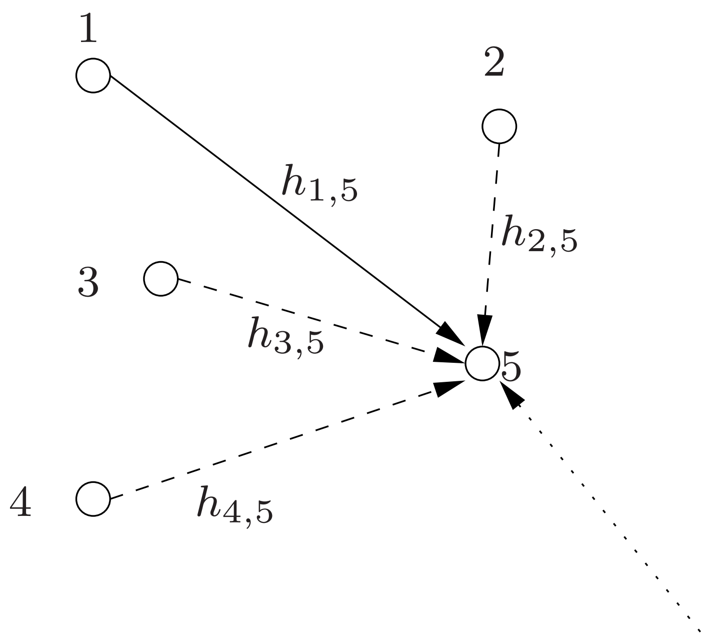
- Problem Statement
  - Successful Communication, Multihop Protocol
- Existing Models and Results
  - Kumar-Gupta Model – Scaling Law of $O(\sqrt{n})$
- New network model – Random connections
  - Random Model – Scaling Law of upto $O(n/\log^c n)$
- Operation and analysis
  - Scheduling, Ensuring Error-Free Communication
- Main result
- Examples and Simulations
- More general models
- Remarks and Summary

Problem Statement



- n nodes, channel $h_{i,j}$ between nodes i and j, k source-destination pairs
- Nodes can relay messages
- Communication using (multiple) hops

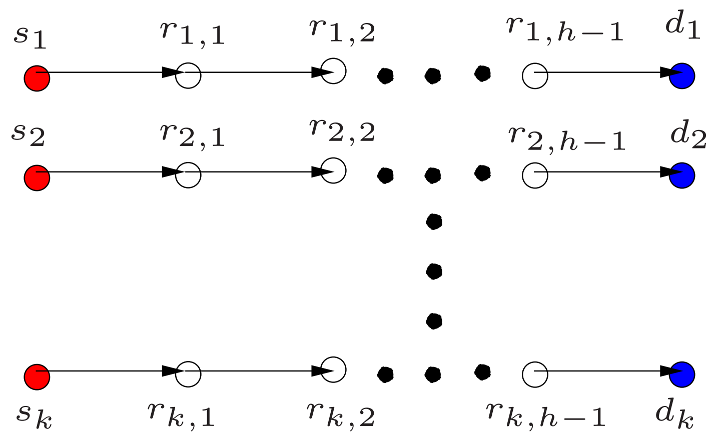
Successful Communication



A node can decode only if the signal-to-interference-plus-noise-ratio (SINR) exceeds some threshold ρ_0. Node 5 can decode only if

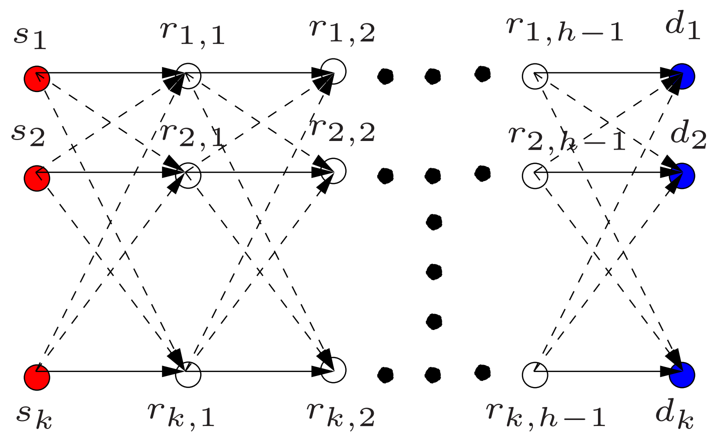
$$\frac{P|h_{1,5}|^2}{\sigma^2 + P(|h_{2,5}|^2| + |h_{3,5}|^2 + |h_{4,5}|^2)} = \frac{P\gamma_{1,5}}{\sigma^2 + P(\gamma_{2,5} + \gamma_{3,5} + \gamma_{4,5})} \geq \rho_0.$$

Scheduling Multiple Hops



- Relay nodes decode and retransmit message
- Nodes cannot transmit and receive simultaneously
  - Therefore non-colliding schedule required
- h hops are used for each communication

Error Free Decoding

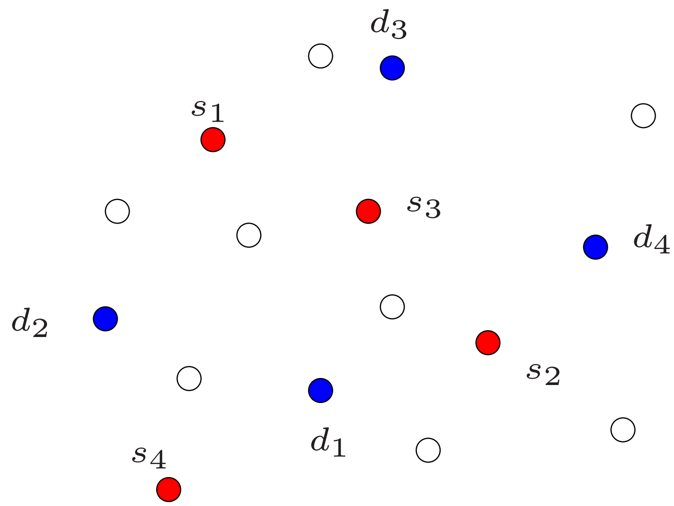


- SINR condition must be fulfilled at every relay node.
- Want to maximize throughput

$$T = (1-\epsilon)\frac{\text{Number of source-destination pairs}}{\text{Number of hops}}\log(1+\rho_0)$$

$$T = (1-\epsilon)\frac{k}{h}\log(1+\rho_0).$$

Existing Results – Distance Based Model



Kumar and Gupta (2000) used a decay law.

