Journal of Combinatorial Theory, Series A 120 (2013) 1440–1449

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Covering and packing for pairs

Yeow Meng Chee [a], Charles J. Colbourn [b], Alan C.H. Ling [c], Richard M. Wilson [d]

[a] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Singapore
[b] Computing, Informatics, and Decision Systems Engineering, Arizona State University, Tempe, AZ 85287-8809, USA
[c] Department of Computer Science, University of Vermont, Burlington, VT 05405, USA
[d] Department of Mathematics, 253-37, California Institute of Technology, Pasadena, CA 91125, USA

ARTICLE INFO

Article history:
Received 24 September 2011
Available online xxxx

Keywords:
Balanced incomplete block design
Pair packing
Pair covering
Group divisible design
Pairwise balanced design

ABSTRACT

When a v-set can be equipped with a set of k-subsets so that every 2-subset of the v-set appears in exactly (or at most, or at least) one of the chosen k-subsets, the result is a balanced incomplete block design (or packing, or covering, respectively). For each k, balanced incomplete block designs are known to exist for all sufficiently large values of v that meet certain divisibility conditions. When these conditions are not met, one can ask for the packing with the most blocks and/or the covering with the fewest blocks. Elementary necessary conditions furnish an upper bound on the number of blocks in a packing and a lower bound on the number of blocks in a covering. In this paper it is shown that for all sufficiently large values of v, a packing and a covering on v elements exist whose numbers of blocks differ from the basic bounds by no more than an additive constant depending only on k.

1. Introduction

Let v, k, and t be integers with $v > k > t \geqslant 2$. Let λ be a positive integer. A (t, λ)-*packing* of *order* v and *blocksize* k is a set V of v elements, and a collection $\mathcal{B}$ of k-element subsets (*blocks*) of V, so that every t-subset of V appears in *at most* λ blocks. A (t, λ)-*covering* of *order* v and *blocksize* k is a set V of v elements, and a collection $\mathcal{B}$ of k-element subsets (*blocks*) of V, so that every t-subset

E-mail addresses: ymchee@ntu.edu.sg (Y.M. Chee), charles.colbourn@asu.edu (C.J. Colbourn), aling@cems.uvm.edu (A.C.H. Ling), rmw@caltech.edu (R.M. Wilson).

0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2013.04.005

of V appears in *at least* λ blocks. When $\lambda = 1$, the simpler notation of t-packing or t-covering is used. When $(V, \mathcal{B})$ is both a (t, λ)-packing and a (t, λ)-covering with blocksize k, it is a t-(v, k, λ) *design*.

A t-(v, k, λ) design, if one exists, has $\lambda\binom{v}{t}/\binom{k}{t}$ blocks. When the required number of blocks is not integral, no such design can exist. Selecting all blocks containing a particular element $x \in V$ and deleting x from each forms the *derived* $(t-1)$-$(v-1, k-1, \lambda)$ design (with respect to x). For a design to exist, evidently the derived design must exist; hence for a t-(v, k, λ) design to exist, $\lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$ must be integral for every $0 \leqslant i \leqslant t$. When these conditions are not all met, one can ask instead for the largest (t, λ)-packing, or for the smallest (t, λ)-covering, of order v and blocksize k. The *Johnson bound* [13] states that such a packing can have no more than

$$\left\lfloor \frac{n}{k} \left\lfloor \cdots \left\lfloor \frac{n-t+2}{k-t+2} \left\lfloor \frac{\lambda(n-t+1)}{k-t+1} \right\rfloor \right\rfloor \cdots \right\rfloor \right\rfloor$$

blocks, while the *Schönheim bound* [21] states that such a covering can have no fewer than

$$\left\lceil \frac{n}{k} \left\lceil \cdots \left\lceil \frac{n-t+2}{k-t+2} \left\lceil \frac{\lambda(n-t+1)}{k-t+1} \right\rceil \right\rceil \cdots \right\rceil \right\rceil$$

blocks. Our main result is that when $t = 2$ and $\lambda = 1$, there exist packings and coverings whose sizes are within a constant of these bounds. Determining when these bounds are met exactly is a challenging question.

In 1963, Erdős and Hanani [9] conjectured that, for fixed k and t, with all blocks of size k, a t-packing on n elements with $(\binom{n}{t}/\binom{k}{t})(1 - o(1))$ blocks and a t-covering on n elements with $(\binom{n}{t}/\binom{k}{t})(1 + o(1))$ blocks both exist. This was proved by Rödl [20], and has spawned a large literature (for example, [10,11,14,15,23]). However, even when $t = 2$, all of these general constructions deviate from the Johnson and Schönheim bounds by an amount that grows as a function of the number of elements. Wilson [25] established that the necessary divisibility conditions for a 2-(v, k, λ) design to exist are asymptotically sufficient (i.e., for fixed k and λ, and sufficiently large v). This provides a different means to establish the Erdős–Hanani conjecture for $t = 2$, but also does not immediately imply that one can find packings or coverings whose sizes are within a constant of the optimal sizes. Wilson [24] earlier considered this more challenging problem for packings, but the solution for the analogous problem for coverings has remained elusive.

We focus on the case when $t = 2$ and $\lambda = 1$ here. Caro and Yuster state stronger results for covering [3] and packing [2] than we prove here. Their approach relies in an essential manner on a strong statement by Gustavsson [12]:

Proposition 1.1. *Let H be a graph with v vertices and h edges, having degree sequence $(d_1, \ldots, d_v)$. Then there exist a constant N_H and a constant $\epsilon_H > 0$, both depending only on H, such that for all $n > N_H$, if G is a graph on n vertices, m edges, and degree sequence $(\delta_1, \ldots, \delta_n)$ so that $\min(\delta_1, \ldots, \delta_n) \geqslant n(1 - \epsilon_H)$, $\gcd(d_1, \ldots, d_v) \mid \gcd(\delta_1, \ldots, \delta_n)$, and $h \mid m$, then G has an edge partition (decomposition) into graphs isomorphic to H.*

We have not been able to verify the proof of Proposition 1.1. Indeed, while the result has been used a number of times in the literature, no satisfactory proof of it appears there. While we expect that the statement is true, we do not think that the proof in [12] is sufficient at this time to employ the statement as a foundation for further results. Therefore we adopt a strategy that is completely independent of Proposition 1.1, and independent of the results built on it.

