

# Importance of Symbol Equity in Coded Modulation for Power Line Communications

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**Abstract**—The use of multiple frequency shift keying modulation with permutation codes addresses the problem of permanent narrowband noise disturbance in a power line communications system. In this paper, we extend this coded modulation scheme based on permutation codes to general codes and introduce an additional new parameter that helps to more precisely capture a code's performance against permanent narrowband noise. As a result, we define a new class of codes, namely, equitable symbol weight codes, which are optimal with respect to this measure. In addition, we demonstrate via simulations that equitable symbol weight codes achieve lower symbol error rates than other codes of the same length and distance over the same alphabet.

## I. INTRODUCTION

Power line communications (PLC) is a technology that enables the transmission of data over high voltage electric power lines. Started in the 1950's in the form of ripple control for load and tariff management in power distribution, this low bandwidth one-way communication system has evolved to a two-way communication system in the 1980's. With the emergence of the Internet in the 1990's, research into broadband PLC gathered pace as a promising technology for Internet access and local area networking, since the electrical grid infrastructure provides "last mile" connectivity to premises and capillarity within premises. Recently, there has been a renewed interest in high-speed narrowband PLC due to applications in sustainable energy strategies, specifically in smart grids (see [1]–[4]).

However, power lines present a difficult communications environment and overcoming permanent narrowband disturbance has remained a challenging problem [5]–[7]. Vinck [5] addressed this problem by showing that multiple frequency shift keying (MFSK) modulation, in conjunction with the use of a permutation code having minimum (Hamming) distance  $d$ , is able to correct up to  $d-1$  errors due to narrowband noise. Since then, more general codes such as constant-composition codes, frequency permutation arrays, and injection codes (see [8]–[23]) have been considered as possible replacements for permutation codes in PLC. Versfeld *et al.* [24], [25] later introduced the notion of 'same-symbol weight' (henceforth, termed as *symbol weight*) of a code as a measure of the capability of a code in dealing with narrowband noise. They also showed empirically that low symbol weight cosets of Reed-Solomon codes outperform normal Reed-Solomon codes in the presence of narrowband noise and additive white Gaussian noise. Infinite families of optimal codes with minimum symbol weight were constructed by Chee *et al.* [26] recently.

Unfortunately, symbol weight alone is not sufficient to capture the performance of a code in dealing with permanent narrowband noise. The purpose of this paper is to extend the analysis of Vinck's coded modulation scheme based on permutation codes (see [5], [27, Subsection 5.2.4]) to general codes. In the process, we introduce an additional new parameter that more precisely captures a code's performance against permanent narrowband noise. This parameter is related to *symbol equity*, the uniformity of frequencies of symbols in each codeword. Codes designed taking into account this new parameter are shown to perform better than general ones.

## II. PRELIMINARIES

For a positive integer  $n$ , the set  $\{1, 2, \dots, n\}$  is denoted by  $[n]$ . For  $X$  a finite set and  $k$  an integer,  $0 \leq k \leq |X|$ , the set of all  $k$ -subsets of  $X$  is denoted  $\binom{X}{k}$ .

Let  $\Sigma$  be a set of  $q$  symbols. A  $q$ -ary code  $\mathcal{C}$  of length  $n$  over the alphabet  $\Sigma$  is a subset of  $\Sigma^n$ . Elements of  $\mathcal{C}$  are called *codewords*. The *size* of  $\mathcal{C}$  is the number of codewords in  $\mathcal{C}$ . For  $i \in [n]$ , the  $i$ th *coordinate* of a codeword  $\mathbf{u}$  is denoted by  $u_i$ , so that  $\mathbf{u} = (u_1, u_2, \dots, u_n)$ .