$$\gamma_{i,j} \propto \frac{1}{(\text{distance}_{i,j})^m}, m > 2 \quad \text{gave} \quad \text{Throughput} \propto \sqrt{n}.$$

- Simultaneous transmission can take place between $O(n)$ nodes and their nearest neighbors
- Two arbitrary nodes are $O(\sqrt{n})$ hops apart

Other Results

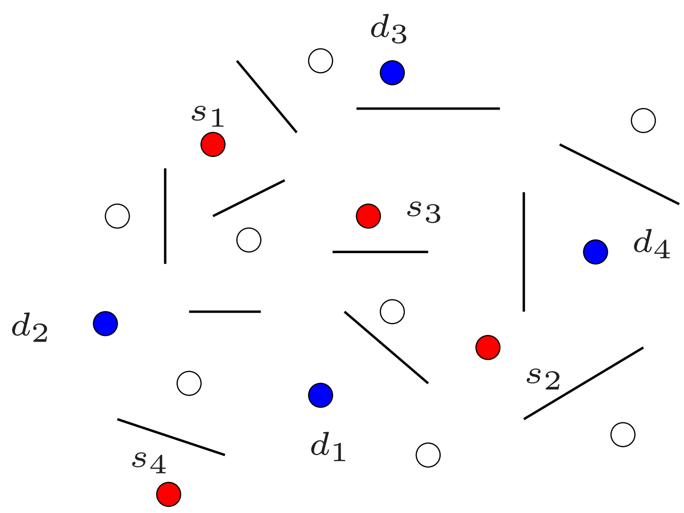
Discouraging since per-user throughput goes down as $\frac{1}{\sqrt{n}}$

- Several other results reinforce this e.g. Léveque and Telatar, Dousse and Thiran, Franceschetti et al.
- Grossglauser and Tse get a constant per-user throughput, but with mobile nodes
- Randomness provides a ray of hope
- Recent work by Miorandi and Altman, Balister et al., Franceschetti et al, Hekmat and van Mieghem shows that randomness introduces properties that are missing in a purely geometric model

Therefore we propose an entirely random model.

New Model – Random Links

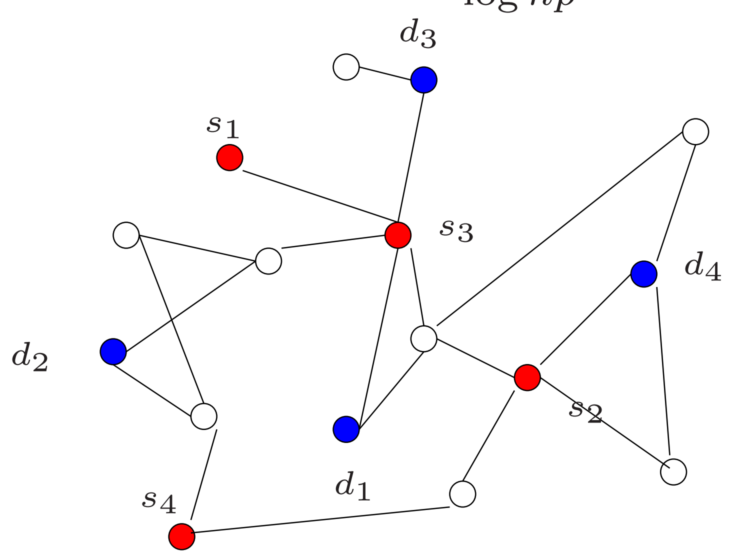
- Strength of connection does not depend on distance
- All connection strengths are identically and independently distributed according to $f_n(\gamma)$



- Model is justified over small area with many obstructions
- Line-of-sight is absent, multipath dominates

Concept of "Good" Edges

- Introduce a parameter β_n. Let $\mathrm{P}(\gamma \geq \beta_n) = Q_n(\beta_n) = p_n$.
- Edges with $\gamma \geq \beta_n$ are "good."
- Obtain $G(n, p)$ by keeping only these and eliminating the rest.
  - $G(n, p)$ is a well-studied random graph model.
  - e.g. diameter is known to be $\frac{\log n}{\log np}$.



Now find schedule of non-colliding paths in this graph.

Scheduling

Theorem 1 *(Broder et al. 1996) Suppose that* $G = G(n, p)$ *and* $p \geq \frac{\log n + \omega_n}{n}$*, where* $\omega_n \to \infty$*. Then, provided k is not greater than* $\alpha n \frac{\log np}{\log n}$*, there are vertex-disjoint paths connecting* s_i *to* d_i *for any set of* k *randomly chosen source-destination pairs.*

- This is guaranteed with probability going to 1 for a positive constant α.
- We can thus establish upto $k = \alpha n \frac{\log np}{\log n}$ non-colliding (in fact, vertex-disjoint) paths. Theorem is tight upto constant.
- Can show that every message makes around $h = \frac{\log n}{\alpha \log np}$ hops.
- Now need to ensure that the probability of error can be made to go to zero.

Probability of Error

$$\begin{aligned} \mathrm{P}(\text{Message } i \text{ fails}) &= \mathrm{P}\Big(\bigcup_{j=1}^{h} \text{SINR at hop } j \le \rho_0\Big) \\ &\le \#\text{ hops } \times \mathrm{P}(\text{SINR at hop } 1 \le \rho_0) \end{aligned}$$

$$\epsilon_n = \mathrm{P}(\text{Message } i \text{ fails}) \le \frac{\log n}{\alpha \log np} \frac{\sigma_\gamma^2/(k-1)}{\left(\frac{P\beta_n - \rho_0\sigma^2}{(k-1)P\rho_0} - \mu_\gamma\right)^2}.$$

Can use Chebyshev bound if $\rho_0 \le \dfrac{P\beta_n}{\sigma^2 + P(k-1)\mu_\gamma}$.

