Additive Asymmetric Quantum Codes

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***Abstract*—We present a general construction of asymmetric quantum codes based on additive codes under the trace Hermitian inner product. Various families of additive codes over $\mathbb{F}_4$ are used in the construction of many asymmetric quantum codes over $\mathbb{F}_4$.**

***Index Terms*—4-circulant codes, additive codes, BCH codes, circulant codes, extremal codes, MacDonald codes, nested codes, quantum codes, quantum Singleton bound, self-orthogonal codes.**

I. Introduction

PREVIOUSLY, most of the works on quantum error-correcting codes were done with the assumption that the channel is symmetric. That is, the various error types were taken to be equiprobable. To be brief, the term *quantum codes* or QECC is henceforth used to refer to quantum error-correcting codes.

Recently, it has been established that, in many quantum mechanical systems, the phase-flip errors happen more frequently than the bit-flip errors or the combined bit-phase flip errors. For more details, [32] can be consulted.

There is a need to design quantum codes that take advantage of this asymmetry in quantum channels. We call such codes asymmetric quantum codes. We require the codes to correct many phase-flip errors but not necessarily the same number of bit-flip errors.

In this paper we extend the construction of asymmetric quantum codes in [35] to include codes derived from classical additive codes under the trace Hermitian inner product.

This work is organized as follows. In Section II, we state some basic definitions and properties of linear and additive codes. Section III provides an introduction to quantum error-correcting codes in general, differentiating the symmetric and the asymmetric cases. In Section IV, a construction of asymmetric QECC based on additive codes is presented.

The rest of the paper focuses on additive codes over $\mathbb{F}_4$. Section V recalls briefly important known facts regarding these codes. A construction of asymmetric QECC from extremal or optimal self-dual additive codes is given in Section VI. A construction from Hermitian self-orthogonal $\mathbb{F}_4$-linear codes is the topic of Section VII. Sections VIII and IX use nested $\mathbb{F}_4$-linear cyclic codes for lengths $n \leq 25$ and nested BCH codes for lengths $27 \leq n \leq 51$, respectively, in the construction. New or better asymmetric quantum codes constructed from nested additive codes over $\mathbb{F}_4$ are presented in Section X, exhibiting the gain of extending the construction to include additive codes. Section XI provides conclusions and some open problems.

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II. Preliminaries

Let p be a prime and $q = p^f$ for some positive integer f. An $[n, k, d]_q$-linear code C of length n, dimension k, and minimum distance d is a subspace of dimension k of the vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q = GF(q)$ with q elements. For a general, not necessarily linear, code C, the notation $(n, M = |C|, d)_q$ is commonly used.

The *Hamming weight* of a vector or a codeword $\mathbf{v}$ in a code C, denoted by $\mathrm{wt}_H(\mathbf{v})$, is the number of its nonzero entries. Given two elements $\mathbf{u}, \mathbf{v} \in C$, the number of positions where their respective entries disagree, written as $\mathrm{dist}_H(\mathbf{u}, \mathbf{v})$, is called the *Hamming distance* of $\mathbf{u}$ and $\mathbf{v}$. For any code C, the *minimum distance* $d = d(C)$ is given by $d = d(C) = \min\{\mathrm{dist}_H(\mathbf{u}, \mathbf{v}) : \mathbf{u}, \mathbf{v} \in C, \mathbf{u} \neq \mathbf{v}\}$. If C is linear, then its closure property implies that $d(C)$ is given by the minimum Hamming weight of nonzero vectors in C.

We follow [30] in defining the following three families of codes according to their duality types.

Definition 2.1: Let $q = r^2 = p^f$ be an even power of an arbitrary prime p with $\overline{x} = x^r$ for $x \in \mathbb{F}_q$. Let n be a positive integer and $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$.

1) $\mathbf{q}^{\mathrm{H}}$ is the family of $\mathbb{F}_q$-linear codes of length n with the Hermitian inner product
$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathrm{H}} := \sum_{i=1}^{n} u_i \cdot v_i^{\sqrt{q}}. \tag{II.1}$$
2) $\mathbf{q}^{\mathrm{H}+}$ **(even)** is the family of trace Hermitian codes over $\mathbb{F}_q$ of length n which are $\mathbb{F}_r$-linear, where $r^2 = q$ is even. The duality is defined according to the trace Hermitian inner product
$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathrm{tr}} := \sum_{i=1}^{n} (u_i \cdot v_i^{\sqrt{q}} + u_i^{\sqrt{q}} \cdot v_i). \tag{II.2}$$
3) $\mathbf{q}^{\mathrm{H}+}$ **(odd)** is the family of trace Hermitian codes over $\mathbb{F}_q$ of length n which are $\mathbb{F}_r$-linear, where $r^2 = q$ is odd. The duality is defined according to the following inner product, which we will still call trace Hermitian inner product,
$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathrm{tr}} := \alpha \cdot \sum_{i=1}^{n} (u_i \cdot v_i^{\sqrt{q}} - u_i^{\sqrt{q}} \cdot v_i) \tag{II.3}$$
where $\alpha \in \mathbb{F}_q \setminus \{0\}$ with $\alpha^r = -\alpha$.

