Decoding Space-Time Codes by absorbing the channel

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The quasi static fading channel

MIMO System

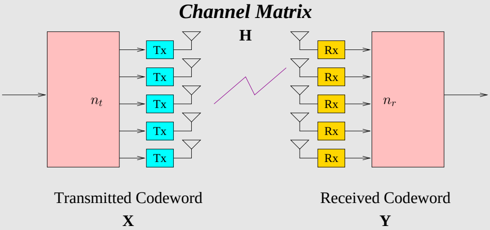

Figure: The Channel Model

The quasi static fading channel

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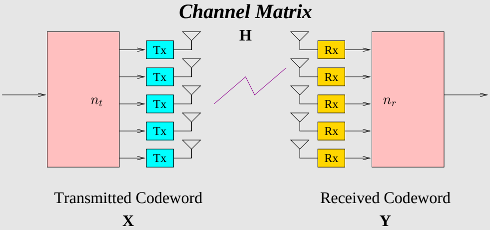

Figure: The Channel Model

- Received signal

$$\boldsymbol{Y}_{n_r \times T} = \boldsymbol{H}_{n_r \times n_t} \cdot \boldsymbol{X}_{n_t \times T} + \boldsymbol{W}_{n_r \times T} \quad (1)$$

  with $\boldsymbol{H}$ perfectly known at the receiver. All matrices have entries in $\mathbb{C}$. $\boldsymbol{W}$ is the noise matrix with i.i.d. Gaussian entries.

- $\boldsymbol{H}$ is assumed constant during the transmission of one codeword.

Decoding

Optimal decoding rule

Find

$$\boxed{\hat{\boldsymbol{X}} = \arg\min \|\boldsymbol{Y} - \boldsymbol{H}\cdot\boldsymbol{X}\|_F^2}$$

where

$$\|\boldsymbol{A}\|_F^2 = \sum_{i,j} \left|a_{ij}\right|^2 = \mathrm{Tr}\left(\boldsymbol{A}\cdot\boldsymbol{A}^\dagger\right)$$

- Decoding is, in general, a computationally hard problem.

Pairwise Error Probability (I)

$$\mathbf{c} = \begin{bmatrix} c_1^1 & c_2^1 & \cdots & \cdots & c_T^1 \\ c_1^2 & c_2^2 & \cdots & \cdots & c_T^2 \\ c_1^3 & c_2^3 & \ddots & \vdots & c_T^3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_1^{n_t} & c_2^{n_t} & \cdots & \cdots & c_T^{n_t} \end{bmatrix} \text{ and } \mathbf{e} = \begin{bmatrix} e_1^1 & e_2^1 & \cdots & \cdots & e_T^1 \\ e_1^2 & e_2^2 & \cdots & \cdots & e_T^2 \\ e_1^3 & e_2^3 & \ddots & \vdots & e_T^3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ e_1^{n_t} & e_2^{n_t} & \cdots & \cdots & e_T^{n_t} \end{bmatrix}$$

are respectively two distinct codewords $(T \geq n_t)$

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Error probability

Union Bound gives

$$P_e \leq \sum_{\mathbf{c} \in \mathscr{C}} \Pr\{\mathbf{c}\} \sum_{\mathbf{e} \neq \mathbf{c}} P(\mathbf{c} \to \mathbf{e})$$

Pairwise Error Probability (II)

- Pairwise error probability for a quasi-static Rayleigh fading channel is upper bounded by

$$P(\mathbf{c} \to \mathbf{e}) \leq \left(\prod_{i=1}^{n_t} \frac{1}{1 + \lambda_i^2 \frac{E_s}{4N_0}} \right)^{n_r} \tag{2}$$

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- **Two criteria**
  - **The rank criterion** : In order to achieve maximum diversity $n_t \cdot n_r$, the matrix $\mathbf{B}(\mathbf{c}, \mathbf{e})$ must be of maximum rank n_t.
  - **The coding advantage** : In order to maximize the coding gain, the quantity

$$\min_{\mathbf{c} \neq \mathbf{e}} \det \mathbf{A}(\mathbf{c}, \mathbf{e}) \quad (3)$$

must be maximized.

Outline

The Golden Code

Golden Code B. Rekaya Viterbo (2005)

Codewords are

$$\mathbf{X} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha\left(z_1 + z_2\varphi\right) & \alpha\left(z_3 + z_4\varphi\right) \\ i \cdot \bar{\alpha}\left(z_3 + z_4\bar{\varphi}\right) & \bar{\alpha}\left(z_1 + z_2\bar{\varphi}\right) \end{pmatrix}$$

with $\varphi = \frac{1+\sqrt{5}}{2}$, $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\varphi$, $\bar{\alpha} = 1 + i - i\bar{\varphi}$ and $z_j \in \mathbb{Z}[i]$.