Put Scheduling and Probability of Error results together to get Main Result.

Main Result

Theorem 2 *Let* $Q_n(x) = \mathrm{P}(\gamma \geq x)$*. Choose any* β_n *such that* $Q_n(\beta_n) = \frac{\log n + \omega_n}{n}$*, where* $\omega_n \to \infty$*. Let* $\mu_\gamma = E\gamma$*,* $\sigma_\gamma^2 = E(\gamma - \mu_\gamma)^2$*. Then there exists a positive constant* α *such that a throughput of*

$$T = (1 - \epsilon_n)\, k_n(\beta_n) \alpha \frac{\log(nQ_n(\beta_n))}{\log n} \log\left(1 + \frac{a_n \beta_n}{\frac{\sigma^2}{P} + (k_n(\beta_n) - 1)\mu_\gamma}\right)$$

is achievable for any positive a_n *such that* $a_n \leq 1$ *and any* $k_n(\beta_n)$ *that satisfy the conditions:*

1.
$$k_n(\beta_n) \leq \alpha n \frac{\log(nQ_n(\beta_n))}{\log n}$$

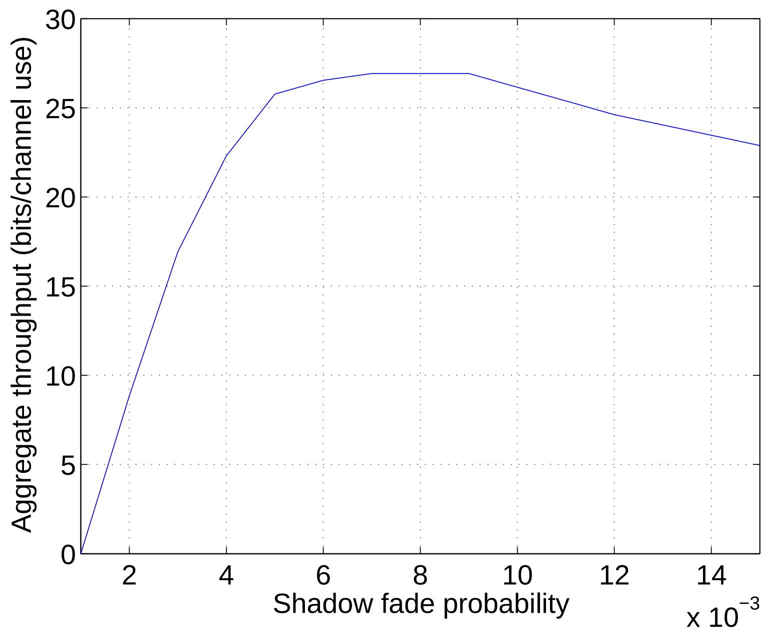
2.
$$\epsilon_n \leq \frac{a_n^2}{\alpha(1 - a_n)^2} \frac{(k_n(\beta_n) - 1)\sigma_\gamma^2}{(\frac{\sigma^2}{P} + (k_n(\beta_n) - 1)\mu_\gamma)^2} \frac{\log n}{\log(nQ_n(\beta_n))} \to 0$$

Heavy Dependence on $f_n(\gamma)$

Distribution	$f_n(\gamma)$	Throughput
Shadow	$(1-p_n)\delta(\gamma)+p_n\delta(\gamma-1)$	$\frac{1}{w_n}\frac{\log^2(\log n)}{\log^3 n}\quad n$
Exponential	$e^{-\gamma}$	$\log n$
Decay	$\frac{4\pi\Delta}{nm}\frac{1}{\gamma^{1+\frac{2}{m}}}, m>2$	$\frac{1}{w_n}\frac{\log^2(\log n)}{(\log n)^{2+m/2}}\quad n$
Heavy Tail	$\frac{c}{1+\gamma^4}$	$\frac{\log\log n}{\log^{4/3} n}\quad n^{1/3}$

Shadow Fading Distribution

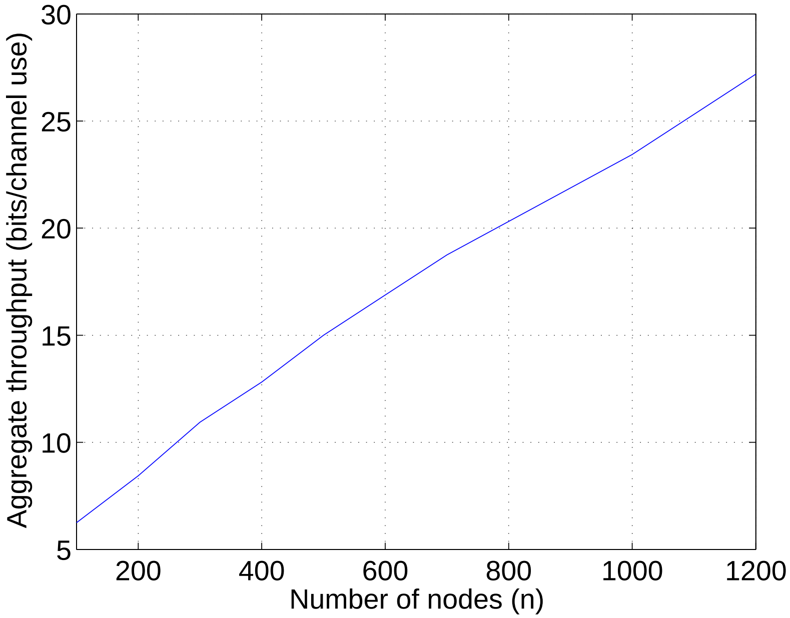
- $f_n(\gamma) = (1 - p_n)\delta(\gamma) + p_n\delta(\gamma - 1)$. Natural choice of $\beta_n = 1$.
- Gives T in terms of p_n. For fixed n, can find optimum $p_n = \frac{\log n + \omega_n}{n}$.