Definition 2.2: A code C of length n is said to be a (classical) additive code if C belongs to either the family $q^{\mathrm{H}+}$ (even) or to the family $q^{\mathrm{H}+}$ (odd).

Let C be a code. Under a chosen inner product $*$, the *dual code* $C^{\perp_*}$ of C is given by

$$C^{\perp_*} := \left\{\mathbf{u} \in \mathbb{F}_q^n : \langle \mathbf{u}, \mathbf{v}\rangle_* = 0 \text{ for all } \mathbf{v} \in C\right\}.$$

Accordingly, for a code C in the family (q^{H})

$$C^{\perp_{\mathrm{H}}} := \left\{\mathbf{u} \in \mathbb{F}_q^n : \langle \mathbf{u}, \mathbf{v}\rangle_{\mathrm{H}} = 0 \text{ for all } \mathbf{v} \in C\right\}$$

and, for a code C in the family $(q^{\mathrm{H}+})$ (even) or $(q^{\mathrm{H}+})$ (odd)

$$C^{\perp_{\mathrm{tr}}} := \left\{\mathbf{u} \in \mathbb{F}_q^n : \langle \mathbf{u}, \mathbf{v}\rangle_{\mathrm{tr}} = 0 \text{ for all } \mathbf{v} \in C\right\}.$$

A code is said to be *self-orthogonal* if it is contained in its dual and is said to be *self-dual* if its dual is itself. We say that a family of codes is *closed* if $(C^{\perp_*})^{\perp_*} = C$ for each C in that family. It has been established [30, Ch. 3] that the three families of codes in Definition 2.1 are closed.

The weight distribution of a code and that of its dual are important in the studies of their properties.

Definition 2.3: The *weight enumerator* $W_C(X,Y)$ of an $(n, M = |C|, d)_q$-code C is the polynomial

$$W_C(X,Y) = \sum_{i=0}^{n} A_i X^{n-i} Y^i, \tag{II.4}$$

where A_i is the number of codewords of weight i in the code C.

The weight enumerator of the Hermitian dual code $C^{\perp_{\mathrm{H}}}$ of an $[n,k,d]_q$-code C is connected to the weight enumerator of the code C via the MacWilliams Equation

$$W_{C^{\perp_{\mathrm{H}}}}(X,Y) = \frac{1}{|C|} W_C(X + (q-1)Y, X - Y). \tag{II.5}$$

In the case of nonlinear codes, we can define a similar notion called the *distance distribution*. The MacWilliams Equation can be generalized to the nonlinear cases as well (see [29, Ch. 5]). From [30, Sect. 2.3] we know that the families $q^{\mathrm{H}+}$ (even) and $q^{\mathrm{H}+}$ (odd) have the same MacWilliams Equation as the family q^{H}. Thus, (II.5) applies to all three families.

Classical codes are connected to many other combinatorial structures. One such structure is the orthogonal array.

Definition 2.4: Let S be a set of q symbols or levels. An orthogonal array A with M runs, n factors, q levels and strength t with index λ, denoted by $OA(M,n,q,t)$, is an $M \times n$ array A with entries from S such that every $M \times t$ subarray of A contains each t-tuple of S^t exactly $\lambda = \frac{M}{q^t}$ times as a row.

The parameter λ is usually not written explicitly in the notation since its value depends on M, q and t. The rows of an orthogonal array are distinct since the purpose of its construction is to minimize the number of runs in the experiment while keeping some required conditions satisfied.

There is a natural correspondence between codes and orthogonal arrays. The codewords in a code C can be seen as the rows of an orthogonal array A and vice versa. The following proposition due to Delsarte (see [14, Th. 4.5]) will be useful in the sequel. Note that the code C in the proposition is a general code. No linearity is required. The duality here is defined over any inner product. For more on how the dual distance is defined for nonlinear codes, we refer to [22, Sec. 4.4].

