Highly nonlinear filter Boolean functions with high algebraic immunity for stream ciphers

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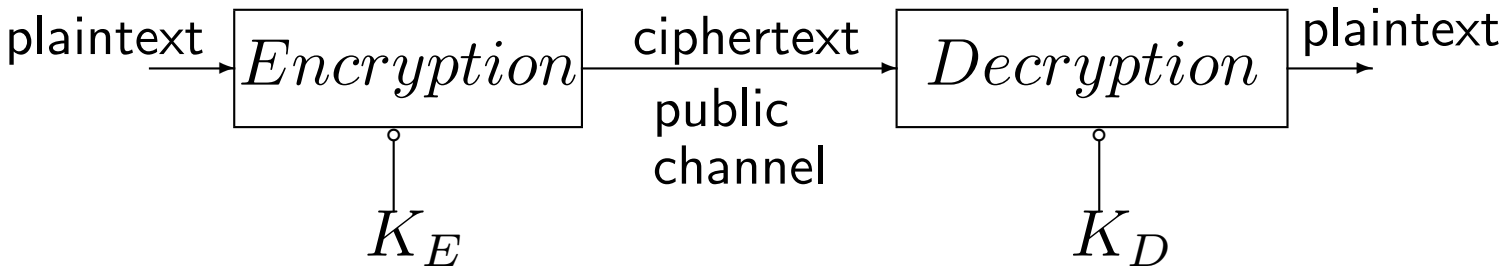
Outline

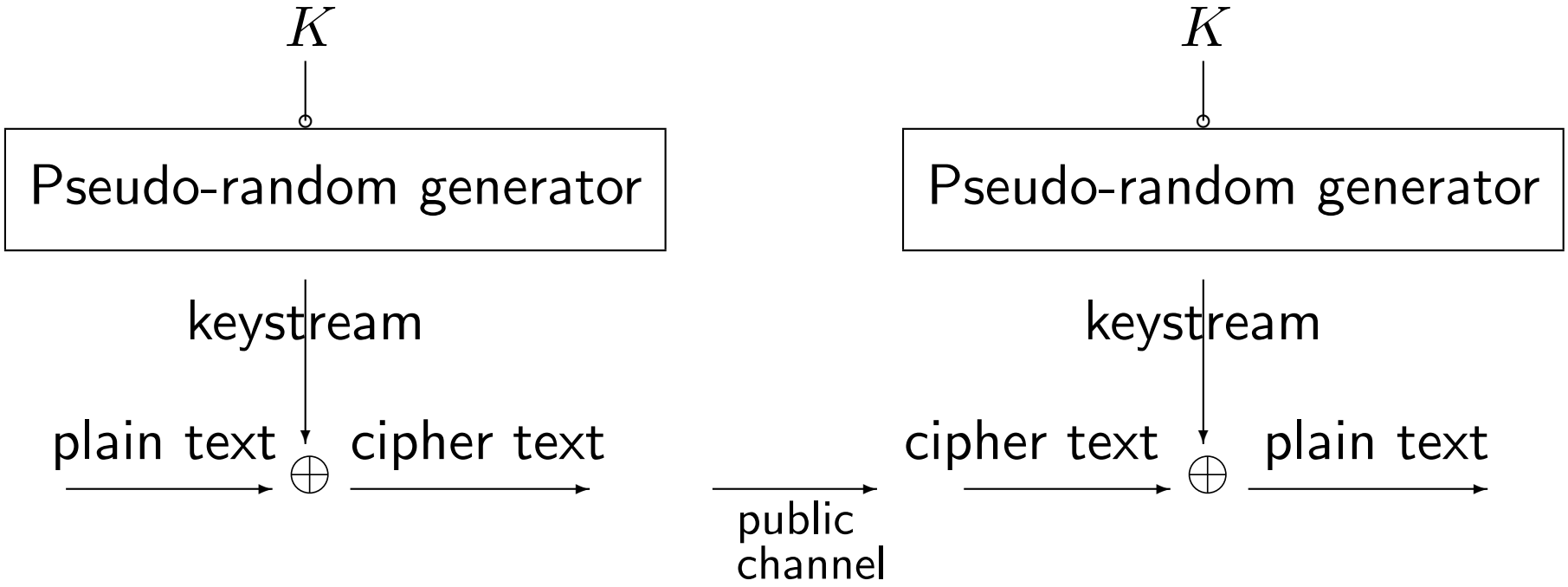
- Preliminaries on stream ciphers and Boolean functions
- Algebraic attacks on stream ciphers and algebraic immunity
- The known Boolean functions with optimal algebraic immununity
- Recent developments