The *composition* of  $\mathbf{u} \in \Sigma^n$  is a vector  $(w_\sigma(\mathbf{u}))_{\sigma \in \Sigma}$ , where  $w_\sigma(\mathbf{u})$  is the number of times the symbol  $\sigma$  appears among the coordinates of  $\mathbf{u}$ , that is,

$$w_\sigma(\mathbf{u}) = |\{i \in [n] : u_i = \sigma\}|.$$

The *symbol weight* of  $\mathbf{u}$  is

$$\text{swt}(\mathbf{u}) = \max_{\sigma \in \Sigma} w_\sigma(\mathbf{u}).$$

A code has *bounded symbol weight*  $r$  if all its codewords have symbol weight at most  $r$ . A code  $\mathcal{C}$  has *constant symbol weight*  $r$  if all its codewords have symbol weight exactly  $r$ . Note that for any  $\mathbf{u} \in \Sigma^n$ , we have  $\text{swt}(\mathbf{u}) \geq \lceil n/q \rceil$ . A code has *minimum symbol weight* if it has constant symbol weight  $\lceil n/q \rceil$ .

An element  $\mathbf{u} \in \Sigma^n$  is said to have *equitable symbol weight* if  $w_\sigma(\mathbf{u}) \in \{\lfloor n/q \rfloor, \lceil n/q \rceil\}$  for all  $\sigma \in \Sigma$ . If all the codewords of  $\mathcal{C}$  have equitable symbol weight, then the code  $\mathcal{C}$  is called an *equitable symbol weight code*. Note that every equitable symbol weight code is also a minimum symbol weight code. The following lemma shows that for any  $\mathbf{u} \in \Sigma^n$  having equitable symbol weight, the number of symbols occurring with frequency  $\lceil n/q \rceil$  in  $\mathbf{u}$  is uniquely determined. Hence, the frequencies of symbols in an equitable symbol weight codeword are as uniformly distributed as possible.

*Lemma 2.1:* Let  $\mathbf{u} \in \Sigma^n$ ,  $r = \lceil n/q \rceil$ , and  $t = qr - n$ . If  $\mathbf{u}$  has equitable symbol weight, then there are  $(q - t)$  symbols each appearing exactly  $r$  times and the remaining  $t$  symbols each appearing exactly  $r - 1$  times in  $\mathbf{u}$ , that is,

$$\begin{aligned} |\{\sigma \in \Sigma : w_\sigma(\mathbf{u}) = r\}| &= q - t, \\ |\{\sigma \in \Sigma : w_\sigma(\mathbf{u}) = r - 1\}| &= t. \end{aligned}$$

*Proof:* Let  $x = |\{\sigma \in \Sigma : w_\sigma(\mathbf{u}) = r\}|$  and  $y = |\{\sigma \in \Sigma : w_\sigma(\mathbf{u}) = r - 1\}|$ . Then the following equations hold:

$$\begin{aligned} x + y &= q, \\ rx + (r - 1)y &= n. \end{aligned}$$

Solving this set of equations gives the lemma.  $\blacksquare$

Consider the usual (Hamming) distance defined on codewords and codes. A  $q$ -ary code of length  $n$  and distance  $d$  is called an  $(n, d)_q$ -code, while a  $q$ -ary code of length  $n$  having bounded symbol weight  $r$  and distance  $d$  is called an  $(n, d, r)_q$ -symbol weight code, and a  $q$ -ary equitable symbol weight code of length  $n$  and distance  $d$  is called an  $(n, d)_q$ -equitable symbol weight code.

### III. CORRECTING NOISE WITH MFSK MODULATION

In coded modulation for power line communications [5], a  $q$ -ary code of length  $n$  is used, whose symbols are modulated using  $q$ -ary MFSK. The receiver demodulates the received signal using an envelope detector to obtain an output, which is then decoded by a decoder.

Four detector/decoder combinations are possible: classical, modified classical, hard-decision threshold, and soft-decision threshold (see [27] for details). A soft-decision threshold detector/decoder requires exact channel state knowledge and is therefore not useful if we do not have channel state knowledge. Henceforth, we only consider the hard-decision threshold detector/decoder here, since it contains more information about the received signal compared to the classical and modified classical ones. We note that in the case of the hard-decision threshold detector/decoder, the decoder used is a *minimum distance decoder*.

Let  $\mathcal{C}$  be an  $(n, d)_q$ -code over alphabet  $\Sigma$ , and let  $\mathbf{u} = (u_1, \dots, u_n)$  be a codeword transmitted over the PLC channel. The received signal (which may contain errors caused by noise) is demodulated to give an output  $\mathbf{v} = (v_1, v_2, \dots, v_n) \in (2^\Sigma)^n$ , where  $2^\Sigma$  denotes the collection of all subsets of  $\Sigma$ . Note that each  $v_i$  is a subset of  $\Sigma$ . The errors that arise from the different types of noises in the channel (see [27, pp. 222–223]) have the following effects on the output of the detector.