Proposition 2.5: [22, Th. 4.9] If C is an $(n, M = |C|, d)_q$ code with dual distance $d^{\perp}$, then the corresponding orthogonal array is an $OA(M,n,q,d^{\perp}-1)$. Conversely, the code corresponding to an $OA(M,n,q,t)$ is an $(n,M,d)_q$ code with dual distance $d^{\perp} \geq t+1$. If the orthogonal array has strength t but not $t+1$, then $d^{\perp}$ is precisely $t+1$.

III. Quantum Codes

We assume that the reader is familiar with the standard error model in quantum error-correction. The essentials can be found, for instance, in [2] and in [15]. For convenience, some basic definitions and results are reproduced here.

Let $\mathbb{C}$ be the field of complex numbers and $\eta = e^{\frac{2\pi\sqrt{-1}}{p}} \in \mathbb{C}$. We fix an orthonormal basis of $\mathbb{C}^q$

$$\{|v\rangle : v \in \mathbb{F}_q\}$$

with respect to the Hermitian inner product. For a positive integer n, let $V_n = (\mathbb{C}^q)^{\otimes n}$ be the n-fold tensor product of $\mathbb{C}^q$. Then V_n has the following orthonormal basis:

$$\left\{|\mathbf{c}\rangle = |c_1 c_2 \ldots c_n\rangle : \mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{F}_q^n\right\} \tag{III.1}$$

where $|c_1 c_2 \ldots c_n\rangle$ abbreviates $|c_1\rangle \otimes |c_2\rangle \otimes \cdots \otimes |c_n\rangle$.

For two quantum states $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ in V_n with

$$|\mathbf{u}\rangle = \sum_{\mathbf{c}\in\mathbb{F}_q^n} \alpha(\mathbf{c})|\mathbf{c}\rangle, \quad |\mathbf{v}\rangle = \sum_{\mathbf{c}\in\mathbb{F}_q^n} \beta(\mathbf{c})|\mathbf{c}\rangle \quad (\alpha(\mathbf{c}), \beta(\mathbf{c}) \in \mathbb{C})$$

the Hermitian inner product of $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ is

$$\langle \mathbf{u}|\mathbf{v}\rangle = \sum_{\mathbf{c}\in\mathbb{F}_q^n} \overline{\alpha(\mathbf{c})}\beta(\mathbf{c}) \in \mathbb{C}$$

where $\overline{\alpha(\mathbf{c})}$ is the complex conjugate of $\alpha(\mathbf{c})$. We say $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ are *orthogonal* if $\langle \mathbf{u}|\mathbf{v}\rangle = 0$.

A quantum error acting on V_n is a unitary linear operator on V_n and has the following form

$$e = X(\mathbf{a})Z(\mathbf{b})$$

with $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{F}_q^n$.

The action of e on the basis (III.1) of V_n is

$$e|\mathbf{c}\rangle = X(a_1)Z(b_1)|c_1\rangle \otimes \cdots \otimes X(a_n)Z(b_n)|c_n\rangle$$

where

$$X(a_i)|c_i\rangle = |a_i + c_i\rangle, \quad Z(b_i)|c_i\rangle = \eta^{T(b_i c_i)}|c_i\rangle$$

with $T: \mathbb{F}_q \to \mathbb{F}_p$ being the trace mapping

$$T(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{m-1}}$$

for $q = p^m$. Therefore

$$e|\mathbf{c}\rangle = \eta^{T(\mathbf{b}\cdot\mathbf{c})}|\,\mathrm{a}+\mathbf{c}\rangle$$

where $\mathbf{b}\cdot\mathbf{c} = \sum_{i=1}^{n} b_i c_i \in \mathbb{F}_q$ is the usual inner product in $\mathbb{F}_q^n$.

For $e = X(\mathbf{a})Z(\mathbf{b})$ and $e' = X(\mathbf{a}')Z(\mathbf{b}')$ with $\mathbf{a}, \mathbf{b}$, and $\mathbf{a}', \mathbf{b}' \in \mathbb{F}_q^n$

$$ee' = \eta^{T(\mathbf{a}\cdot\mathbf{b}'-\mathbf{a}'\cdot\mathbf{b})}e'e.$$

Hence, the set

$$E_n = \left\{\eta^\lambda X(\mathbf{a})Z(\mathbf{b}) | 0 \le \lambda \le p-1, \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n\right\}$$

forms a (nonabelian) group, called the *error group* on V_n.