In the remainder of the paper, we first recall relevant known results. Then in Section 3, we determine the possible structure of optimal packings and coverings, in order to determine what can remain uncovered in a packing, and what must be covered more than once in a covering. This is done in general for packings and coverings with a single hole, in order to limit any deviation from the desired bound to the manner in which a (fixed size) hole is filled. In Section 4, the most important part of the proof is established, namely that in each congruence class, one finite example can be produced. Finally in Section 5, these single examples are shown to form the required ingredients to establish asymptotic existence.

2. Background

To proceed more formally, we require a number of definitions and preliminary results from combinatorial design theory; related background material can be found in [1,22]. A *balanced incomplete block design* (*BIBD*) is a 2-(v,k,λ) design. Balanced incomplete block designs have been extensively studied because of their central role in numerous applications in experimental design, coding and information theory, communications, and connections with fundamental topics in algebra, finite geometry, number theory, and combinatorics (see [5,7] for examples). The general divisibility conditions (stated for general t earlier) require that $\lambda\binom{v}{2} \equiv 0 \pmod{\binom{k}{2}}$ and $\lambda(v-1) \equiv 0 \pmod{k-1}$.

A *group divisible design* $(V,\mathcal{G},\mathcal{B})$ is a finite set V of *elements* or *points*; a partition $\mathcal{G}=\{G_1,\dots,G_s\}$ of V (*groups*); and a set $\mathcal{B}$ of subsets of V (*blocks*), with the property that every 2-subset of V lying within a group appears in no block, while every 2-subset of V with elements from different groups appears in exactly λ blocks. When K is a set of positive integers for which $|B|\in K$ whenever $B\in\mathcal{B}$, the design is a (K,λ)*-GDD*. When $\lambda=1$, we write simply K*-GDD*. Its *order* is $|V|$, its *index* is λ, and its *type* is $\sigma_1^{u_1}\cdots\sigma_\ell^{u_\ell}$ when the multiset of group sizes $\{|G_i|:\ 1\leqslant i\leqslant s\}$ is the same as the multiset formed by including u_j copies of σ_j when $\sigma_j\neq 0$, for all $1\leqslant j\leqslant \ell$. We write (k,λ)-GDD (or k-GDD when $\lambda=1$) when $K=\{k\}$. A *transversal design* $\mathrm{TD}_\lambda(k,n)$ is a (k,λ)-GDD of type n^k. We write $\mathrm{TD}(k,n)$ when $\lambda=1$. A transversal design is *idempotent* if its element set is $\{1,\dots,k\}\times\{1,\dots,n\}$, and its block set contains $\{\{(i,j):\ 1\leqslant i\leqslant k\}:\ 1\leqslant j\leqslant n\}$. A *pairwise balanced design* with blocksizes K and order v ((K,λ)*-PBD* of order v) is a (K,λ)-GDD of type 1^v; we write K-PBD when $\lambda=1$. Then a balanced incomplete block design ((k,λ)*-BIBD*) is a (k,λ)-PBD; we write k-BIBD when $\lambda=1$.

An *incomplete pairwise balanced design* of *order* v with *holesize* h, *blocksizes* K, and *index* λ is a triple $(V,H,\mathcal{B})$ for which $|V|=v$, $|H|=h$, $H\subseteq V$, $\mathcal{B}$ contains a set of subsets of V for which $|B|\in K$ whenever $B\in\mathcal{B}$, and for every pair of distinct elements $x,y\in V$, the number of blocks in $\{\{x,y\}\subset B\in\mathcal{B}\}$ is 0 if $\{x,y\}\subseteq H$ and λ otherwise. The notation (K,λ)*-IPBD*(v,h) is used; we may omit λ when it is 1, and write k instead of K when $K=\{k\}$.

Let K be a set of positive integers, each at least 2. Then define $\alpha(K)=\gcd\{k-1:\ k\in K\}$ and $\beta(K)=\gcd\{\binom{k}{2}:\ k\in K\}$.

Wilson establishes a crucial asymptotic existence result:

Theorem 2.1. *(See [25].) Let K be a set of integers, each at least 2. Let λ be a positive integer. For all sufficiently large n satisfying $\lambda(n-1)\equiv 0 \pmod{\alpha(K)}$ and $\lambda\binom{n}{2}\equiv 0 \pmod{\beta(K)}$, there exists a (K,λ)-PBD of order n. In particular for $K=\{k\}$, when $\lambda(n-1)\equiv 0 \pmod{k-1}$, $\lambda\binom{n}{2}\equiv 0 \pmod{\binom{k}{2}}$, and n is sufficiently large, there exists a (k,λ)-BIBD of order n.*

Colbourn and Rödl prove a variant that we use:

Theorem 2.2. *(See [6].) Let $\varepsilon>0$. Let $K=\{k_1,\dots,k_m\}$ be a set of block sizes. Let $\{p_1,\dots,p_m\}$ be non-negative numbers with $\sum_{i=1}^m p_i=1$. For all sufficiently large v satisfying $v-1\equiv 0 \pmod{\alpha(K)}$ and $\binom{v}{2}\equiv 0 \pmod{\beta(K)}$, there is a K-PBD of order v in which, for each $1\leqslant i\leqslant m$, the fraction of pairs appearing in blocks having size k_i is in the range $[p_i-\varepsilon,p_i+\varepsilon]$.*

A stronger version of Theorem 2.2 is given in [26], and a variant for resolvable designs appears in [8].