1,000 nodes. Simulations $p^* \approx 0.008$. Theory $\frac{\log(1000)}{1000} \approx 0.0069$.

Shadow Fading Distribution – Scaling Law

- For $p_n = \frac{\log n + \omega_n}{n}$, $h = \frac{\log n}{\log(\log n + \omega_n)}$, $T = \frac{1}{w_n} \frac{\log^2(\log n)}{\log^3 n} \; n$



$p = \frac{2 \log n}{n}$, $100 - 1200$ nodes.

Density obtained from Decay Law

Consider a single node transmitting in a network with a distance-based decay law, say $\frac{1}{(\text{distance})^m}$. The marginal distribution of received powers can be shown to be

$$f_n(\gamma) = \frac{4\pi\Delta}{nm}\frac{1}{\gamma^{1+\frac{2}{m}}}, \gamma \in \left[\left(\frac{2\pi\Delta}{n+2\pi\Delta d^2}\right)^{m/2}, \frac{1}{d^m}\right], m > 0.$$

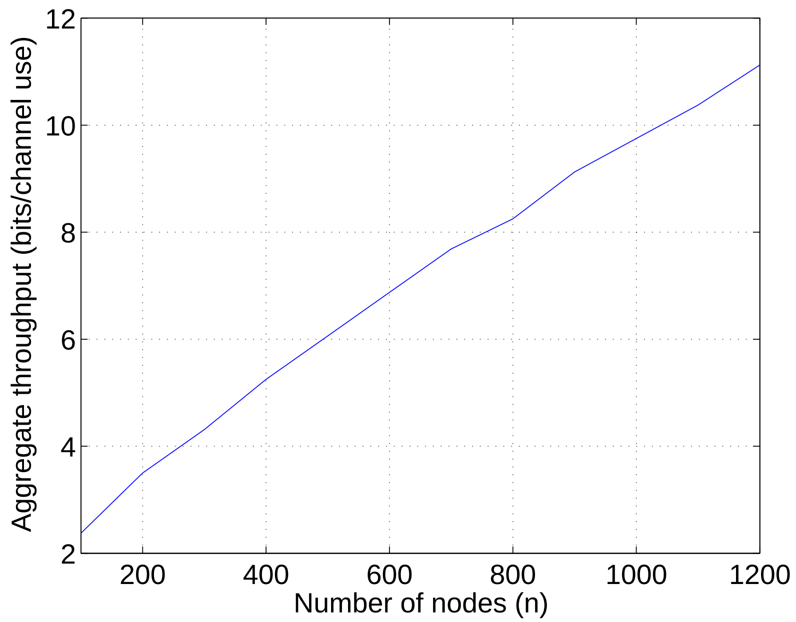
Achievable throughputs scale like

$$T = \begin{cases} \frac{\log(\log n+\omega_n)}{\log n(\log n+\omega_n)^{m/2}} \; n^{m/2} & m < 2 \\ \frac{\log(\log n+\omega_n)}{\log^2 n(\log n+\omega_n)} \; n & m = 2 \\ \frac{1}{w_n}\frac{\log^2(\log n+\omega_n)}{\log^2 n(\log n+\omega_n)^{m/2}} \; n & m > 2. \end{cases}$$

Significantly better than $O(\sqrt{n})$ predicted by Kumar-Gupta (2000) for $m > 2$.

Density obtained from Decay Law

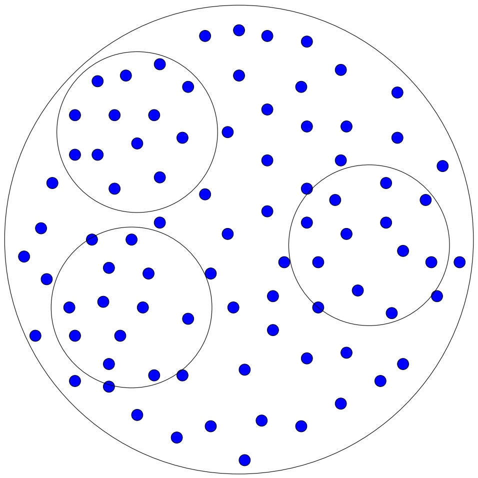
- $f_n(\gamma) = \frac{4\pi\Delta}{nm}\frac{1}{\gamma^{1+\frac{2}{m}}}$. Optimum choice of β_n is $\frac{2\pi\Delta}{(\log n+\omega_n)^{m/2}}$
- $p_n = \frac{\log n+\omega_n}{n}$, $h = \frac{\log n}{\log(\log n+\omega_n)}$, $T = \frac{1}{w_n}\frac{\log^2(\log n+\omega_n)}{\log^2 n(\log n+\omega_n)^{3/2}}\ n$



$m = 3$, $d = 1$, $\Delta = 1$. Throughput increases almost linearly.