Preliminaries on stream ciphers and Boolean functions

Ciphers (cryptography) :



Synchronous stream ciphers :



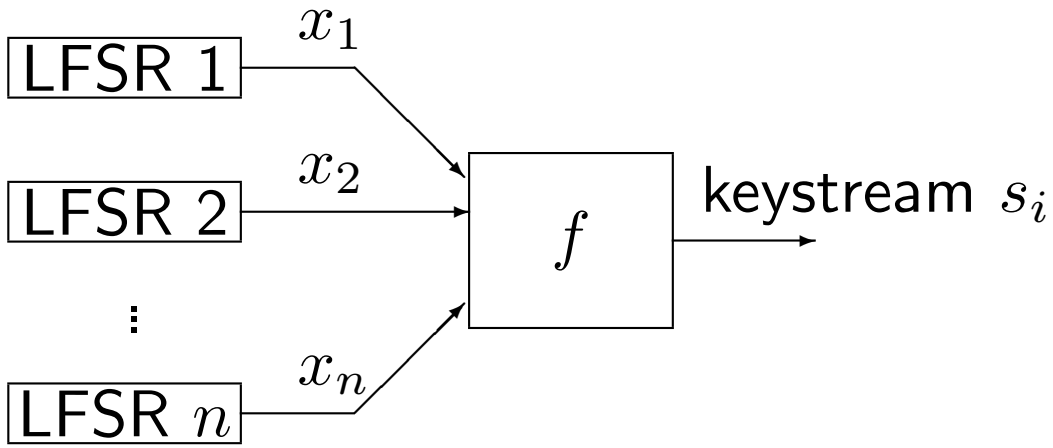
Every PRG consists in a linear part (for efficiency) and a nonlinear part (for robustness).

Boolean functions $f : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$ are often used in the nonlinear part.

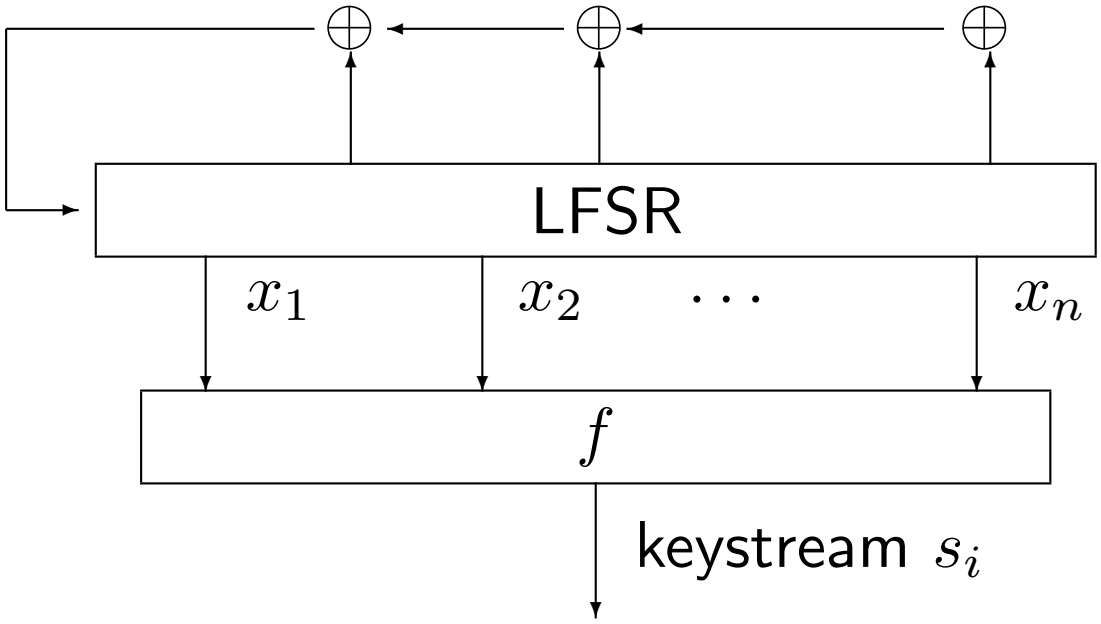
There exist **two theoretical models** for their use in the pseudo-random generators (PRG) of Synchronous stream ciphers.