Perhaps the most powerful generalization of Theorem 2.1 is due to Lamken and Wilson [16]. We introduce this next. Let $K_n^{(r,\lambda)}$ be a complete digraph on n vertices with exactly λ edges of color i joining any vertex x to any vertex y for every color i in a set of r colors. A family $\mathcal{F}$ of subgraphs of $K_n^{(r,\lambda)}$ is a *decomposition* of $K_n^{(r,\lambda)}$ if every edge $e\in E(K_n^{(r,\lambda)})$ belongs to exactly one member in $\mathcal{F}$. Given a family Φ of edge-r-colored digraphs, a Φ-decomposition of $K_n^{(r,\lambda)}$ is a decomposition $\mathcal{F}$ such that every graph $F\in\mathcal{F}$ is isomorphic to some graph $G\in\Phi$. For a vertex x of an edge-r-colored digraph G, the degree-vector of x is the $2r$-vector $d(x)=(\mathrm{in}_1(x),\mathrm{out}_1(x),\mathrm{in}_2(x),\mathrm{out}_2(x),\dots,\mathrm{in}_r(x),\mathrm{out}_r(x))$, where $\mathrm{in}_j(x)$ and $\mathrm{out}_j(x)$ denote the indegree and outdegree of vertex x in the spanning subgraph of G by edges of color j, respectively, for $1 \leqslant j \leqslant r$. We denote by $\alpha(G)$ the greatest common divisor of the integers t such that the $2r$-vector $(t, t, \dots, t)$ is an integral linear combination of the vectors $d(x)$ as x ranges over the vertex set $V(G)$ of G. Equivalently, $\alpha(G)$ is the smallest positive integer t_0 such that $(t_0, t_0, \dots, t_0)$ is an integral linear combination of the vectors $\{d(x)\}$. Let Φ be a family of simple edge-r-colored digraphs and let $\alpha(\Phi)$ denote the greatest common divisor of the integers t such that the $2r$-vector $(t, t, \dots, t)$ is an integral linear combination of the vectors $\{d(x)\}$ as x ranges over all vertices of all graphs in Φ. For each graph $G \in \Phi$, let $\mu(G) = (m_1, m_2, \dots, m_r)$, where m_i is the number of edges of color i in G. We denote by $\beta(\Phi)$ the greatest common divisor of the integers m such that $(m, m, \dots, m)$ is an integral linear combination of the vectors $\{\mu(G)\colon\ G \in \Phi\}$. Equivalently, $\beta(\Phi)$ is the smallest positive integer m_0 such that $(m_0, m_0, \dots, m_0)$ is an integral linear combination of the vectors $\{\mu(G)\}$. A graph $G_0 \in \Phi$ is *useless* when it cannot occur in any Φ-decomposition of $K_n^{(r,\lambda)}$. Φ is *admissible* when no member of Φ is useless.

Theorem 2.3. *(See [16].) Let Φ be an admissible family of simple edge-r-colored digraphs. For all sufficiently large n satisfying $\lambda(n-1) \equiv 0 \pmod{\alpha(\Phi)}$ and $\lambda n(n-1) \equiv 0 \pmod{\beta(\Phi)}$, a Φ-decomposition of $K_n^{(r,\lambda)}$ exists.*

Theorem 2.3 has numerous consequences for the existence of various classes of combinatorial designs. Building on Theorem 2.3, Liu establishes the following:

Theorem 2.4. *(See [17].) Let K be a set of integers, each at least 2. Let m and λ be positive integers. For all sufficiently large n satisfying $\lambda m(n-1) \equiv 0 \pmod{\alpha(K)}$ and $\lambda m^2\binom{n}{2} \equiv 0 \pmod{\beta(K)}$, there exists a (K, λ)-GDD of order m^n.*

Mohácsy and Ray-Chaudhuri prove a result for a fixed number of groups when the index is 1.

Theorem 2.5. *(See [18,19].) Let k and u be integers with $u \geqslant k \geqslant 2$. For all sufficiently large m satisfying $m(u-1) \equiv 0 \pmod{k-1}$ and $m^2u(u-1) \equiv 0 \pmod{k(k-1)}$, there exists a k-GDD of type m^u.*

This subsumes a classical result of Chowla, Erdős, and Straus:

Theorem 2.6. *(See [4].) Let $k \geqslant 2$ be an integer. For all sufficiently large m, there exists a TD(k, m).*

3. Packings, coverings, and the optima

We use known asymptotic existence results to treat asymptotic existence of packings and coverings in the cases that a k-BIBD does not exist. We require further definitions, to extend packings and coverings to have a ‘hole’.

A *packing with blocksize k with a hole* $(V, H, \mathcal{B})$ is a set V of elements, a subset (*hole*) $H \subset V$, and a set $\mathcal{B}$ of k-subsets of V, so that for every $\{x, y\} \subset V$, $\{x, y\} \not\subset H$, there is *at most* one $B \in \mathcal{B}$ with $\{x, y\} \subset B$; when $\{x, y\} \subset H$, there is no block $B \in \mathcal{B}$ with $\{x, y\} \subset B$. The *leave* Γ of $(V, \mathcal{B})$ is a graph with vertex set V; pair $\{x, y\}$ appears as an edge if and only if $\{x, y\} \not\subseteq H$ and is not a subset of any block of $\mathcal{B}$.

A *covering with blocksize k with a hole* $(V, H, \mathcal{B})$ is a set V of elements, a subset $H \subset V$, and a set $\mathcal{B}$ of k-subsets of V, so that for every $\{x, y\} \subset V$, $\{x, y\} \not\subset H$, there is *at least* one $B \in \mathcal{B}$ with $\{x, y\} \subset B$. The *excess* Γ of $(V, \mathcal{B})$ is a multigraph with vertex set V; the number of times pair $\{x, y\}$ appears as an edge is exactly λ_{xy} when $\{x, y\} \subset H$, and $\lambda_{xy} - 1$ otherwise, where λ_{xy} is the number of blocks of $\mathcal{B}$ that contain $\{x, y\}$.

A *packing with blocksize k* $(V, \mathcal{B})$ is a packing with blocksize k with a hole $(V, \emptyset, \mathcal{B})$, and a *covering with blocksize k* $(V, \mathcal{B})$ is a covering with blocksize k with a hole $(V, \emptyset, \mathcal{B})$. A *maximum packing with blocksize k* is a packing with blocksize k $(V, \mathcal{B})$ with the most blocks among all packings with blocksize k on $|V|$ elements; equivalently, its leave has the fewest edges. A *minimum covering with blocksize*

k is a covering with blocksize k $(V, \mathcal{B})$ with fewest blocks among all coverings with blocksize k on $|V|$ elements; equivalently, its excess has the fewest edges.