More general models

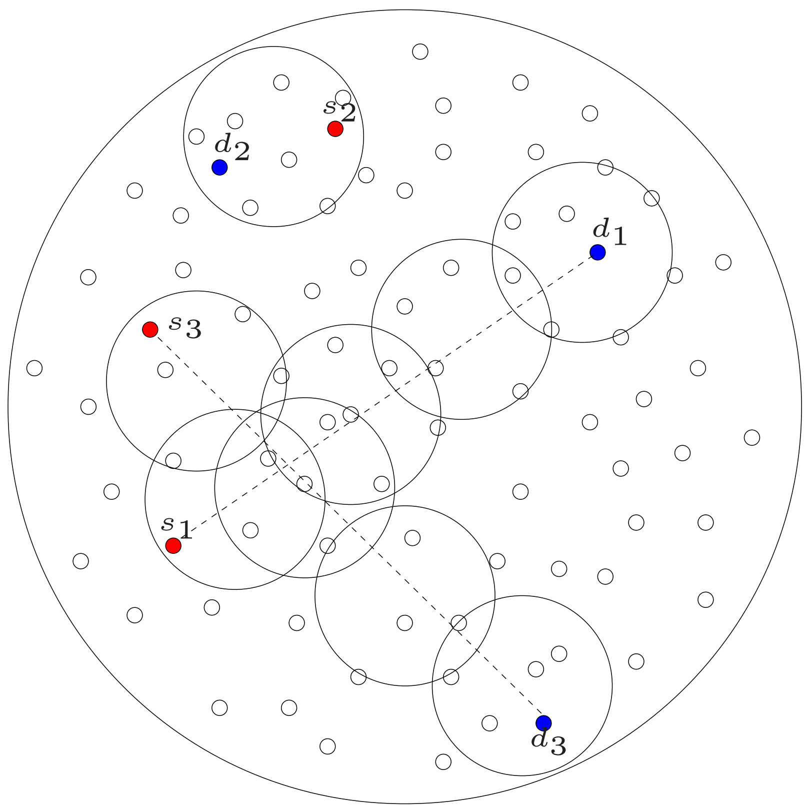
The random model does not incorporate distance effects.



- Globally: channel strength decays with distance
- Locally: channel strengths are drawn i.i.d. from a distribution

$$p_x(\gamma) = \begin{cases} f(\gamma) & \text{if } x \le r \\ \frac{\mu_\gamma r^m}{x^m} \exp(-\gamma \frac{x^m}{\mu_\gamma r^m}) & \text{if } x > r \end{cases}.$$

Throughput for the Two-Scale Model



Example:

Decay law: $\frac{1}{\text{dist}^m}$ Distribution: $\frac{1}{(1+\gamma)^t}, t > 2$

Throughput: $\frac{1}{\log^2 n} \, n^{\frac{1}{t-1}}$

Multiscale and mixture models

- Three-scale model – can use ideas similar to the two-scale model

$$p_x(\gamma) = \begin{cases} f(\gamma) & \text{if } x \le r_1 \\ \frac{r_2 - x}{r_2 - r_1} f(\gamma) + \frac{x - r_1}{r_2 - r_1} \frac{\mu_\gamma r_2^m}{x^m} \exp(-\gamma \frac{x^m}{\mu_\gamma r_2^m}) & \text{if } r_1 < x \le r_2 \\ \frac{\mu_\gamma r_2^m}{x^m} \exp(-\gamma \frac{x^m}{\mu_\gamma r_2^m}) & \text{if } x > r_2 \end{cases} .$$

- Mixture model – need to develop new approach

$$p_x(\gamma) = \frac{R - x}{R} f(\gamma) + \frac{x}{R} \frac{1}{x^m} \exp(-\gamma x^m)$$

Remarks and future work

- Developed decentralized implementation schemes
- Upperbounds on the achievable throughput using multihop protocol
  - Information-theoretic upperbounds?
- How do adhoc networks compare with other (e.g. cellular) networks?
  - Spectral efficiency, power efficiency, operation

Summary

Incorporating randomness gives significantly more encouraging scaling laws than distance-based model.

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Capacity of Wireless Erasure Networks

1. A. F. Dana, R. Gowaikar, R. Palanki, B. Hassibi, M. Effros, "Capacity of wireless erasure networks," *IEEE Transactions on Information Theory, March 2006*.

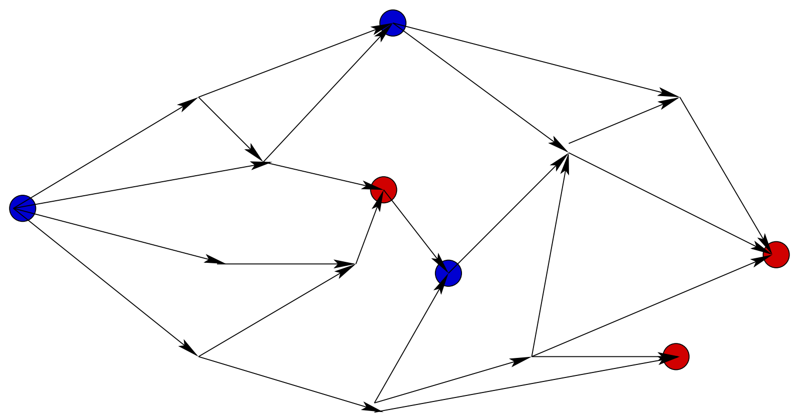
2. R. Gowaikar, A. F. Dana, R. Palanki, B. Hassibi, M. Effros, "On the capacity of wireless erasure relay networks," *Proc. IEEE ISIT 2004.*

3. A. F. Dana, R. Gowaikar, B. Hassibi, M. Effros, M. Médard, "Should we break a wireless network into sub-networks?" *Proc. 41st Annual Allerton Conf. on Communication, Control, and Computing, 2003*.

Outline for this section

- The Multicast Problem and known results
- New network model and operation
- Cuts and Cut Capacities
- Main Result – Capacity Result for Erasure Wireless Networks
- Sketch of the proof
  - Achievability and Converse
- Remarks and practical implementation
- Future work and Summary