Definition 3.1: For a quantum error $e = \eta^\lambda X(\mathbf{a})Z(\mathbf{b}) \in E_n$, we define the *quantum weight* $w_Q(e)$, the X*-weight* $w_X(e)$ and the Z*-weight* $w_Z(e)$ of e by

$$\begin{aligned} w_Q(e) &= |\{i : 1 \le i \le n, (a_i, b_i) \ne (0,0)\}| \\ w_X(e) &= |\{i : 1 \le i \le n, a_i \ne 0\}| \\ w_Z(e) &= |\{i : 1 \le i \le n, b_i \ne 0\}|. \end{aligned}$$

Thus, $w_Q(e)$ is the number of qudits where the action of e is nontrivial by $X(a_i)Z(b_i) \ne I$ (identity) while $w_X(e)$ and $w_Z(e)$ are, respectively, the numbers of qudits where the X-action and the Z-action of e are nontrivial. We are now ready to define the distinction between symmetric and asymmetric quantum codes.

Definition 3.2: A q-ary quantum code of length n is a subspace Q of V_n with dimension $K \ge 1$. A quantum code Q of dimension $K \ge 2$ is said to detect $d-1$ qudits of errors for $d \ge 1$ if, for every orthogonal pair $|\mathbf{u}\rangle, |\mathbf{v}\rangle$ in Q with $\langle\mathbf{u}|\mathbf{v}\rangle = 0$ and every $e \in E_n$ with $w_Q(e) \le d-1$, $|\mathbf{u}\rangle$ and $e|\mathbf{v}\rangle$ are orthogonal. In this case, we call Q a *symmetric* quantum code with parameters $((n, K, d))_q$ or $[[n, k, d]]_q$, where $k = \log_q K$. Such a quantum code is called *pure* if $\langle\mathbf{u}|e|\mathbf{v}\rangle = 0$ for any $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ in Q and any $e \in E_n$ with $1 \le w_Q(e) \le d-1$. A quantum code Q with $K = 1$ is assumed to be pure.

Let d_x and d_z be positive integers. A quantum code Q in V_n with dimension $K \ge 2$ is called an asymmetric quantum code with parameters $((n, K, d_z/d_x))_q$ or $[[n, k, d_z/d_x]]_q$, where $k = \log_q K$, if Q detects $d_x - 1$ qudits of X-errors and, at the same time, $d_z - 1$ qudits of Z-errors. That is, if $\langle\mathbf{u}|\mathbf{v}\rangle = 0$ for $|\mathbf{u}\rangle, |\mathbf{v}\rangle \in Q$, then $\langle\mathbf{u}|e|\mathbf{v}\rangle = 0$ for any $e \in E_n$ such that $w_X(e) \le d_x - 1$ and $w_Z(e) \le d_z - 1$. Such an asymmetric quantum code Q is called *pure* if $\langle\mathbf{u}|e|\mathbf{v}\rangle = 0$ for any $|\mathbf{u}\rangle, |\mathbf{v}\rangle \in Q$ and $e \in E_n$ such that $1 \le w_X(e) \le d_x - 1$ and $1 \le w_Z(e) \le d_z - 1$. An asymmetric quantum code Q with $K = 1$ is assumed to be pure.

Remark 3.3: An asymmetric quantum code with parameters $((n, K, d/d))_q$ is a symmetric quantum code with parameters $((n, K, d))_q$, but the converse is not true since, for $e \in E_n$ with $w_X(e) \le d-1$ and $w_Z(e) \le d-1$, the weight $w_Q(e)$ may be bigger than $d-1$.

Given any two codes C and D, let the notation $\mathrm{wt}(C \setminus D)$ denote $\min\{\mathrm{wt}_H(\mathbf{u}(\ne \mathbf{0})) : \mathbf{u} \in (C \setminus D)\}$. The analogue of the well-known CSS construction (see [11]) for the asymmetric case is known.

Proposition 3.4: [32, Lemma 3.1] Let C_x, C_z be linear codes over $\mathbb{F}_q^n$ with parameters $[n, k_x]_q$, and $[n, k_z]_q$ respectively. Let $C_x^\perp \subseteq C_z$. Then there exists an $[[n, k_x + k_z - n, d_z/d_x]]_q$ asymmetric quantum code, where $d_x = \mathrm{wt}(C_x \setminus C_z^\perp)$ and $d_z = \mathrm{wt}(C_z \setminus C_x^\perp)$.

The resulting code is said to be *pure* if, in the above construction, $d_x = d(C_x)$ and $d_z = d(C_z)$.