- 1) Let  $1 \leq e \leq q$ . If  $e$  narrowband noise errors occur, then there is a set of  $e$  symbols contained in every  $v_i$ , that is,  $|\cap_{i=1}^n v_i| \geq e$ .
- 2) Let  $1 \leq e \leq q$ . If  $e$  signal fading errors occur, then there are  $e$  symbols, none of which appears in any  $v_i$ , that is,  $(\cup_{i=1}^n v_i) \cap \Gamma = \emptyset$  for some  $\Gamma \in \binom{\Sigma}{e}$ .
- 3) Let  $1 \leq e \leq n$ . If  $e$  impulse noise errors occur, then there is a set  $\Pi \in \binom{[n]}{e}$  of  $e$  positions such that  $v_i = \Sigma$  for all  $i \in \Pi$ .

- 4) Let  $1 \leq e \leq n(q - 1)$ . If  $e$  insertion errors occur, then there is a set  $\Omega \in \binom{[n] \times \Sigma}{e}$  such that for each  $(i, \sigma) \in \Omega$ ,  $v_i$  contains  $\sigma$  and  $\sigma \neq u_i$ .
- 5) Let  $1 \leq e \leq n$ . If  $e$  deletion errors occur, then there is a set  $\Pi \in \binom{[n]}{e}$  of  $e$  positions that that  $v_i$  does not contain  $u_i$  for all  $i \in \Pi$ .

Both insertion and deletion errors are due to *background noise*.

*Example 3.1:* The same detector output can arise from different combinations of error types. Suppose  $\mathbf{u} = (1, 2, 3, 4)$ . A signal fading error of symbol 1 and a deletion error at position 1 would each result in the same detector output of  $\mathbf{v} = (\emptyset, \{2\}, \{3\}, \{4\})$ .

For  $\mathbf{u} \in \Sigma^n$  and  $\mathbf{v} \in (2^\Sigma)^n$ , define<sup>1</sup>

$$d(\mathbf{u}, \mathbf{v}) = |\{i : u_i \notin v_i\}|.$$

We also extend the definition of distance so that for  $\mathcal{C} \subseteq \Sigma^n$ , we have  $d(\mathcal{C}, \mathbf{v}) = \min_{\mathbf{u} \in \mathcal{C}} d(\mathbf{u}, \mathbf{v})$ . Given  $\mathbf{v} \in (2^\Sigma)^n$ , a minimum distance decoder (for a code  $\mathcal{C}$ ) outputs a codeword  $\mathbf{u} \in \mathcal{C}$  which has the smallest distance to  $\mathbf{v}$ , that is, a minimum distance decoder returns an element of

$$\arg \min_{\mathbf{u} \in \mathcal{C}} d(\mathbf{u}, \mathbf{v}) := \{\mathbf{u} \in \mathcal{C} : d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}', \mathbf{v}) \ \forall \mathbf{u}' \in \mathcal{C}\}.$$

Below, we study the conditions under which a minimum distance decoder outputs the correct codeword, that is, when  $\arg \min_{\mathbf{u} \in \mathcal{C}} d(\mathbf{u}, \mathbf{v}) = \{\mathbf{u}\}$ . This is equivalent to saying that the decoder correctly outputs  $\mathbf{u}$  if and only if  $d(\mathcal{C} \setminus \{\mathbf{u}\}, \mathbf{v}) > d(\mathbf{u}, \mathbf{v})$ .