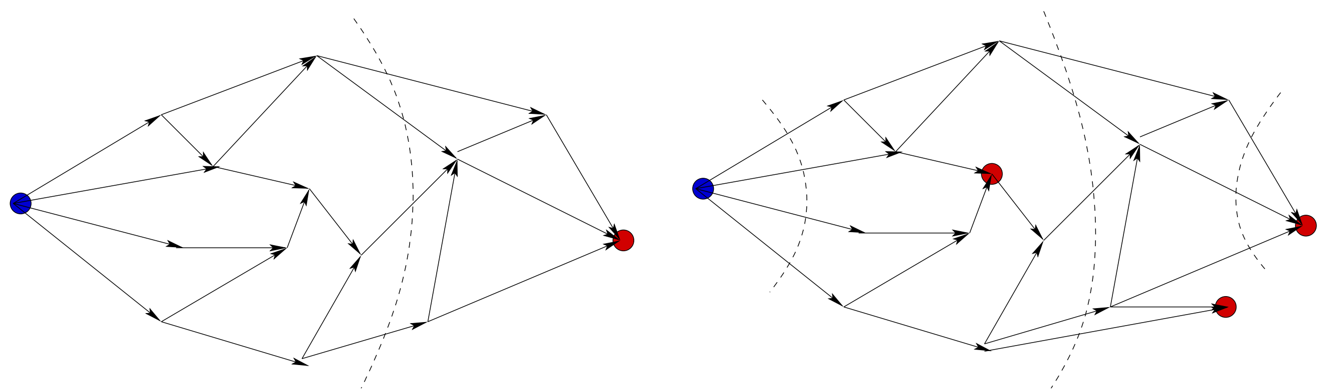
The Multicast Problem



- Multiple sources with information
- Multiple destinations requesting subsets of information
- Each link is a channel
- Can all requests be satisfied? What is the capacity region?
- What strategies get you to capacity? What delays are expected?

Review of Existing Results

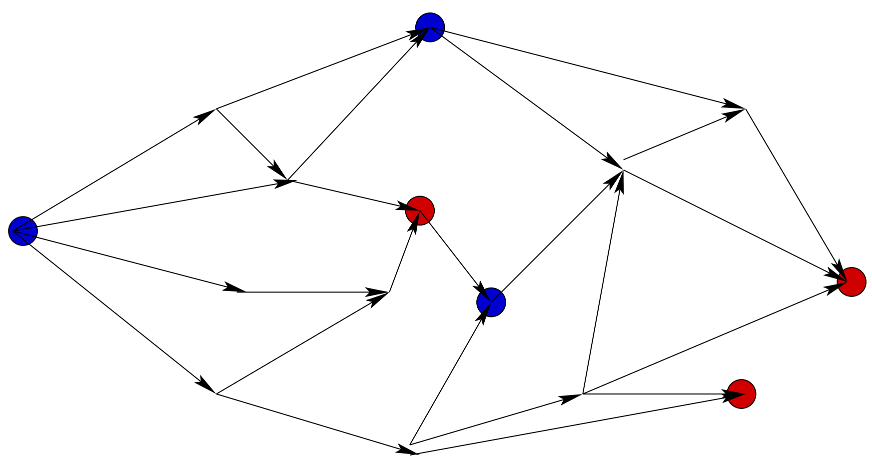
- General multi-terminal networks are unsolved though min-cut outer bounds are well known.
- For a wireline network, having a single sender and a single receiver, a max-flow min-cut capacity result exists.



- In addition, we now know how to achieve these min-cut bounds in some special cases of the multicast problem. (Ahlswede, Cai, Li, Yeung '00; Koetter, Médard '03)

Missing features

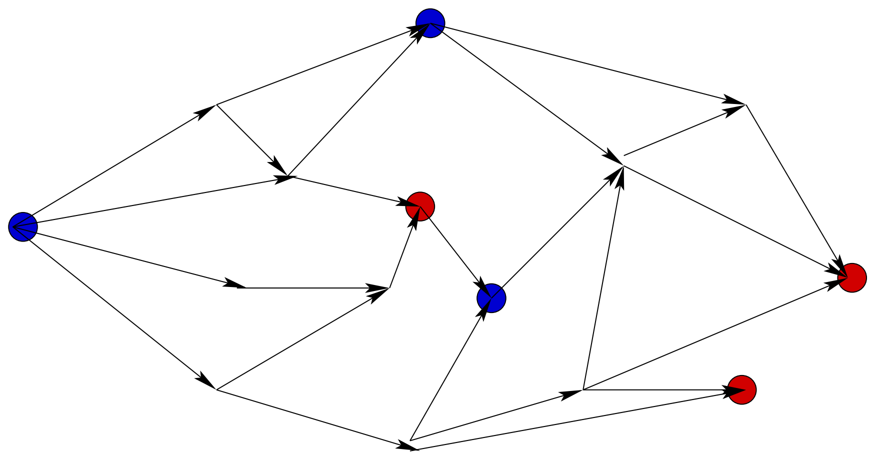
The wireline model does not incorporate broadcast at transmitting nodes and interference at nodes with multiple access – both of these are part of many practical systems.



Need to come up with a model that incorporates these features but is still tractable.

A simple yet realistic model

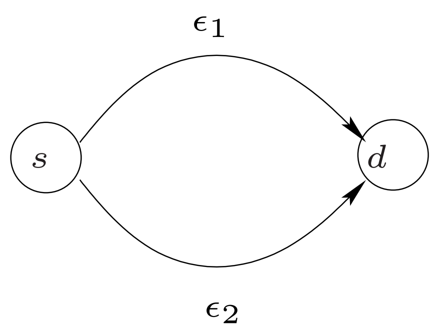
- Assume that all the links are erasure channels
  - Valid in systems with ARQ-like mechanisms



- Broadcast – same message on all outgoing links
- Interference – erasures patterns may be correlated
  - Errors on one link cause errors on another

Examples – Some simple networks

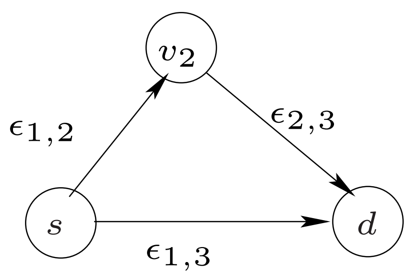
1.

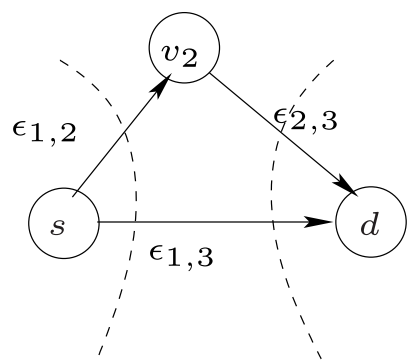


$$C = 1 - \epsilon_1 + 1 - \epsilon_2$$

$$C = 1 - \epsilon_1 \epsilon_2$$

2.