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The two layers of **X** (the two diagonals) can be vectorized,

$$\mathrm{vec}\mathbf{X_1} = \begin{pmatrix} x_{11} \\ x_{22} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha & \alpha\varphi \\ \bar{\alpha} & \bar{\alpha}\bar{\varphi} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\mathrm{vec}\mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{21} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha & \alpha\varphi \\ i\bar{\alpha} & i\bar{\alpha}\bar{\varphi} \end{pmatrix} \cdot \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}$$

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Remark the transform which maps (z_1, z_2) onto the first layer

$$\underline{U} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \alpha\varphi \\ \bar{\alpha} & \bar{\alpha}\bar{\varphi} \end{bmatrix}$$

Number i isolates the first layer from the second one so that minimum determinant is not zero.

Minimum determinant

We obtain

$$\boxed{\delta_{\min} = \min_{\mathbf{X}\neq 0} |\det \mathbf{X}|^2 = \frac{1}{5}}$$

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Problems

- Symbols z_i are, in the “real life”, QAM symbols (finite subset of $\mathbb{Z}[i]$)
- The Golden code is computationally hard to decode.

Division Algebra

The Algebra of the Golden Code

The Golden Algebra $\mathscr{A}$ (quaternion algebra) has elements

$$\boldsymbol{A} = \begin{bmatrix} s_1 + \theta s_2 & s_3 + \theta s_4 \\ i s_3 + \bar{\theta} s_4 & s_1 + \bar{\theta} s_2 \end{bmatrix}$$

with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$ and s_l, $l = 1 \ldots 4$ are elements of the field $\mathbb{Q}(i)$.

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- Every non zero element in $\mathscr{A}$ has an inverse since

$$\det \boldsymbol{A} = (2+i)\left[N(s_1 + \theta s_2) - iN\left(s_3 + \theta s_4\right)\right] \neq 0$$

In fact, i is not a norm of $\mathbb{Q}\left(i, \sqrt{5}\right)$.

Code defined on an order

The Order of the Golden Code

The Golden Order $\mathcal{O}_{\mathscr{A}}$ has elements

$$\boldsymbol{O} = \begin{bmatrix} s_1 + \theta s_2 & s_3 + \theta s_4 \\ i s_3 + \bar{\theta} s_4 & s_1 + \bar{\theta} s_2 \end{bmatrix}$$

with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$ and s_l, $l = 1 \ldots 4$ are elements of $\mathbb{Z}[i]$. $\mathcal{O}_{\mathscr{A}}$ is a maximal order.

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Golden Code

A codeword $\boldsymbol{X}$ of the Golden code is

$$\boldsymbol{X} = \begin{bmatrix} \alpha(s_1 + \theta s_2) & \alpha(s_3 + \theta s_4) \\ i\bar{\alpha}(s_3 + \bar{\theta} s_4) & \bar{\alpha}(s_1 + \bar{\theta} s_2) \end{bmatrix}$$

with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$, $\overline{\alpha} = 1 + i - i\overline{\theta}$ and s_l, $l = 1 \ldots 4$ are the information symbols carved from $\mathbb{Z}[i]$.

The group of units $\mathcal{O}^+$

Group of units of $\mathcal{O}_{\mathcal{A}}$

The group of units $\mathcal{O}^\times$ is the group of elements $\boldsymbol{O}$ of the order with determinant equal to a unit in $\mathbb{Z}[i]$, i.e.

$$\det \boldsymbol{O} \in \{\pm 1, \pm i\}$$

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Subgroup $\mathcal{O}^+$

The subgroup $\mathcal{O}^+$ of $\mathcal{O}^\times$ is

$$\mathcal{O}^+ = \left\{\boldsymbol{O} \in \mathcal{O}_{\mathcal{A}} \,|\, \det \boldsymbol{O} = 1\right\}$$

Outline

The MIMO Channel is a Unit

Received Signal

Received signal is

$$\boldsymbol{Y}_{2\times 2} = \boldsymbol{H}_{2\times 2} \cdot \boldsymbol{X}_{2\times 2} + \boldsymbol{W}_{2\times 2}$$

where $\boldsymbol{X}$ is a Golden Code codeword.

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- Set $\tilde{Y} = \frac{1}{\sqrt{\det H}} Y$ and write

$$\boxed{\tilde{Y} = \tilde{H} \cdot X + \tilde{W}}$$

with $\tilde{H} \in SL_2(\mathbb{C})$. In fact, we could restrict to

$$PSL_2(\mathbb{C}) = SU_2(\mathbb{C}) \setminus SL_2(\mathbb{C})$$

since $\tilde{H}$ can be known up to a unitary transform.

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  since $\tilde{H}$ can be known up to a unitary transform.