Both use Linear Feedback Shift Registers in the linear part :

Combiner model :



Filter model



In both models, f must be balanced to avoid distinguishing attacks.

Two representations of Boolean functions :

- *The Algebraic Normal Form* (ANF) :

$$f(x_1, \cdots, x_n) = \sum_{I \subseteq \{1,\ldots,n\}} a_I \left(\prod_{i \in I} x_i \right), \; a_I \in \mathbb{F}_2.$$

The ANF exists and is unique.

The algebraic degree is the degree of the ANF.
It must be large because of Berlekamp-Massey and Rønjom-Helleseth attacks.

Affine functions : sums of linear and constant, that is : $\deg \leq 1$.
Notation : $a_1\, x_1 + \cdots + a_n\, x_n = a \cdot x\ ; \quad a \in \mathbb{F}_2^n$.

- *The univariate representation (the trace representation)* :

- The vector space $\mathbb{F}_2^n$ is endowed with the structure of the field $\mathbb{F}_{2^n}$.
Any function $f : \mathbb{F}_{2^n} \mapsto \mathbb{F}_{2^n}$ admits the unique representation :

$$f(x) = \sum_{j=0}^{2^n-1} a_j\, x^j; \quad a_j,\, x \in \mathbb{F}_{2^n}.$$

- f is Boolean if and only if :

$$a_0, a_{2^n-1} \in \mathbb{F}_2 \text{ and } a_{2j} = (a_j)^2, \forall j \in \mathbb{Z}/(2^n-1)\mathbb{Z}.$$

Hence :

$f(x) = tr(P(x))$, where $tr(x) = x + x^2 + x^{2^2} + \cdots + x^{2^{n-1}}$.

Then the algebraic degree equals : $\max\{w_2(j);\ j$ s.t. $a_j \neq 0\}$, where $w_2(j)$ is the Hamming weight of the binary expansion of j.

Affine functions $tr(ax) + \epsilon$, $a \in \mathbb{F}_2^n$, $\epsilon \in \mathbb{F}_2$.

The *Walsh transform* of a Boolean function :

$$\widehat{\mathrm{f}}(a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+a\cdot x} \text{ or } \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+tr(ax)}.$$

The *Hamming distance* between two functions :

$$d_H(f,g) = w_H(f+g) = |\{x \in \mathbb{F}_2^n \,/\, f(x) \neq g(x)\}.$$

The *nonlinearity* of a Boolean function f is the minimum Hamming distance from f to affine functions and equals :

$$nl(f) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} |\widehat{\mathrm{f}}(a)|.$$

The nonlinearity nl is upper bounded by $2^{n-1} - 2^{n/2-1}$ (covering radius bound). This maximum is achieved by bent functions.

The nonlinearity nl must be large to prevent the system from the Meier-Staffelbach fast correlation attack and its variants.

Balancedness, high algebraic degree and large nonlinearity was considered as roughly sufficient for the filter model of pseudo-random generator before the introduction of algebraic attacks.

An additional criterion in the case of the combiner model : to resist the Siegenthaler correlation attack, the function should be resilient of a high order.