Let  $d' = d(\mathcal{C} \setminus \{\mathbf{u}\}, \mathbf{u})$ . Since  $\mathcal{C}$  has distance  $d$ , we have  $d' \geq d$ . Observe the following:

- Let  $1 \leq e \leq n$ . If  $e$  impulse noise errors occur, then

$$d(\mathbf{u}, \mathbf{v}) = 0 \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathbf{u}\}, \mathbf{v}) \geq d' - e.$$

- Let  $1 \leq e \leq n(q - 1)$ . If  $e$  insertion errors occur, then

$$d(\mathbf{u}, \mathbf{v}) = 0 \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathbf{u}\}, \mathbf{v}) \geq d' - e.$$

- Let  $1 \leq e \leq n$ . If  $e$  deletion errors occur, then

$$d(\mathbf{u}, \mathbf{v}) = e \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathbf{u}\}, \mathbf{v}) = d'.$$

For errors due to narrowband noise and signal fading, we define the function  $E_C : [q] \rightarrow [n]$  by

$$E_C(e) = \max_{\Gamma \in \binom{\Sigma}{e}} \max_{\mathbf{c} \in \mathcal{C}} \left\{ \sum_{\sigma \in \Gamma} w_\sigma(\mathbf{c}) \right\}.$$

The quantity  $E_C(e)$  measures the maximum number of coordinates over all codewords that can be affected by  $e$  narrowband noise and/or fading errors. The following is now immediate:

- Let  $1 \leq e \leq q$ . If  $e$  narrowband noise errors occur, then

$$d(\mathbf{u}, \mathbf{v}) = 0 \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathbf{u}\}, \mathbf{v}) \geq d' - E_C(e).$$

- Let  $1 \leq e \leq q$ . If  $e$  signal fading errors occur, then

$$d(\mathbf{u}, \mathbf{v}) \leq E_C(e) \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathbf{u}\}, \mathbf{v}) = d'.$$

<sup>1</sup>Identify  $\mathbf{c} \in \Sigma^n$  with  $(\{c_1\}, \{c_2\}, \dots, \{c_n\}) \in (2^\Sigma)^n$ , so that  $d(\mathbf{u}, \mathbf{c})$  gives the Hamming distance of  $\mathbf{u}$  and  $\mathbf{c}$ .

Hence, if we denote by  $e_{\text{NBD}}$ ,  $e_{\text{SFD}}$ ,  $e_{\text{IMP}}$ ,  $e_{\text{INS}}$ , and  $e_{\text{DEL}}$  the number of errors due to narrowband noise, signal fading, impulse noise, insertion, and deletion, respectively, we have

$$\begin{aligned} d(u, v) &\leq e_{\text{DEL}} + E_{\mathcal{C}}(e_{\text{SFD}}), \\ d(\mathcal{C} \setminus \{u\}, v) &\geq d' - e_{\text{IMP}} - e_{\text{INS}} - E_{\mathcal{C}}(e_{\text{NBD}}). \end{aligned}$$

Now,

$$\begin{aligned} d(u, v) - d(\mathcal{C} \setminus \{u\}, v) &\leq (e_{\text{DEL}} + E_{\mathcal{C}}(e_{\text{SFD}})) - (d' - e_{\text{IMP}} - e_{\text{INS}} - E_{\mathcal{C}}(e_{\text{NBD}})) \\ &\leq e_{\text{DEL}} + e_{\text{IMP}} + e_{\text{INS}} + E_{\mathcal{C}}(e_{\text{SFD}}) + E_{\mathcal{C}}(e_{\text{NBD}}) - d'. \end{aligned} \quad (1)$$

Under the condition

$$e_{\text{DEL}} + e_{\text{IMP}} + e_{\text{INS}} + E_{\mathcal{C}}(e_{\text{SFD}}) + E_{\mathcal{C}}(e_{\text{NBD}}) < d,$$

the inequality (1) reduces to  $d(u, v) < d(\mathcal{C} \setminus \{u\}, v)$ , which implies correct decoding.

On the other hand, if

$$e_{\text{DEL}} + e_{\text{IMP}} + e_{\text{INS}} + E_{\mathcal{C}}(e_{\text{SFD}}) + E_{\mathcal{C}}(e_{\text{NBD}}) \geq d,$$

say  $e_{\text{IMP}} = d$ , and  $u, w \in \mathcal{C}$  is such that  $d(u, w) = d$  (since  $\mathcal{C}$  has distance  $d$ ,  $u, w$  must exist), then  $d' = d(\mathcal{C} \setminus \{u\}, u) = d$ , and we have  $d(u, v) - d(\mathcal{C} \setminus \{u\}, v) \leq d - d' = 0$ . In this case, the correctness of the decoder output cannot be guaranteed. We therefore have the following theorem.