- Suppose $\tilde{H}$ is a unit in $\mathcal{O}^+$ up to a left unitary transform, then $\tilde{H} \cdot X$ is a new codeword $\tilde{X}$, absorption of the channel by the code,

  $$\boxed{\tilde{Y} = \tilde{X} + \tilde{W}}$$

  and $X = \tilde{H}^{-1} \cdot \tilde{X}$.

The Multiplicative Error (1)

- $\tilde{\boldsymbol{H}}$ is no more a unit. We write $\tilde{\boldsymbol{H}} = \boldsymbol{E} \cdot \boldsymbol{U}$ where $\boldsymbol{U}$ is a unit.

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Fact

Two problems to solve

The Multiplicative Error (2)

Unit Search

With a ZF detector used, minimizing the noise variance after ZF is equivalent to

$$\tilde{\boldsymbol{U}} = \arg\min_{\boldsymbol{U}\in\mathcal{O}^+} \left\| \tilde{\boldsymbol{H}} \cdot \boldsymbol{U}^{-1} \right\|_F$$

where $\|\boldsymbol{A}\|_F$ is the Frobenius norm of $\boldsymbol{A}$.

- Find an algorithm that can search for $\tilde{\boldsymbol{U}}$ with a low complexity.

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Detection

- Find then a detector that can use this algebraic reduction
- Performance ?

Outline

Action of $SL_2(\mathbb{C})$

The Hyperbolic Space $\mathbb{H}_3$

Space

$$\boxed{\mathbb{H}_3 = \{(z, r), z \in \mathbb{C}, r \in \mathbb{R}, r > 0\}}$$

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- $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $g \in SL_2(\mathbb{C})$

Action on $\mathbb{H}_3$

$$g((z,r)) = \left(z^\star, r^\star\right) \quad \text{with} \quad \begin{cases} z^\star = \dfrac{(az+b)\left(\bar{c}\bar{z}+\bar{d}\right)+a\bar{c}r^2}{|cz+d|^2+|c|^2 r^2} \\ r^\star = \dfrac{r}{|cz+d|^2+|c|^2 r^2} \end{cases}$$

- We make our group act over $J = (0,1)$

Hyperbolic distance

Hyperbolic distance

Define the hyperbolic dictance on $\mathbb{H}_3$

$$\boxed{\cosh \rho \left(P, P'\right) = 1 + \frac{d\left(P, P'\right)^2}{2rr'}}$$

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Hyperbolic vs Frobenius

$$\forall g \in SL_2(\mathbb{C}), \|g\|_F^2 = 2\cosh \rho\left(J, g(J)\right)$$

The fundamental domain

- $\mathcal{O}^+$ is a discrete subgroup of $SL_2(\mathbb{C})$. The action of $\mathcal{O}^+$ on J generates a tesselation of $\mathbb{H}_3$.
- The tesselation defines, for each element g of $\mathcal{O}^+$, a hyperbolic polyhedron

$$\mathcal{P}_g = \left\{x \in \mathbb{H}_3 \,|\, \rho\left(x, g(J)\right) \leq \rho\left(x, g'(J)\right), \forall g' \neq g\right\}$$

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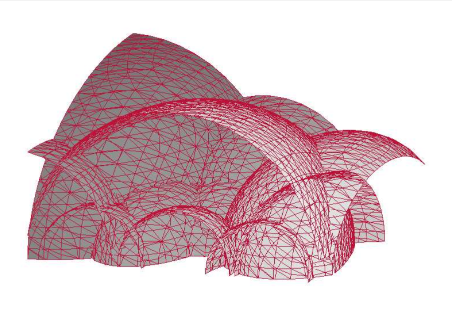
Fundamental Domain

$\mathcal{P}_I$ (Dirichlet polyhedron) is the fundamental domain of $\mathcal{O}^+$



The generators

8 generators for $\mathcal{O}^+$

$$u_1 = \begin{pmatrix} i\theta & 0 \\ 0 & i\bar{\theta} \end{pmatrix} \quad u_2 = \begin{pmatrix} i & 1+i \\ i-1 & i \end{pmatrix} \quad u_3 = \begin{pmatrix} \theta & 1+i \\ i-1 & \bar{\theta} \end{pmatrix}$$

$$u_4 = \begin{pmatrix} \theta & -1-i \\ -i+1 & \bar{\theta} \end{pmatrix} \quad u_5 = \begin{pmatrix} 1+i & 1+i\bar{\theta} \\ i(1+i\theta) & 1+i \end{pmatrix} \quad u_6 = \begin{pmatrix} 1+i & 1+i\theta \\ i(1+i\bar{\theta}) & 1+i \end{pmatrix}$$

$$u_7 = \begin{pmatrix} 1-i & \bar{\theta}+i \\ i(\theta+i) & 1-i \end{pmatrix} \quad u_8 = \begin{pmatrix} 1-i & \theta+i \\ i(\bar{\theta}+i) & 1-i \end{pmatrix}$$

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Word problem

Each element in $\mathcal{O}^+$ can be written by using eight letters (u_i)

Outline

Reduction Algorithm (I)

- Using the fundamental domain (generators), the aim is to find

$$\tilde{\boldsymbol{U}} = \arg \min_{\boldsymbol{U} \in \mathcal{O}^+} \left\| \tilde{\boldsymbol{H}} \cdot \boldsymbol{U}^{-1} \right\|_F$$

  by using an iterative process.