Algebraic attacks on stream ciphers and algebraic immunity

Algebraic attacks : *Principle* (Shannon) :

-Find equations with the key bits as unknowns
-Solve the system of these equations.

For stream ciphers (combiner model and filter model) :
- denote by $(s_0, \ldots, s_{N-1})$ the initial state of the linear part of the pseudo-random generator ;
- there exists a linear automorphism L and a linear mapping L' s.t.

$$s_i = f(L' \circ L^i(s_0, \ldots, s_{N-1})).$$

Problem of the general algebraic attack :

Highly non-linear equations with many unknowns.

But with stream ciphers we can have many equations $\longrightarrow$

over-defined system.

One can then linearize the system (or use Gröbner bases).

However the number of unknowns is then much too large.

Courtois-Meier : If one can find $g \neq 0$ and h of low degrees such that $fg = h$, then the equation $s_i = f(L' \circ L^i(s_0, \ldots, s_{N-1}))$ implies the low degree equation :

$$s_i \; g(L' \circ L^i(s_0, \ldots, s_{N-1})) = h(L' \circ L^i(s_0, \ldots, s_{N-1}))$$

and the degree of the nonlinear system and the number of unknowns in the related linear system decrease.

Algebraic immunity :

A necessary and sufficient condition for existence of low degree $g \neq 0$ and h such that $fg = h$ (Meier-Pasalic-C.C.) :
there exists $g \neq 0$ of low degree such that $fg = 0$ or $(f + 1)g = 0$.

Definition : a function g such that $fg = 0$ is called an $annihilator$. The $algebraic\ immunity\ AI(f)$ is the minimum degree of the nonzero annihilators of f and of those of $f + 1$.

We have : $AI(f) \leq \deg(f)$ and $AI(f) \leq \left\lceil \frac{n}{2} \right\rceil$.

In practical situation, $AI(f)\ must\ be\ greater\ than\ or\ equal\ to\ 7$.

Hence we need $n \geq 13$ and in fact $n \approx 20$.

A variant by Courtois of algebraic attacks, called "fast algebraic attack" needs the existence of $g \neq 0$ and h such that $fg = h$, where only g has low degree and h has reasonable degree.

It needs more data.

The known Boolean functions with optimal algebraic immununity

Until recently, two classes existed :

- The majority function defined (for every n) by :

$$f(x) = 1 \text{ iff } w_H(x) \geq n/2.$$

and its generalizations by Dalai et al., Bracken, C.C... ;

- An iterative construction (Dalai-Gupta-Maitra), n even.

In both cases, the functions have high degree but *insufficient nonlinearity* and bad resistance to Fast Algebraic Attacks (Dalai, Gupta, Maitra, Armknecht, C.C., Gaborit, Meier, Ruatta...).

A recently found infinite class of balanced functions with optimal algebraic immunity :

Definition

Let $n \geq 2$ and α a primitive element of the field $\mathbb{F}_{2^n}$.
We denote by f the Boolean function on $\mathbb{F}_{2^n}$ whose support is $\{0, 1, \alpha, \cdots, \alpha^{2^{n-1}-2}\}$.

Theorem (Feng, Liao, Yang - C.C., Feng)

The function f defined above has optimal algebraic immunity $\lceil n/2 \rceil$.

Proof (sketch) :

Let $g(x) = \sum_{j=0}^{2^n-1} a_j \, x^j$ be a non-zero annihilator of $f + 1$.

g is a codeword of a Reed-Solomon code of designed distance $2^{n-1}+1$.

Hence $|\{j \,/\, a_j \neq 0\}| \geq 2^{n-1} + 1$ and $\deg(g) \geq \lceil \frac{n}{2} \rceil$.

Algebraic degree (C.C., Feng) : f has degree $n-1$ (optimal).