*Theorem 3.1:* Let  $\mathcal{C}$  be an  $(n, d)_q$ -code over alphabet  $\Sigma$ . Let  $e_{\text{DEL}}, e_{\text{IMP}}, e_{\text{INS}} \in [n]$  and  $e_{\text{NBD}}, e_{\text{SFD}} \in [q]$ . Then  $\mathcal{C}$  is able to correct  $e_{\text{NBD}}$  narrowband noise errors,  $e_{\text{SFD}}$  signal fading errors,  $e_{\text{IMP}}$  impulse noise errors,  $e_{\text{INS}}$  insertion errors, and  $e_{\text{DEL}}$  deletion errors if and only if

$$e_{\text{DEL}} + e_{\text{IMP}} + e_{\text{INS}} + E_{\mathcal{C}}(e_{\text{SFD}}) + E_{\mathcal{C}}(e_{\text{NBD}}) < d.$$

Therefore, the parameters  $n$ ,  $q$ ,  $d$ , and  $r$  (symbol weight) of a code are insufficient to characterize the total error-correcting capability of a code in a PLC system, since  $E_{\mathcal{C}}$  cannot be specified by  $n$ ,  $q$ ,  $d$ , and  $r$  alone. We now introduce an additional new parameter that together with  $n$ ,  $q$ , and  $d$ , more precisely captures the error-correcting capability of a code for PLC.

*Definition 3.1:* Let  $\mathcal{C}$  be a code of distance  $d$ . The *narrowband noise and signal fading error-correcting capability* of  $\mathcal{C}$  is

$$c(\mathcal{C}) = \min\{e : E_{\mathcal{C}}(e) \geq d\}.$$

From Theorem 3.1 we infer that a code  $\mathcal{C}$  can correct up to  $c(\mathcal{C}) - 1$  narrowband noise and signal fading errors. In general, for codes  $\mathcal{C}$  with bounded symbol weight  $r$ , we have  $\lceil d/r \rceil \leq c(\mathcal{C}) \leq \min\{d, q\}$ . However, the gap between the upper and lower bounds can be large. Furthermore, the lower bound can be attained, giving codes of low resilience against narrowband noise, as is shown in the following example.

*Example 3.2:* The code

$$\mathcal{C} = \left\{ \underbrace{(1, \dots, 1)}_{r \text{ times}}, 2, 3, 4 \dots q, \underbrace{(2, \dots, 2)}_{r \text{ times}}, 1, 3, 4 \dots q \right\}$$

is a  $(q + r - 1, r + 1, r)_q$ -symbol weight code with narrowband noise and signal fading error-correcting capability  $c(\mathcal{C}) = \lceil d/r \rceil = 2$ .

In the next section, we provide a tight upper bound for  $c(\mathcal{C})$  and demonstrate that equitable symbol weight codes attain this upper bound.

#### IV. $E_{\mathcal{C}}$ AND EQUITABLE SYMBOL WEIGHT CODES

If  $\mathcal{C}$  is a code of length  $n$  with bounded symbol weight  $r$ , then  $E_{\mathcal{C}}(1) = r$ , and  $E_{\mathcal{C}}(e) \geq \min\{n, r + e - 1\}$  for  $e > 1$ . If  $\mathcal{C}$  is restricted to more specific classes of codes,  $E_{\mathcal{C}}$  can be determined precisely. In the following,  $\mathcal{C}$  is a  $q$ -ary code of length  $n$  over  $\Sigma = [q]$ .

- 1) When  $q|n$ , we have  $E_{\mathcal{C}}(e) = ne/q$  for all  $e \in [q]$  if and only if  $\mathcal{C}$  is a frequency permutation array.
- 2) When  $n \leq q$ ,

$$E_{\mathcal{C}}(e) = \begin{cases} e, & \text{for all } e \in [n] \\ n, & \text{otherwise} \end{cases}$$

if and only if  $\mathcal{C}$  is an injection code. In particular, when  $q = n$ , this gives  $E_{\mathcal{C}}(e) = e$  for all  $e \in [q]$  if and only if  $\mathcal{C}$  is a permutation code.