- The optimal remaining error

$$\boldsymbol{E} = \tilde{\boldsymbol{H}} \cdot \tilde{\boldsymbol{U}}^{-1}$$

  is inside the fundamental domain of $\mathcal{O}^+$.

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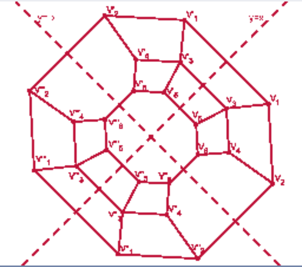

Reduction Algorithm (II)

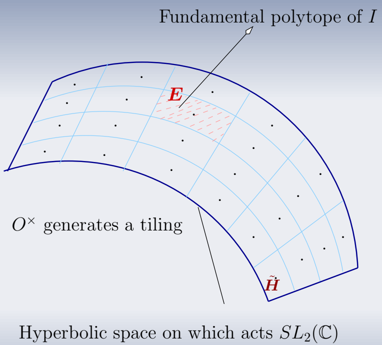

Figure: The algorithm

Reduction Algorithm (II)

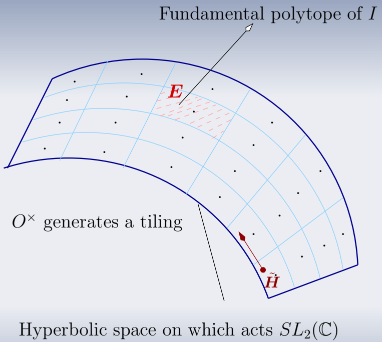

Figure: The algorithm

Reduction Algorithm (II)

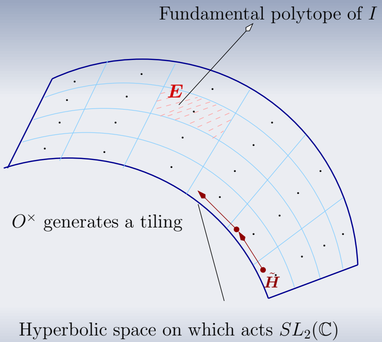

Figure: The algorithm

Reduction Algorithm (II)

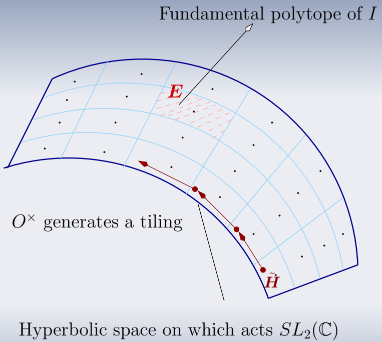

Figure: The algorithm

Reduction Algorithm (II)

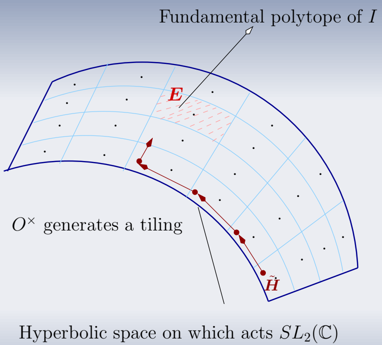

Figure: The algorithm

Reduction Algorithm (II)

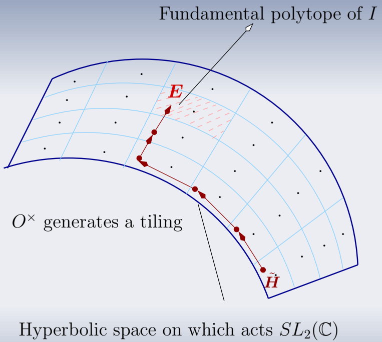

Figure: The algorithm

Simulation Results

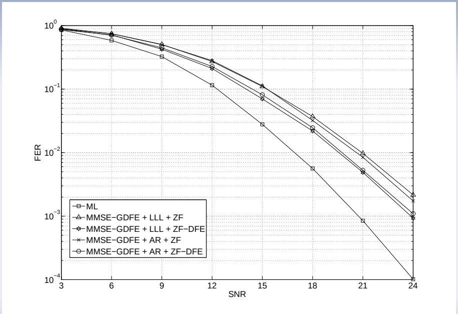

Figure: Simulation Results 8 bits pcu

Figure: Thank you for your attention !!!