Nonlinearity (C.C., Feng) :

$$nl(f) \geqslant 2^{n-1} - \frac{2^{\frac{n}{2}+1}}{\pi} \ln\left(\frac{4(2^n-1)}{\pi}\right) - 1 \;\sim\; 2^{n-1} - \frac{\ln 2}{\pi}\, n\, 2^{\frac{n}{2}+1}.$$

Exact values of the nonlinearity for f :

n	6	7	8	9	10	11
Best nl known before	22	48	98	196	400	798
The exact values of nl	*24*	*54*	*112*	*232*	*478*	*980*
upper bound $2^{n-1} - 2^{n/2-1}$	28	58	120	244	496	1001

Immunity of f against fast algebraic attacks

• $\deg(g) = 1$: for $n \leq 12$, no functions g and h exist such that $f\,g = h$ and $\deg(h) < n - 1$ if n odd and $\deg(h) < n - 2$ if n even.

• $\deg(g) > 1$: for $n \leq 9$, no function such that $f\,g = h$, $\deg(g) \leq n/2$ and $\deg(g) + \deg(h) < n - 1$ exist.

- The instance with $n = 9$ turns out to be optimal : no function such that $f\,g = h$, $\deg(g) \leq n/2$ and $\deg(g) + \deg(h) < n$ exist. This is the first time where a function with optimal immunity against FAA's can be observed.

- The problem of proving the good behavior of f against FAA for every n is open.

The problem of computing the output to the function (with help of G. Hanrot and J. Detrey) :

The complexity of computing $f(x)$ is same as for the discrete log. But n is "small".

The complexity is lower when n is even.

We can then use the Pohlig-Hellman method, with tables for the discrete log for the sizes $2^{n/2} - 1$ and $2^{n/2} + 1$.

The time for computation will be very reasonable (1 bit per cycle) but this will need about 200,000 transistors for $n = 20$.

It is possible to reduce the number of transistors by cutting in three pieces instead of two :

$$2^{18} - 1 = 27 * 73 * 133 \; ; \quad 2^{20} - 1 = 41 * 93 * 275.$$

This reduces the number of transistors to 40,000 for $n = 20$.

Recent developments

Definition (Z. Tu and Y. Deng - IACR ePrint archive)

$$(x,y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n};\ f^{\#}(x,y) = f(xy^{2^n-2}) = f\left(\frac{x}{y}\right),\ \text{with } \frac{x}{0} = 0.$$

Theorem (Z. Tu and Y. Deng) *up to a conjecture*

The function $f^{\#}$ has optimal algebraic immunity n.

Proof (sketch) :

Let $h(x,y) = \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} a_{i,j}\ x^i y^j$ be a non-zero annihilator of $f^{\#}+1$ with $\max\{w_2(i)+w_2(j);\ a_{i,j} \neq 0\} \leq n-1$.

We have $h(\gamma\, y, y) = 0$ for every $y \in \mathbb{F}_{2^n}^*$ and every $\gamma \in \{\alpha^{2^{n-1}-1}, \cdots, \alpha^{2^n-2}\}$.

For every $y \in \mathbb{F}_{2^n}^*$, $h(\gamma\, y, y)$ equals :

$$\sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} a_{i,j}\ \gamma^i y^{i+j} = \sum_{t=0}^{2^n-2} \left(\sum_{i=0}^{2^n-2} a_{i,t-i}\ \gamma^i \right) y^t,$$

where $t-i$ is taken modulo 2^n-1.

Hence $\sum_{i=0}^{2^n-2} a_{i,t-i}\,\gamma^i = 0$ for every t.

The BCH bound implies then, for every t, that :
- either $a_{i,t-i} = 0$ for every i,
- or $card\,\{i \,/\, a_{i,t-i} \neq 0\} \geq 2^{n-1} + 1$.

Conjecture (checked by Z. Tu and Y. Deng til $n = 29$) :

$\forall n \geq 1, \forall t \in \mathbb{Z}/(2^n - 1)\mathbb{Z}$:

$$card\,\{i \in \mathbb{Z}/(2^n - 1)\mathbb{Z} \,|\, w_2(i) + w_2(t - i) \leq n - 1\} \leq 2^{n-1}.$$

Nonlinearity :

$$nl(f^{\#}) = 2^{2n-1} - 2^{n-1}$$

($f^{\#}$ has best possible nonlinearity ; it is bent).