- 3) Let  $(c_1, c_2, \dots, c_q) \in [n]^q$  with  $c_1 \geq c_2 \geq \dots \geq c_q$ . If for each  $u \in \mathcal{C}$ , we have  $w_j(u) = c_j$  for all  $j \in [q]$ , then  $E_{\mathcal{C}}(e) = \sum_{i=1}^e c_i$  for all  $e \in [q]$ . Such a code is a constant-composition code.
- 4) If  $\mathcal{C}$  is an equitable symbol weight code, then from Lemma 2.1,

$$E_{\mathcal{C}}(e) = \begin{cases} re, & \text{if } e \leq q - t \\ r(q - t) + (e - q + t)(r - 1), & \text{otherwise,} \end{cases}$$

where  $r = \lceil n/q \rceil$  and  $t = qr - n$ .

For  $c(\mathcal{C})$  to be large,  $E_{\mathcal{C}}$  must be slow growing. We seek codes  $\mathcal{C}$  for which  $E_{\mathcal{C}}$  is as slow growing as possible. Fix  $n, q$ , and let  $\mathcal{F}_{n,q}$  be the (finite) family of functions

$$\mathcal{F}_{n,q} = \{E_{\mathcal{C}} : \mathcal{C} \text{ is a } q\text{-ary code of length } n\}.$$

If  $f \in \mathcal{F}_{n,q}$ , then  $f$  is a monotone increasing function with  $f(q) = n$ . Define the order  $\preceq$  on  $\mathcal{F}_{n,q}$  so that  $f \preceq g$  if either  $f(e) = g(e)$  for all  $e \in [q]$ , or there exists<sup>2</sup>  $e' \in [q]$  such that  $f(e) = g(e)$  for all  $e \leq e' - 1$  and  $f(e') < g(e')$ .

*Proposition 4.1:* Let  $f_{n,q}^* : [q] \rightarrow [n]$  be defined by

$$f_{n,q}^*(e) = \begin{cases} re, & \text{if } 1 \leq e \leq q - t \\ r(q - t) + (e - q + t)(r - 1), & \text{otherwise,} \end{cases}$$

where  $r = \lceil n/q \rceil$  and  $t = qr - n$ . Then  $f_{n,q}^*$  is the unique least element in  $\mathcal{F}_{n,q}$  with respect to the total order  $\preceq$ .

*Proof:* Since  $\preceq$  is total, it suffices to establish that  $f_{n,q}^* \preceq f$  for all  $f \in \mathcal{F}_{n,q}$ , and that  $f_{n,q}^* \in \mathcal{F}_{n,q}$ .

Let  $f = E_{\mathcal{C}} \in \mathcal{F}_{n,q}$ , where  $\mathcal{C}$  is a  $q$ -ary code of length  $n$  over the alphabet  $[q]$ . Let  $u \in \mathcal{C}$ . By permuting symbols

<sup>2</sup>Note that when  $e' = 1$ , the statement is vacuously true.

if necessary, we may assume that  $w_1(\mathbf{u}) \geq w_2(\mathbf{u}) \geq \dots \geq w_q(\mathbf{u})$ . We show that for all  $e \in [q]$ ,

$$\sum_{i=1}^e w_i(\mathbf{u}) \geq f_{n,q}^*(e). \quad (2)$$

This would then imply  $E_C(e) \geq f_{n,q}^*(e)$  for all  $e \in [q]$ , and consequently  $f \succeq f_{n,q}^*$ .

Suppose on the contrary that  $\sum_{i=1}^e w_i(\mathbf{u}) < f_{n,q}^*(e)$  for some  $e \in [q]$ . If  $e \leq q - t$ , then we have  $\sum_{i=1}^e w_i(\mathbf{u}) < re$  and  $r - 1 \geq w_e(\mathbf{u}) \geq w_j(\mathbf{u})$  for  $j \geq e + 1$ . Hence,

$$n = \sum_{i=1}^q w_i(\mathbf{u}) < re + (q - e)(r - 1) = qr - q + e \leq qr - t = n,$$

a contradiction.