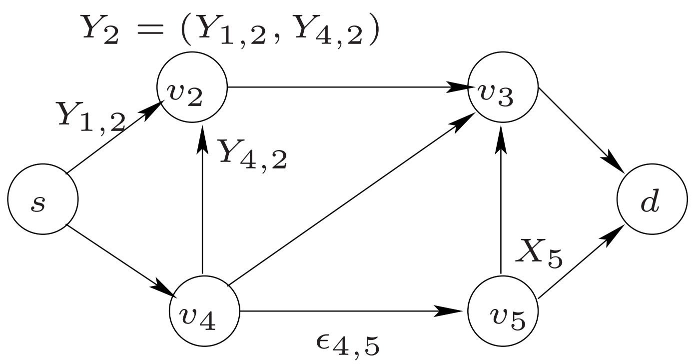


$$C = \min\{1 - \epsilon_{1,2} + 1 - \epsilon_{1,3}, 1 - \epsilon_{2,3} + 1 - \epsilon_{1,3}\}$$

$$C = ?$$

Model for Erasure Wireless Network

- $G = (V, E)$ directed, acyclic graph
- Edges are memoryless, erasure channels with instantaneous transmission
- Broadcast at transmission



source $s = v_1$, destination $d = v_{|V|}$

transmitted signals X_i, received signals Y_i

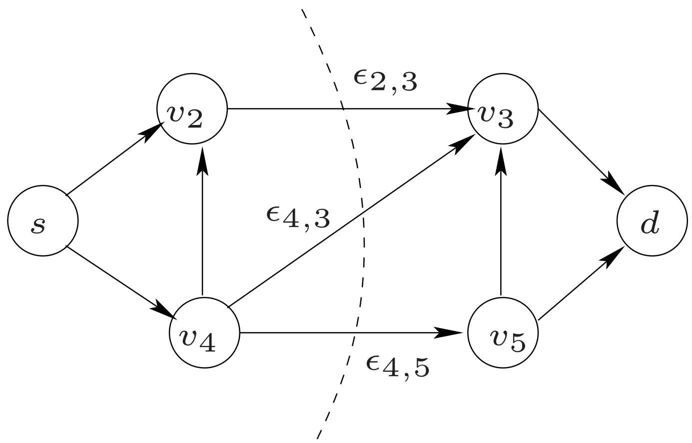
Capacity of the network : Preliminaries

- Communication in blocks of n packets
- Node v_i uses encoding function f_i on what it receives to determine what it transmits
- Capacity – maximum rate for which average probability of error goes to zero

$s - d$ cut and Cutset

- An $s - d$ cut : a partition of V into subsets $V_s \ni s$ and $V_d \ni d$
- Cutset $E(V_s)$: the set of edges going across the cut

$$E(V_s) = \{(v_i, v_j) | (v_i, v_j) \in E, v_i \in V_s, v_j \in V_d\}$$

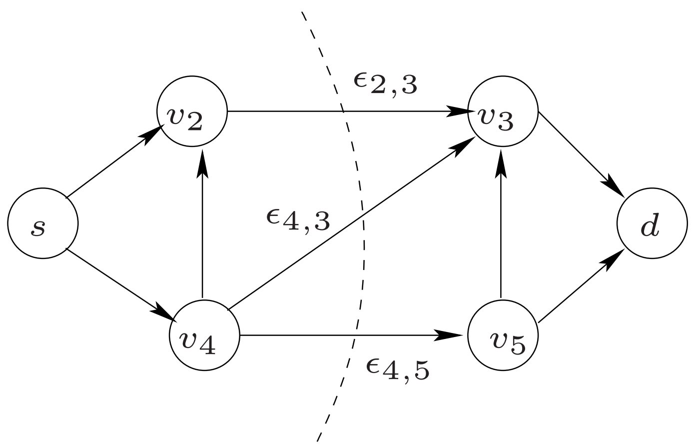


$s - d$ cut given by $V_s = \{s, v_2, v_4\}$
$E(V_s) = \{(v_2, v_3), (v_4, v_3), (v_4, v_5)\}$

Cut Capacity

Cut capacity $W(V_s)$: the 'value' of a cut

$$W(V_s) = \sum_{i:(v_i,v_j)\in E(V_s)} \left(1 - \prod_{j:(v_i,v_j)\in E(V_s)} \epsilon_{i,j} \right)$$



$$W(V_s) = 1 - \epsilon_{2,3} + 1 - \epsilon_{4,3}\epsilon_{4,5}$$

Main Result

Assume that the decoders know erasure locations from across the network

Theorem 3 *The capacity of the erasure wireless network with a single source and a single destination is given by the value of the cut with the minimum value.*

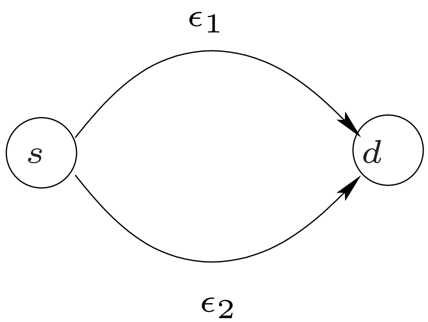
$$C = \min_{V_s} W(V_s)$$

Theorem 4 *The capacity of the erasure wireless network with a single source and multiple destinations (say $d_1, d_2, \cdots d_r$) is the smallest of the capacities to the individual destinations.*

$$C = \min_{d_i} \min_{V_{s,i}:\ an\ s-d_i\ cut} W(V_{s,i})$$

Back to the original examples

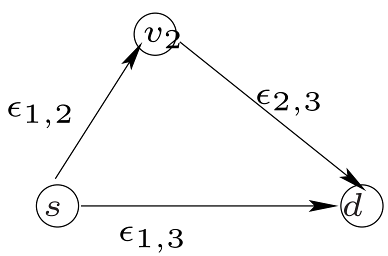
1.