But this function has (low) degree n and it is not balanced.

A balanced version of f :

$$f^{\#'}(x,y) = \begin{cases} f\left(\frac{x}{y}\right) \text{ if } y \neq 0 \\ f(x) \text{ if } y = 0 \end{cases}$$

This function has optimal algebraic immunity as well and is balanced. Its degree equals $2n-1$ and $nl(f^{\#'}) \geq 2^{2n-1} - 2^{n-1} - n\,2^{n/2}\ln 2 - 1$.

But (C.C. ePrint Archive) $f^{\#'}$ differs from $f^{\#}$ only when $x = 0$.

Hence for every linear Boolean function ℓ over $\mathbb{F}_{2^n}$, the function :

$$\ell(x)f^{\#'}(x,y) = \ell(x)f^{\#}(x,y)$$

has algebraic degree at most $n+1$.
This is almost the worst case for the resistance to FAA of a $2n$-variable function of algebraic immunity n.

Trying to repair :

Let $f^{\#''}(x,y) = f(x/y) + 1_E(x,y)$ where

$$E = \{(0, u_\emptyset)\} \cup \left\{(\alpha^i u_i, u_i);\ i \in \{2^{n-1}, \cdots, 2^n - 2\}\right\}; \quad u_i \in \mathbb{F}_{2^n}^*$$

where $< e + E >= \mathbb{F}_{2^n}^2$, for every vector e.
- $f^{\#''}(x,y)$ is balanced.
- It has also optimal AI :
let $h(x,y) = \sum_{i=0}^{2^n-2} \sum_{j=0}^{2^n-2} a_{i,j}\ x^i y^j$ be a non-zero annihilator of $f^{\#''} + 1$ with $\max\{w_2(i) + w_2(j);\ a_{i,j} \neq 0\} \leq n - 1$.

For every $y \in \mathbb{F}_{2^n}^*$, we have $h(0, y) = 0$ except maybe for $y = u_\emptyset$ and $h(\gamma\, y, y) = 0, \forall \gamma = \alpha^i, i \in \{2^{n-1}, \cdots, 2^n - 2\}$, except maybe for $y = u_i$. The rest of the proof is similar as for $f^{\#'}$.

The nonlinearity of $f^{\#''}$ satisfies $nl(f^{\#''}) \geq 2^{2n-1} - 2^n$.

Computer investigations show that $f^{\#''}$ can have an optimal algebraic degree $2n - 1$ and behave well against FAA. But this last fact seems true only for small values of n, unfortunately.

Conclusion

There exists only one infinite class of functions which potentially satisfies all the necessary criteria for being used as a filter function.

But proving its good behavior is a twofold open problem.

Finding such proof or discovering new classes provably satisfying all the necessary criteria is vital for the future of the filter model.

Announcement : Next SETA conference

Sequences and Their Applications

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PC co-chairs : A. Pott ; C. Carlet

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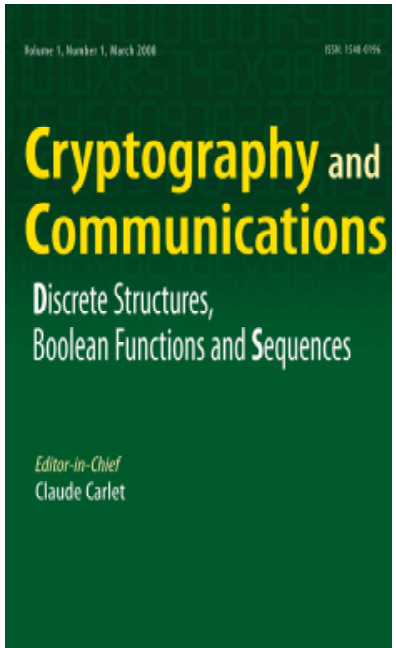

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