Similarly, when  $e > q - t$ , we have  $\sum_{i=1}^e w_i(\mathbf{u}) < r(q - t) + (e - q + t)(r - 1)$  and  $r - 1 \geq w_e(\mathbf{u}) \geq w_j(\mathbf{u})$  for  $j \geq e + 1$ . Hence,

$$\begin{aligned} n &= \sum_{i=1}^q w_i(\mathbf{u}) \\ &< r(q - t) + (e - q + t)(r - 1) + (q - e)(r - 1) \\ &= qr - t = n, \end{aligned}$$

also a contradiction.

The proposition then follows by noting that  $f_{n,q}^* \in \mathcal{F}_{n,q}$ , since  $E_C = f_{n,q}^*$  when  $\mathcal{C}$  is a  $q$ -ary equitable symbol weight code of length  $n$ . ■

*Corollary 4.1:*  $\mathcal{C}$  is a  $q$ -ary equitable symbol weight code of length  $n$  if and only if  $E_C = f_{n,q}^*$ .

*Proof:* If  $\mathcal{C}$  is a  $q$ -ary equitable symbol weight code of length  $n$ , we have already determined that  $E_C = f_{n,q}^*$ . Hence, it only remains to show that  $E_C = f_{n,q}^*$  implies  $\mathcal{C}$  is a  $q$ -ary equitable symbol weight code of length  $n$ . Let  $\mathbf{u} \in \mathcal{C}$  and we follow the notations in the proof of Proposition 4.1. Equality holds in (2) if and only if  $w_i(\mathbf{u}) = r$  for  $1 \leq i \leq q - t$  and  $w_i(\mathbf{u}) = r - 1$ , otherwise. That is,  $\mathbf{u}$  has equitable symbol weight. Hence,  $\mathcal{C}$  is an equitable symbol weight code. ■

It follows that an equitable symbol weight code  $\mathcal{C}$  gives  $E_C$  of the slowest growth rate. This is the desired condition for correcting as many narrowband noise and signal fading errors as possible.

We end this section with a tight upper bound on  $c(\mathcal{C})$ .

*Corollary 4.2:* Let  $\mathcal{C}$  be an  $(n, d)_q$ -code. Then

$$c(\mathcal{C}) \leq \min \{e : f_{n,q}^*(e) \geq d\},$$

and equality is achieved when  $\mathcal{C}$  has equitable symbol weight.

*Proof:* Let  $c' = \min \{e : f_{n,q}^*(e) \geq d\}$ . Observe that

$$E_C(c') \geq f_{n,q}^*(c') \geq d.$$

Hence, by minimality of  $c(\mathcal{C})$ , we have  $c(\mathcal{C}) \leq c'$ . The second part of the statement follows from Corollary 4.1. ■

The results in this section establish that an equitable symbol weight code has the best narrowband noise and signal fading error-correcting capability, among codes of the same distance and symbol weight.

## V. SIMULATION RESULTS

In this section, we show via simulations the difference in performance between equitable symbol weight codes and (non-equitable) minimum symbol weight codes in the presence of narrowband noise and signal fading. More specifically, we consider the following codes<sup>3</sup>.

- $\mathcal{C}_{\text{SW}}$  : (25, 24, 2)<sub>17</sub>-symbol weight code of size 51,
- $\mathcal{C}_{\text{ESW}}$  : (25, 24)<sub>17</sub>-equitable symbol weight code of size 51,
- $\mathcal{D}_{\text{SW}}$  : (17, 17, 2)<sub>16</sub>-symbol weight code of size 16, and
- $\mathcal{D}_{\text{ESW}}$  : (17, 17)<sub>16</sub>-equitable symbol weight code of size 16.

We show that  $\mathcal{C}_{\text{ESW}}$  and  $\mathcal{D}_{\text{ESW}}$  achieve lower symbol error rates as compared to  $\mathcal{C}_{\text{SW}}$  and  $\mathcal{D}_{\text{SW}}$ , respectively, in a PLC channel with varying degrees of narrowband noise and signal fading levels.

