Extending Brickell-Davenport Theorem to Non-Perfect Secret Sharing Schemes

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a joint work with
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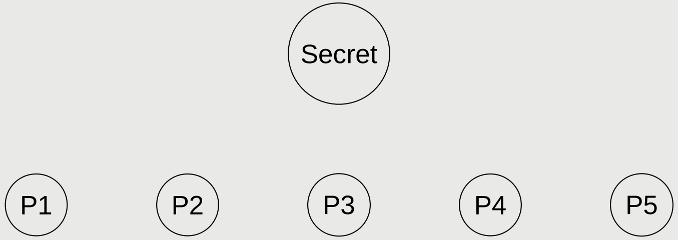
Program

Secret Sharing Scheme

A method to protect a secret



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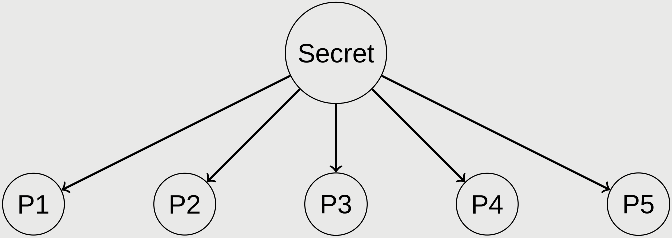

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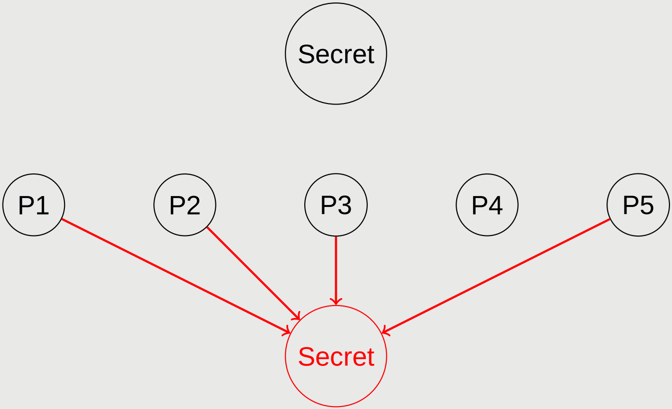

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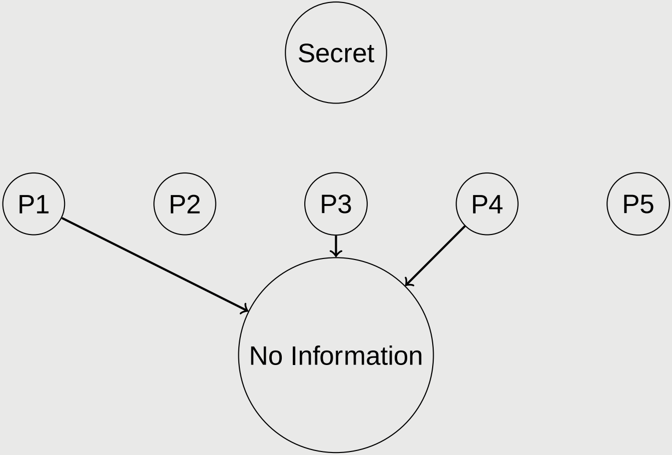

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- Electronic elections
- Electronic biddings
- Distributed signatures
- Network Coding
- Database access
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Multiparty computation protocols

If is desirable to have schemes with homomorphic properties whose shares are small in comparison with the secret

Definition of a Secret Sharing Scheme

A secret sharing scheme on the set $P = \{p_1, \ldots, p_n\}$ of participants is a mapping

$$\begin{aligned} \Pi\colon E &\to E_0 \times E_1 \times \cdots \times E_n \\ x &\mapsto (\pi_0(x), \pi_1(x), \ldots, \pi_n(x)) \end{aligned}$$

together with a probability distribution on E where

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Σ is perfect if $\overline{\mathcal{A}} = \mathcal{B}$ ($we define \overline{\mathcal{A}} = \mathcal{P}(P) \setminus \mathcal{A}$).

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$$h(A) = \frac{H(E_A)}{H(E_{p_0})}.$$

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If $\mathcal{S}$ is defined from Σ, then $\Gamma_{p_0}(\mathcal{S})$ is the access structure of Σ.

Schemes and Polymatroids (II)

For every scheme Σ, the value

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Lemma

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In particular, $\sigma(\Sigma) \geq 1$.

The best possible situation for a perfect scheme is that $h(\{i\}) = 1$ for every $i \in P$. In this case, we say that Σ is ideal.
Its access structure is called ideal as well.

Ideal Schemes and Matroids

A matroid $M = (Q, h)$ is a polymatroid in which

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Theorem

The ports of representable matroids admit ideal secret sharing schemes.

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Since the uniform matroid is representable, their matroid ports admit ideal schemes.

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There are some previous works in this direction:

- Kurosawa et al'94
- Pailier'98

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Example:
Some protocols in multiparty computation need:

- efficient schemes
- sets of size less than t are forbidden
- big sets are authorized
- a solution: ramp schemes and other non-perfect schemes (Chen, Cramer, de Haan, Cascudo'08)

Generalized Matroid Ports

Definition

Let $M = (P \cup R, h)$ be a matroid. The generalized port of the matroid M at the set R is the access structure $\Gamma_R(M) = (\mathcal{A}, \mathcal{B})$, where

- $A \in \mathcal{A}$ iff $h(A \cup R) = h(A) + h(R)$
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The access structure of the ramp scheme is a generalized matroid port:

- Consider the uniform matroid M of dimension t on $P \cup R$, with $|R| = k$
- The access structure coincides with $\Gamma_R(M)$

Bounds on the Complexity(I)

Lemma

Let Σ be an secret sharing scheme with access structure $\Gamma = (\mathcal{A}, \mathcal{B})$. Let

$$k = \min\{|B \setminus A| \,:\, B \in \mathcal{B}, A \in \mathcal{A}, A \subseteq B\}$$

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We need additional conditions

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Define $h(A|B) = h(A \cup B) - h(B)$ for every $A, B \subseteq Q$

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Definition

A scheme is ideal if $\alpha(\mathcal{S}) = \beta(\mathcal{S}) = \max_{x \in P} h(\{x\})$.

Extension of Brickell-Davenport Theorem

Theorem (Brickell-Davenport)

Let $\mathcal{S} = (Q, h)$ be a polymatroid with $h(\{p_0\}) = 1$ such that $\Gamma_{p_0}(\mathcal{S})$ is perfect and

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It improves the connection found by Kurosawa et al'94. Our result is more general

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Theorem

The generalized ports of representable matroids are ideal access structures

Open Problems and Future Work

We already have used this result to characterize some families of ideal non-perfect access structures.

Some open problems and interesting topics are

- construction of ideal non-perfect schemes with homomorphic properties
- construction of efficient schemes for interesting access structures
- characterization of ideal non-perfect access structures
- bounds on the complexity of generalized matroid ports

Thank you