Suppose that $(V, H, \mathcal{B})$ is a packing with blocksize k with a hole, with $v = |V|$, $h = |H|$, and $n = |V \setminus H|$. Let x be a vertex in $V \setminus H$. The number of pairs on V that contain x is congruent to $v-1$ modulo $k-1$. The number containing x that appear in blocks of $\mathcal{B}$ is congruent to 0 modulo $k-1$. Hence x has degree congruent to $v-1$ modulo $k-1$ in the leave. When the hole is nonempty, elements in the hole have degrees congruent to n modulo $k-1$ in the leave. By the same token, in the excess of a covering with blocksize k with a hole, x has degree congruent to $-(v-1)$ modulo $k-1$; elements in the hole have degrees congruent to $-n$ modulo $k-1$.

We employ specific types of packings and coverings with holes in which the leave or excess has all vertices in the hole of degree 0. For an integer $n \equiv 0 \pmod{k(k-1)}$ and an integer $h \geqslant 1$, let $\delta \equiv h-1 \pmod{k-1}$ and $\Delta \equiv -(h-1) \pmod{k-1}$ with $0 \leqslant \delta, \Delta < k-1$. Then an *optimum packing with blocksize k with a hole*, k-OP$(n+h, h)$, is a packing with blocksize k on $n+h$ elements whose leave has degree δ on each vertex not in the hole, and 0 on each vertex in the hole; and an *optimum covering with blocksize k with a hole*, k-OC$(n+h, h)$, is a covering with blocksize k on $n+h$ elements whose excess has degree Δ on each vertex not in the hole, and 0 on each vertex in the hole. When $h \equiv 1 \pmod{k-1}$, $\delta = \Delta = 0$. In this case, a k-OP(v, h) and a k-OC(v, h) are the same, and are equivalent to a k-IPBD(v, h).

In any packing with blocksize k on $v = n+h$ elements with $n \equiv 0 \pmod{k(k-1)}$, no vertex can have degree smaller than δ in the leave; and in any covering with blocksize k on $v = n+h$ elements with $n \equiv 0 \pmod{k(k-1)}$, no vertex can have degree smaller than Δ in the excess. Indeed, choosing ℓ and Λ so that $\ell \leqslant v-1 \leqslant \Lambda$; $\ell, \Lambda \equiv v-1 \pmod{k-1}$; $\Lambda - \ell < k-1$; and $\ell = \Lambda$ when $v \equiv 1 \pmod{k-1}$, every packing with blocksize k on v elements contains at most $\ell_{v,k} = \frac{\ell v}{k(k-1)}$ blocks, while every covering with blocksize k on v elements contains at least $L_{v,k} = \frac{\Lambda v}{k(k-1)}$ blocks. Then $\ell_{v,k}$ is at least the Johnson bound, and $\Lambda_{v,k}$ is at most the Schönheim bound.

The purpose of this paper is to prove the following two results.

Theorem 3.1. *There is a constant p_k such that for all $v \geqslant k$, the number of blocks in a maximum packing with blocksize k on v elements is at least $\ell_{v,k} - p_k$ and at most $\ell_{v,k}$.*

Theorem 3.2. *There is a constant a_k such that for all $v \geqslant k$, the number of blocks in a minimum covering with blocksize k on v elements is at least $L_{v,k}$ and at most $L_{v,k} + a_k$.*

We establish these results in a number of steps. Treating an arbitrary but fixed value of k, in Section 4, we show that for every c satisfying $0 \leqslant c < k(k-1)$, there exist positive integers $n_c \equiv 0 \pmod{k(k-1)}$ and $h_c \equiv c \pmod{k(k-1)}$ so that a k-OP$(n_c + h_c, h_c)$ exists; we also show that for every c satisfying $0 \leqslant c < k(k-1)$, there exist positive integers $m_c \equiv 0 \pmod{k(k-1)}$ and $\ell_c \equiv c \pmod{k(k-1)}$ so that a k-OC$(m_c + \ell_c, \ell_c)$ exists. This provides a single example for optimal packings and coverings with a hole in every congruence class modulo $k(k-1)$. In Section 5, we use these results to establish that there exist integers κ_k and u_k, depending only on k, so that whenever $v \geqslant \kappa_k$, there exists an $h \leqslant u_k$ for which a k-OP(v, h) exists, and there also exists an $\ell \leqslant u_k$ for which a k-OC(v, ℓ) exists. From this, because u_k is fixed and independent of v, we establish Theorems 3.1 and 3.2 by filling the holes. The crucial step, particularly for coverings, is producing one example in each congruence class. We treat this next.

4. One example in each congruence class

In the case when $h \equiv 1 \pmod{k-1}$, a k-OP(v, h) and a k-OC(v, h) coincide with a k-IPBD(v, h), so we treat this situation first; subsequently the packing and covering cases differ.

4.1. Packing and covering: $v \equiv 1 \pmod{k-1}$

An *incomplete transversal design* ITD$(k, n+\phi; \phi)$ is a set V of $k(n+\phi)$ elements, of which $k\phi$ form a *hole* H. The elements are partitioned into k *groups* $G_1, \ldots, G_k$ so that $|G_i \cap H| = \phi$ for $1 \leqslant i \leqslant k$.

This set is equipped with a set of k-subsets (*blocks*) with the property that every pair of elements that appears in a group or appears in the hole H appears in no block, and every other pair appears in exactly one block.