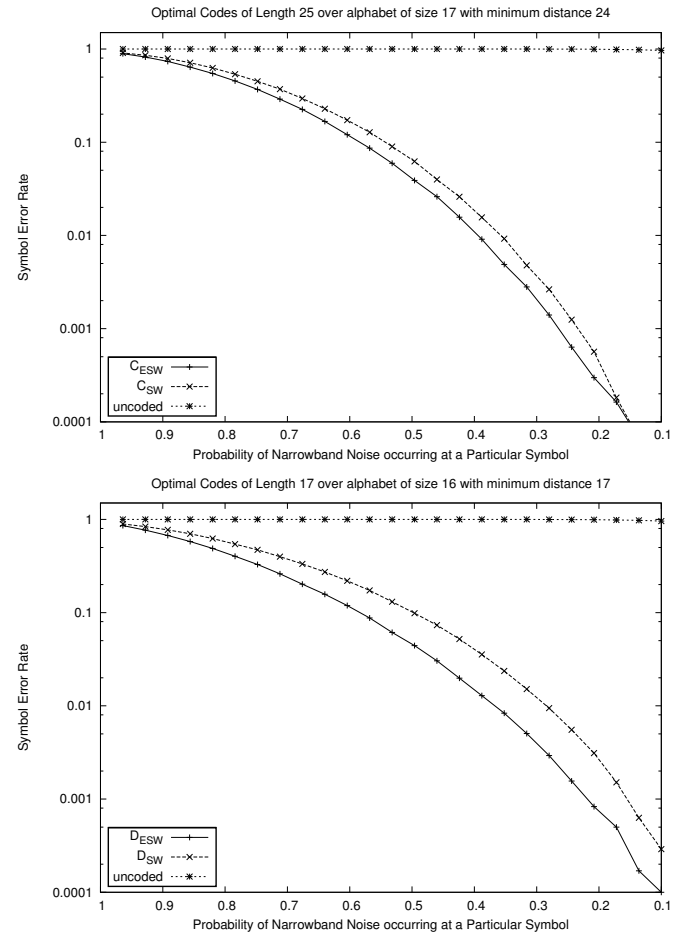


Fig. 1. Comparison of various optimal codes in a simulated PLC channel.

The setup is as follows. Let  $p$  be a real number between 0 and 1. We simulate a PLC channel with the following characteristics:

- 1) for each  $\sigma \in \Sigma$ , a narrowband noise or signal fading error occurs at symbol  $\sigma$  with probability  $p$ ,

<sup>3</sup>Due to space constraints, we only made comparisons between codes of different lengths. More simulation results are given in the full paper.

- 2) for each  $i \in [n]$ , an impulse noise error occurs at coordinate  $i$  with probability 0.1, and
- 3) for each  $(\sigma, i) \in \Sigma \times [n]$ , an insertion/deletion error occurs at symbol  $\sigma$ , coordinate  $i$  with probability 0.1.

These errors occur independently.

We choose  $10^5$  random codewords (with repetition) from each code to transmit through the simulated PLC channel. At the receiver, we decode detector output  $v$  to codeword  $u'$ . The number of symbols in error is then  $d(u', u)$  and the *symbol error rate* is the ratio of the total number of symbols in error to the total number of symbols transmitted. For uncoded communication,  $10^5$  random codewords were chosen from  $\Sigma^n$  and errors introduced. Here, no decoding is performed and the number of symbols in error is given by  $|\{i : v_i \neq \{u_i\}\}|$ .

The results of the simulation are displayed in Fig. 1. Equitable symbol weight codes  $C_{\text{ESW}}$  and  $D_{\text{ESW}}$  achieve lower symbol error rates compared to the minimum symbol weight codes  $C_{\text{SW}}$  and  $D_{\text{SW}}$ , respectively.

## VI. CONCLUSION

We have introduced a new code parameter that captures the error-correcting capability of a code with respect to narrowband noise and signal fading. Equitable symbol weight codes are shown to be optimal with respect to this parameter when code length, alphabet size and distance are fixed. We also provide simulations that show equitable symbol weight codes to achieve lower symbol error rates as compared to their non-equitable counterparts. These results motivate the study of equitable symbol weight codes as a viable option to handle narrowband noise and signal fading in a PLC channel.

We have also constructed infinite classes of optimal equitable symbol weight codes. This will appear in the full paper.

## ACKNOWLEDGMENT

Research of the authors is supported in part by the Singapore National Research Foundation under Research Grant NRF-CRP2-2007-03. C. Wang is also supported in part by NSFC under Grant 10801064. The authors also thank the anonymous reviewers whose valuable comments improved the presentation of the paper.

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