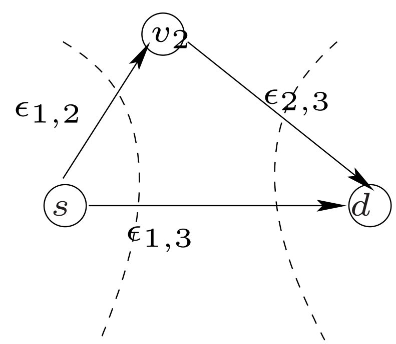


$$C = 1 - \epsilon_1 + 1 - \epsilon_2$$

$$C = 1 - \epsilon_1 \epsilon_2$$

2.



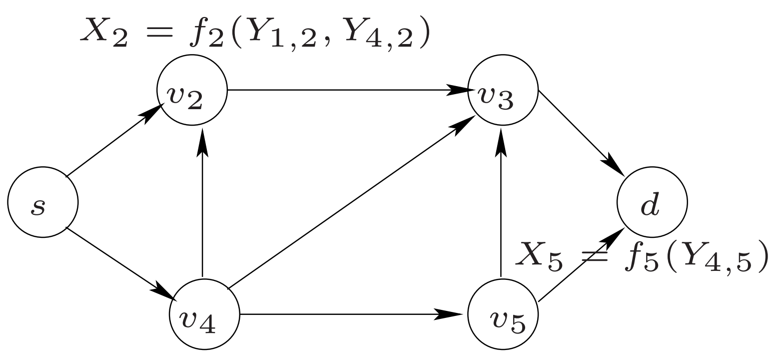


$$C = \min\{1 - \epsilon_{1,2} + 1 - \epsilon_{1,3}, 1 - \epsilon_{2,3} + 1 - \epsilon_{1,3}\}$$

$$C = \min\{1 - \epsilon_{1,2}\epsilon_{1,3}, 1 - \epsilon_{2,3} + 1 - \epsilon_{1,3}\}$$

Sketch of Proof

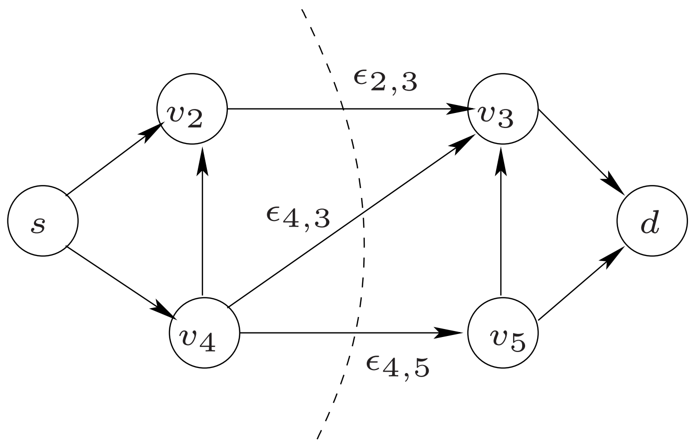
- Encoding functions f_i are chosen randomly and are known to destinations
- Destinations know erasure locations, can simulate the network for every codeword
- The message for which simulated output matches what the destination actually observed is the decoder output



The network appears deterministic to the destinations.

Achievability and Converse

Invoke ideas of typical sets, randomness of encoding functions etc.



Probability of error across this cut $= 2^{n(R-(1-\epsilon_{2,3}+1-\epsilon_{4,3}\epsilon_{4,5}))}$

Therefore define cut capacity as $W(V_s) = 1 - \epsilon_{2,3} + 1 - \epsilon_{4,3}\epsilon_{4,5}$

Now, if $R < W(V_s)$, the probability of error goes to zero.

Converse : Similar to standard outer-bound results

Tools used : Fano's Inequality, Data Processing Inequality

Some Remarks

- Some more multicast results go through.
  - Multiple sources, multiple destinations, all destinations want all sources
  - Single source with multiple processes, multiple destinations needing disjoint subsets of processes from source
- Note that we do not have to perform channel and network coding separately to reach capacity. In fact, making each link or sub-network error-free is demonstrably sub-optimal.
- Perhaps the correct approach is to ask "What sort of side-information should the destinations have?"

Linear encoding and practical implementation

- Choosing all encoding functions to be linear also takes us to capacity
- Now the decoder only has to solve a linear systems of equations
- This takes only polynomial time (for a full-rank system)
- Practical implementations involving rateless codes have been proposed (Lun et al.)

Future work

- General multicast problems e.g. k-pairs problem
- Multiple, possibly correlated information sources
- Capacity when the decoder does not know erasure locations
- Networks with other channels – DMCs, AWGN etc.

Summary

Obtained capacity for a class of wireless erasure networks for several multicast settings

Other contributions

- Practical operating schemes for erasure and Gaussian networks
  - Nodes are permitted to only forward or decode
  - Proposed a (decentralized) greedy algorithm that determines the optimal operation for each node

R. Gowaikar, A. F. Dana, B. Hassibi, M. Effros, "Practical Schemes for Wireless Network Operation," *IEEE Trans. Comm., March 2007*

- Decoding in multiple antenna systems
  - Proposed the Increasing Radii Algorithm that allows a trade-off between computational complexity of decoding and error performance
  - Speeds up the decoder by a factor of 50 in some cases, with negligible loss of performance

R. Gowaikar, B. Hassibi, "Statistical Pruning for near-maximum likelihood decoding," *IEEE Trans. Sig. Pro., June 2007.*

Future Directions

- Real-time communications esp. for control applications
  - Noisy measurements have to be transmitted over noisy channels
  - Highly accurate information has to be received in a very short amount of time
- Computations in a network
  - Nodes have partial information about some underlying process
  - Computing and communicating functions of noisy and distributed data
- World without wires
  - Local and global repercussions

$\boxed{\square}$