Lemma 4.1. *Let $k \geqslant 2$ be an integer. Let $0 \leqslant \phi \leqslant k$. For all sufficiently large n, an ITD$(k, n+\phi; \phi)$ exists.*

Proof. Using Theorem 2.6, choose ω so that a TD$(k+1, \omega)$, a TD$(k+1, \omega+1)$, a TD$(k+1, \omega+2)$, and a TD$(k+1, \omega+3)$ all exist. Delete one group in each to form an *idempotent* TD(k, v) for each $v \in \{\omega, \omega+1, \omega+2, \omega+3\}$. For n sufficiently large, there is an $\{\omega+1, \omega+2, \omega+3\}$-PBD of order $n+\omega+1$ containing a block of size $\omega+1$ by Theorem 2.2. (Because $\alpha(\{\omega+1, \omega+2, \omega+3\}) = 1$ and $\beta(\{\omega+1, \omega+2, \omega+3\}) = 1$, this follows by choosing $0 < \varepsilon < \frac{1}{4}$ and choosing the fraction of pairs in blocks of size $\omega+1$ to be 2ε.) Delete all but ϕ elements from a block of size $\omega+1$, and remove the block of size ϕ making a hole, to form an $\{\omega, \omega+1, \omega+2, \omega+3\}$-IPBD$(n+\phi, \phi)$. Give every element weight k, and use the idempotent TDs to inflate all blocks. The $k\phi$ elements arising from the ϕ elements of hole in the IPBD form the hole of the ITD. $\square$

Lemma 4.2. *Let h be an integer for which $h \equiv 1 \pmod{k-1}$ and $k \leqslant h \leqslant k(k-1)+1$. Then there exist infinitely many integers γ for which a k-IPBD$(\gamma k(k-1)+h, h)$ exists.*

Proof. Let $\phi = \frac{h-k}{k-1}$. Choose γ so that a k-BIBD of order $\gamma(k-1)+k$ and an ITD$(k, \gamma(k-1)+\phi; \phi)$ both exist. (Use Lemma 4.1 for the existence of the ITD.) Start with the ITD on the elements of V having a hole on the elements in $H \subset V$. Add $k-\phi$ new elements N_∞. For $1 \leqslant i \leqslant k$, let N_i consist of the ϕ elements in the ith group of the ITD that appear in H. Place on the elements of the ith group, together with N_∞, the blocks of a copy of the k-BIBD, omitting a block on the elements of $N_i \cup N_\infty$. On the $\gamma k(k-1) + \phi(k-1) + k = \gamma k(k-1) + h$ elements of $V \cup N_\infty$, all pairs are covered except those within the hole on elements $N_\infty \cup \bigcup_{i=1}^{k} N_i$ of size $h = \phi(k-1)+k$. $\square$

Corollary 4.3. *Whenever $c \equiv 1 \pmod{k-1}$ and $0 \leqslant c < k(k-1)$, there are infinitely many integers n_c and h_c with $n_c \equiv 0 \pmod{k(k-1)}$ and $h_c \equiv c \pmod{k(k-1)}$ so that a k-OP(n_c+h_c, h_c) exists.*

Proof. Set $h_c = c$ if $c \geqslant k$, and $h_c = k(k-1)+1$ if $c = 1$. Apply Lemma 4.2 with $h = h_c$, and set $n_c = \gamma k(k-1)$. $\square$

The same argument establishes:

Corollary 4.4. *Whenever $c \equiv 1 \pmod{k-1}$ and $0 \leqslant c < k(k-1)$, there are infinitely many integers m_c and ℓ_c with $m_c \equiv 0 \pmod{k(k-1)}$ and $\ell_c \equiv c \pmod{k(k-1)}$ so that a k-OC$(m_c+\ell_c, \ell_c)$ exists.*

4.2. Packing: $v \not\equiv 1 \pmod{k-1}$

Lemma 4.5. *For every integer c satisfying $0 \leqslant c < k(k-1)$, there exist an $n_c \equiv 0 \pmod{k(k-1)}$ and an $h_c \equiv c \pmod{k(k-1)}$ for which a k-OP(n_c+h_c, h_c) exists.*

Proof. When $c > 0$, write $c = s(k-1)+d$ with $1 \leqslant d < k$. When $c = 0$, set $s = d = k-1$. If $d = 1$, apply Lemma 4.3. Otherwise choose $\alpha \equiv 1 \pmod{k(k-1)}$ and $N > \alpha$ so that

$N \equiv \alpha \pmod{k-1}$;
a k-GDD of type d^α exists (Theorem 2.4);
an α-BIBD of order N exists (Theorem 2.1); and
an ITD$(k, d(N-\alpha)+s, s)$ exists (Lemma 4.1).

Treat the α-BIBD as an α-GDD of type $1^{N-\alpha}\alpha^1$ by removing a block, and inflate using the k-GDD of type d^α to form a k-GDD of type $d^{N-\alpha}(d\alpha)^1$. Adjoin $d\alpha - s$ infinite elements to the

ITD$(k, d(N-\alpha)+s, s)$. On each group together with the infinite elements, place a copy of the k-GDD of type $d^{N-\alpha}(d\alpha)^1$, aligning the group of size $d\alpha$ on the s elements in the intersection of the group and the hole of the ITD, together with the $d\alpha - s$ infinite elements. The result is a k-GDD of type $d^{k(N-\alpha)}(d\alpha + s(k-1))^1$. Treat this as a packing. On the $dk(N-\alpha)$ points not in the large hole, the leave has degree $d-1$, so the result is a k-OP$(n_c + h_c, h_c)$ with $n_c = dk(N-\alpha)$ and $h_c = d\alpha + s(k-1)$. Because $dk \equiv 0 \pmod{k}$ and $N - \alpha \equiv 0 \pmod{k-1}$, $n_c \equiv 0 \pmod{k(k-1)}$. Because $d\alpha \equiv d \pmod{k(k-1)}$, $h_c \equiv c \pmod{k(k-1)}$. □

4.3. Covering: $v \not\equiv 1 \pmod{k-1}$

We employ some further, more specialized, combinatorial objects to treat coverings for the remaining congruence classes.

Let V be a set of elements; $\mathcal{B}$ be a set of k-subsets of V; $\mathcal{G} = \{G_1, \ldots, G_r\}$ be a partition of V, and $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a partition of V. Suppose that $|G_i \cap H_j| = \mu$ for all $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant t$. Further suppose that for every 2-subset $\{x, y\} \subset V$, either $\{x, y\} \in \left(\bigcup_{i=1}^{r} \binom{G_i}{2}\right) \cup \left(\bigcup_{j=1}^{t} \binom{H_j}{2}\right)$, or there is exactly one $B \in \mathcal{B}$ with $\{x, y\} \subset B$, but not both. Then $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$ is a *double group divisible design* with *blocksize* k (k-DGDD) of *type* $(\mu^r)^t$. A *holey transversal design* with blocksize k (k-HTD) of *type* μ^r is a k-DGDD of type $(\mu^r)^k$.

Theorem 4.6. *Let $k \geqslant 2$ be an integer. For all sufficiently large r, there exists a k-HTD of type 2^r.*

Proof. Choose $K = \{x_1, \ldots, x_s\}$ so that $\alpha(K) = \beta(K) = 1$, and so that for each $1 \leqslant i \leqslant s$, x_i is large enough to ensure that Theorem 2.6 yields a TD$(k+1, x_i)$. Remove one group (and rename elements as needed) to form an idempotent TD(k, x_i). When r is large enough, Theorem 2.4 yields a K-GDD $(V, \mathcal{G}, \mathcal{B})$ of type 2^r with groups $\mathcal{G} = \{G_1, \ldots, G_r\}$. The elements of the k-HTD to be formed are $V \times \{0, \ldots, k-1\}$. For each block $B \in \mathcal{B}$, on the elements $B \times \{0, \ldots, k-1\}$, align the k groups on $\{B \times \{i\}\colon 0 \leqslant i < k\}$ to place the blocks of an idempotent TD$(k, |B|)$. In the resulting design, one set of groups is formed by $V \times \{i\}$ for $0 \leqslant i < k$, the other by $G_j \times \{0, \ldots, k-1\}$ for $1 \leqslant j \leqslant r$. □

Theorem 4.7. *Let $k \geqslant 2$ be an integer. For all sufficiently large integers r and t satisfying $t - 1 \equiv 0 \pmod{k-1}$ and $\binom{t}{2} \equiv 0 \left(\bmod \binom{k}{2}\right)$, there exists a k-DGDD of type $(2^r)^t$.*

Proof. Apply Theorem 2.1 to form a k-BIBD $(V, \mathcal{B})$ with t elements. Apply Theorem 4.6 to form a k-HTD of type 2^r. To form the k-DGDD, use elements $V \times \{a, b\} \times \{1, \ldots, r\}$. For every $B \in \mathcal{B}$, place a copy of the HTD on $B \times \{a, b\} \times \{1, \ldots, r\}$, aligning groups of size $2k$ on $B \times \{a, b\} \times \{i\}$ for $1 \leqslant i \leqslant r$, and groups of size $2r$ on $\{x\} \times \{a, b\} \times \{1, \ldots, r\}$ for $x \in B$. □

The key construction follows:

Theorem 4.8. *Let t, r, y be positive integers so that $r \equiv 0 \pmod{k(k-1)}$, $t \equiv 1 \pmod{k(k-1)}$, and $y \not\equiv 2 \pmod{k-1}$. Suppose that there exist*

(1) *a k-DGDD of type $(2^r)^t$;*
(2) *a k-BIBD on $2t + k - 2$ elements;*
(3) *a k-OC$(2r + y, y)$.*

Then there is a k-OC$(2rt + k - 2 + y, 2r + y + k - 2)$.

Proof. Let $V = \{a_{i,j}, b_{i,j}\colon 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant t\}$ be the elements of the k-DGDD, with groups aligned so that $G_i = \{a_{i,j}, b_{i,j}\colon 1 \leqslant j \leqslant t\}$ and $H_j = \{a_{i,j}, b_{i,j}\colon 1 \leqslant i \leqslant r\}$. Let $\mathcal{B}$ be its set of blocks. Adjoin a set C of $k-2$ new elements. For $1 \leqslant i \leqslant r$, on $C \cup G_i$, form a k-BIBD on $2t + k - 2$ elements, aligning a block on $C \cup \{a_{it}, b_{it}\}$; then delete that block, and call the resulting set of blocks $\mathcal{D}_i$. Adjoin a set R

with y further new elements. For $1 \leqslant j < t$, on $R \cup H_j$ place a k-OC$(2r+y, y)$ with the hole aligned on R, whose block set is $\mathcal{E}_j$.

We consider the design on the $2rt+k-2+y$ elements $V \cup R \cup C$ with block set $\mathcal{B} \cup \bigcup_{i=1}^{r} \mathcal{D}_i \cup \bigcup_{j=1}^{t-1} \mathcal{E}_j$. All blocks have size k because each ingredient contains only blocks of size k. First we show that the design is a covering with a hole on $R \cup C \cup H_t$. Two elements in the hole do not appear together in a block. An element from $G_i \cap H_j$ with $j < t$ appears in a block with each element of $C \cup (G_i \cap H_t)$ in $\mathcal{D}_i$; it appears in a block with each element of R in $\mathcal{E}_j$; and it appears with each element of $H_t \setminus G_i$ in a block of $\mathcal{B}$. Consider two distinct elements $x \in G_i \cap H_j$ and $y \in G_m \cap H_n$ with $j, n < t$. If $i = m$ and $j = n$, then $\{x, y\} = \{a_{i,j}, b_{i,j}\}$ appears in a block of $\mathcal{D}_i$ (and also in at least one block of $\mathcal{E}_j$). If $i \neq m$ and $j = n$, then $\{x, y\}$ appears in at least one block of $\mathcal{E}_j$. If $i = m$ and $j \neq n$, then $\{x, y\}$ appears in one block of $\mathcal{D}_i$. If $i \neq m$ and $j \neq n$, then $\{x, y\}$ appears in one block of $\mathcal{B}$. Hence the design is a covering with a hole on $R \cup C \cup H_t$.

Secondly, we establish that it has the correct excess degrees to be an optimal covering with a hole, a k-OC. The design has $2rt+k-2+y$ elements. Because $r \equiv 0 \pmod{k-1}$, the number of elements satisfies $2rt+k-2+y \equiv y-1 \pmod{k-1}$. The hole has $2r+k-2+y$ elements. Because $r \equiv 0 \pmod{k-1}$, the number of elements in the hole satisfies $2r+k-2+y \equiv y-1 \pmod{k-1}$. Let $\bar{y} \equiv -(y-1) \pmod{k-1}$ with $0 \leqslant \bar{y} < k-1$. We must show that every element not in the hole has degree $\bar{y}+1$ in the excess, and every element in the hole has degree 0 in the excess. We treat elements in the hole first. Each element of C appears only in blocks $\{\mathcal{D}_i\colon 1 \leqslant i \leqslant r\}$. It appears in $r(t-1)$ pairs to be covered, and appears in $r(t-1)/(k-1)$ blocks, with $(t-1)/(k-1)$ blocks arising in each of $\{\mathcal{D}_i\colon 1 \leqslant i \leqslant r\}$ because this was constructed from a BIBD. Each element of R appears only in blocks $\{\mathcal{E}_j\colon 1 \leqslant j < t\}$. Because elements of R have excess degree 0 in the k-OC$(2r+y, y)$ forming $\mathcal{E}_j$, they have excess degree 0 in the union. Each element of H_t appears only in blocks of $\mathcal{B}$, and has excess degree 0. Now consider an element $x \in G_i \cap H_j$, with $j \neq t$ so that x is not in the hole. Then x appears in elements of $\mathcal{B}$, $\mathcal{D}_i$ and $\mathcal{E}_j$. It appears in $2(r-1)(t-1)/(k-1)$ blocks of $\mathcal{B}$, because it arises from the DGDD. It appears in $(2t+k-3)/(k-1)$ blocks of $\mathcal{D}_i$, because it arises from a BIBD. Now in $\mathcal{E}_j$, x is not in the hole of the k-OC$(2r+y, y)$, and hence it arises in $(2r-1+y+\bar{y})/(k-1)$ blocks. So in total x appears in $\frac{1}{k-1}(2rt+k-2+y+\bar{y})$ blocks, and because it appears in $(2rt+k-2+y)-1$ pairs, its excess degree is $\bar{y}+1$.

Because $2r+y+k-2 \equiv y-1 \pmod{k-1}$ and $y \not\equiv 2 \pmod{k-1}$, the result is the k-OC$(2rt+k-2+y, 2r+y+k-2)$. $\square$

Corollary 4.9. *For each $0 \leqslant c < k(k-1)$, there exist integers m_c and ℓ_c with $m_c \equiv 0 \pmod{k(k-1)}$ and $\ell_c \equiv c \pmod{k(k-1)}$ for which a k-OC$(m_c+\ell_c, \ell_c)$ exists.*

Proof. Let t_0 and r_0 be integers with $t_0 \equiv 1 \pmod{k(k-1)}$ and $t_0 > 1$ so that whenever $r \geqslant r_0$, $t \geqslant t_0$, and $t \equiv 1 \pmod{k(k-1)}$,

(1) there is a k-DGDD of type $(2^r)^t$ (apply Theorem 4.7), and
(2) there is a k-BIBD on $2t+k-2$ $(\equiv k \pmod{k(k-1)})$ elements (apply Theorem 2.1).

When $c \equiv 1 \pmod{k-1}$, apply Corollary 4.4 to choose one k-OC$(m_c+\ell_c, \ell_c)$ with $m_c \equiv 0 \pmod{k(k-1)}$ and $m_c \geqslant r_0$. In general, when a k-OC$(m_c+\ell_c, \ell_c)$ with $m_c \equiv 0 \pmod{k(k-1)}$ and $m_c \geqslant r_0$ exists, Theorem 4.8 produces a k-OC$(m_c t_0+k-2+\ell_c, m_c+\ell_c+k-2)$. Set $m_{c+k-2 \bmod k(k-1)} = m_c(t_0-1)$, which exceeds r_0 and is a multiple of $k(k-1)$. Set $\ell_{c+k-2 \bmod k(k-1)} = m_c+\ell_c+k-2 \equiv c+k-2 \pmod{k(k-1)}$. Then $k-2$ applications of Theorem 4.8 handle all congruence classes. $\square$

5. Asymptotic existence

Our next task is to handle not just one example for hole size in each congruence class modulo $k(k-1)$, but to extend to all sufficiently large orders.

Theorem 5.1. *Let $k \geqslant 2$ be an integer. Then there are constants κ_k and u_k so that whenever $v \geqslant \kappa_k$, there is a k-OP(v, h) and a k-OC(v, h) with $h \leqslant u_k$.*

Proof. By Corollary 4.9, for $0 \leqslant c < k(k-1)$ there are integers $m_c \equiv 0 \pmod{k(k-1)}$ and $\ell_c \equiv c \pmod{k(k-1)}$ for which a k-OC$(m_c + \ell_c, \ell_c)$ exists. By Lemma 4.5, for $0 \leqslant c < k(k-1)$ there are integers $n_c \equiv 0 \pmod{k(k-1)}$ and $h_c \equiv c \pmod{k(k-1)}$ for which a k-OP$(n_c + h_c, h_c)$ exists. Set $u_k = \max\{n_c + h_c, m_c + \ell_c\colon\ 0 \leqslant c < k(k-1)\}$.

Using Theorem 2.4, choose an integer x for which k-GDDs of type $(k(k-1))^{x'}$ exist for all $x' \in \{x, x+1, x+2, x+3\}$. Again using Theorem 2.4, choose an integer r for which an $\{x+1, x+2, x+3\}$-GDD of type p^g exists for all $1 \leqslant p \leqslant u_k$ and all $g \geqslant r$. Then set $\kappa_k = rk(k-1) + u_k$, a constant depending only on k.

We develop the remainder of the proof for packings; that for coverings parallels it very closely. Let $v \geqslant \kappa_k$ be an integer, and write $v = \phi k(k-1) + c$ with $0 \leqslant c < k(k-1)$. Write $v - h_c = gn_c + d$ so that $d \equiv 0 \pmod{k(k-1)}$ and $d < n_c$. Let $n' = n_c/(k(k-1))$ and $d' = d/(k(k-1))$.

Construct a k-GDD of type $n_c^g d^1$ as follows. Form an $\{x+1, x+2, x+3\}$-GDD of type $(n')^{g+1}$. Delete all but d' elements in one group to form an $\{x, x+1, x+2, x+3\}$-GDD of type $(n')^g (d')^1$. Inflate using weight $k(k-1)$, employing k-GDDs of type $(k(k-1))^{x'}$ for $x' \in \{x, x+1, x+2, x+3\}$, to form a k-GDD of type $n_c^g d^1$. Then add h_c new elements, and place a k-OP$(n_c + h_c, h_c)$ on each group of size n_c together with the h_c new elements, aligning the hole on these h_c elements. The result is a k-OP$(v, h_c + d)$, and $h_c + d \leqslant u_k$ as required. $\square$

Proof of Theorem 3.1. When $v < \kappa_k$, a maximum packing with blocksize k contains at least $\ell_{v,k} - \binom{\kappa_k}{2}$ blocks, and $\binom{\kappa_k}{2}$ is a constant. When $v \geqslant \kappa_k$, form a k-OP(v, h) with $h \leqslant u_k$, which has at least $\ell_{v,k} - \binom{u_k}{2}$ blocks and $\binom{u_k}{2}$ is a constant. $\square$

Proof of Theorem 3.2. When $v < \kappa_k$, a minimum covering with blocksize k requires at most $\binom{\kappa_k}{2}$ blocks, which is a constant. When $v \geqslant \kappa_k$, form a k-OC(v, h) with $h \leqslant u_k$, which has at most $L_{v,k}$ blocks. A covering on h points in which every block contains some pair that is covered only once has at most $\binom{u_k}{2}$ blocks, which is a constant independent of v. Use this to fill the hole. $\square$

6. Conclusion

For $t = 2$, our results establish that the elementary Johnson and Schönheim bounds are essentially the correct ones, in that the respective optima cannot differ from them by more than an additive constant. Unless this constant can be shown to be quite small, the specific value obtained for the constant is not of particular interest. Without recourse to Proposition 1.1 or a similar statement, we see no way at present to obtain differences from the bounds that are bounded by a quantity as small as (say) k in general, although it is plausible that such bounds hold.

Acknowledgment

We thank an anonymous referee for helpful comments on the presentation.

References

[1] T. Beth, D. Jungnickel, H. Lenz, Design Theory, vol. I, second edition, Encyclopedia Math. Appl., vol. 69, Cambridge University Press, Cambridge, 1999.
[2] Y. Caro, R. Yuster, Packing graphs: the packing problem solved, Electron. J. Combin. 4 (1) (1997), Research Paper 1, approx. 7 pp. (electronic).
[3] Y. Caro, R. Yuster, Covering graphs: the covering problem solved, J. Combin. Theory Ser. A 83 (2) (1998) 273–282.
[4] S. Chowla, P. Erdős, E.G. Straus, On the maximal number of pairwise orthogonal Latin squares of a given order, Canad. J. Math. 12 (1960) 204–208.
[5] C.J. Colbourn, J.H. Dinitz, D.R. Stinson, Applications of combinatorial designs to communications, cryptography, and networking, in: Surveys in Combinatorics, Canterbury, 1999, in: London Math. Soc. Lecture Note Ser., vol. 267, Cambridge Univ. Press, Cambridge, 1999, pp. 37–100.
[6] C.J. Colbourn, V. Rödl, Percentages in pairwise balanced designs, Discrete Math. 77 (1–3) (1989) 57–63.
[7] C.J. Colbourn, P.C. van Oorschot, Applications of combinatorial designs in computer science, ACM Comput. Surv. 21 (2) (1989) 223–250.
[8] P. Dukes, A.C.H. Ling, Asymptotic existence of resolvable graph designs, Canad. Math. Bull. 50 (4) (2007) 504–518.

[9] P. Erdős, H. Hanani, On a limit theorem in combinatorial analysis, Publ. Math. Debrecen 10 (1963) 10–13.
[10] P. Frankl, V. Rödl, Near perfect coverings in graphs and hypergraphs, European J. Combin. 6 (4) (1985) 317–326.
[11] D.A. Grable, More-than-nearly-perfect packings and partial designs, Combinatorica 19 (2) (1999) 221–239.
[12] T. Gustavsson, Decompositions of large graphs and digraphs with high minimum degree, PhD thesis, Dept. of Mathematics, Univ. of Stockholm, 1991.
[13] S.M. Johnson, A new upper bound for error-correcting codes, IRE Trans. IT-8 (1962) 203–207.
[14] A.V. Kostochka, V. Rödl, Partial Steiner systems and matchings in hypergraphs, Random Structures Algorithms 13 (3–4) (1998) 335–347.
[15] N.N. Kuzjurin, On the difference between asymptotically good packings and coverings, European J. Combin. 16 (1) (1995) 35–40.
[16] E.R. Lamken, R.M. Wilson, Decompositions of edge-colored complete graphs, J. Combin. Theory Ser. A 89 (2) (2000) 149–200.
[17] J. Liu, Asymptotic existence theorems for frames and group divisible designs, J. Combin. Theory Ser. A 114 (3) (2007) 410–420.
[18] H. Mohácsy, The asymptotic existence of group divisible designs of large order with index one, J. Combin. Theory Ser. A 118 (7) (2011) 1915–1924.
[19] H. Mohácsy, D.K. Ray-Chaudhuri, An existence theorem for group divisible designs of large order, J. Combin. Theory Ser. A 98 (1) (2002) 163–174.
[20] V. Rödl, On a packing and covering problem, European J. Combin. 6 (1) (1985) 69–78.
[21] J. Schönheim, On coverings, Pacific J. Math. 14 (1964) 1405–1411.
[22] D.R. Stinson, Combinatorial Designs, Springer-Verlag, New York, 2004.
[23] V.H. Vu, New bounds on nearly perfect matchings in hypergraphs: higher codegrees do help, Random Structures Algorithms 17 (1) (2000) 29–63.
[24] R.M. Wilson, The construction of group divisible designs and partial planes having the maximum number of lines of a given size, in: Proc. Second Chapel Hill Conf. on Combinatorial Mathematics and its Applications, Univ. North Carolina, Chapel Hill, NC, 1970, Univ. North Carolina, Chapel Hill, NC, 1970, pp. 488–497.
[25] R.M. Wilson, An existence theory for pairwise balanced designs. III. Proof of the existence conjectures, J. Combin. Theory Ser. A 18 (1975) 71–79.
[26] R.M. Wilson, The proportion of various graphs in graph designs, in: R.A. Brualdi, S. Hedayat, H. Kharaghani, G. Khosrovshahi, S. Shahriari (Eds.), Combinatorics and Graphs: The Twentieth Anniversary Conference of IPM Combinatorics, American Mathematical Society, Providence, RI, 2010, pp. 251–255.