IV. Asymmetric QECC From Additive Codes

The following result has been established recently:

Theorem 4.1: [35, Th. 3.1]

1) There exists an asymmetric quantum code with parameters $((n, K, d_z/d_x))_q$ with $K \ge 2$ if and only if there exist K nonzero mappings

$$\varphi_i : \mathbb{F}_q^n \to \mathbb{C} \text{ for } 1 \le i \le K \tag{IV.1}$$

satisfying the following conditions: for each d such that $1 \le d \le \min\{d_x, d_z\}$ and partition of $\{1, 2, \ldots, n\}$,

$$\begin{cases} \{1, 2, \ldots, n\} = A \cup X \cup Z \cup B, \\ |A| = d-1, \quad |B| = n + d - d_x - d_z + 1, \\ |X| = d_x - d, \quad |Z| = d_z - d, \end{cases} \tag{IV.2}$$

and each $\mathbf{c}_A, \mathbf{c}'_A \in \mathbb{F}_q^{|A|}$, $\mathbf{c}_Z \in \mathbb{F}_q^{|Z|}$ and $\mathbf{a}_X \in \mathbb{F}_q^{|X|}$, we have the equality

$$\sum_{\substack{\mathbf{c}_X \in \mathbb{F}_q^{|X|}, \\ \mathbf{c}_B \in \mathbb{F}_q^{|B|}}} \overline{\varphi_i(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B)}\varphi_j(\mathbf{c}'_A, \mathbf{c}_X - \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B) = \begin{cases} 0 & \text{for } i \ne j \\ I(\mathbf{c}_A, \mathbf{c}'_A, \mathbf{c}_Z, \mathbf{a}_X) & \text{for } i = j \end{cases} \tag{IV.3}$$

where $I(\mathbf{c}_A, \mathbf{c}'_A, \mathbf{c}_Z, \mathbf{a}_X)$ is an element of $\mathbb{C}$ which is independent of i. The notation $(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B)$ represents the rearrangement of the entries of the vector $\mathbf{c} \in \mathbb{F}_q^n$ according to the partition of $\{1, 2, \ldots, n\}$ given in (IV.2).

2) Let (φ_i, φ_j) stand for $\sum_{\mathbf{c} \in \mathbb{F}_q^n} \overline{\varphi_i(\mathbf{c})}\varphi_j(\mathbf{c})$. There exists a pure asymmetric quantum code with parameters $((n, K \ge 1, d_z/d_x))_q$ if and only if there exist K nonzero mappings φ_i as shown in (IV.1) such that
   - φ_i are linearly independent for $1 \le i \le K$, i.e., the rank of the $K \times q^n$ matrix $(\varphi_i(\mathbf{c}))_{1 \le i \le K, \mathbf{c} \in \mathbb{F}_q^n}$ is K; and
   - for each d with $1 \le d \le \min\{d_x, d_z\}$, a partition in (IV.2) and $\mathbf{c}_A, \mathbf{a}_A \in \mathbb{F}_q^{|A|}, \mathbf{c}_Z \in \mathbb{F}_q^{|Z|}$ and $\mathbf{a}_X \in \mathbb{F}_q^{|X|}$, we have the equality

$$\sum_{\substack{\mathbf{c}_X \in \mathbb{F}_q^{|X|}, \\ \mathbf{c}_B \in \mathbb{F}_q^{|B|}}} \overline{\varphi_i(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B)}\varphi_j(\mathbf{c}_A + \mathbf{a}_A, \mathbf{c}_X + \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B) = \begin{cases} 0 & \text{for } (\mathbf{a}_A,\ \mathbf{a}_X) \ne (\mathbf{0},\ \mathbf{0}) \\ \frac{(\varphi_i, \varphi_j)}{q^{d_z - 1}} & \text{for } (\mathbf{a}_A,\ \mathbf{a}_X) = (\mathbf{0},\ \mathbf{0}). \end{cases} \tag{IV.4}$$

The following result is due to Keqin Feng and Long Wang. It has, however, never appeared formally in a published form before. Since it will be needed in the sequel, we present it here with a proof.

Proposition 4.2: (K. Feng and L. Wang) Let a, b be positive integers. There exists an asymmetric quantum code Q with parameters $((n, K, a/b))_q$ if and only if there exists an asymmetric quantum code Q' with parameters $((n, K, b/a))_q$. Q' is pure if and only if Q is pure.

Proof: We begin by assuming the existence of an $((n, K, a/b))_q$ asymmetric quantum code Q. Let φ_i with $1 \leq i \leq K$ be the K mappings given in Theorem 4.1. Define the following K mappings

$$\begin{aligned}\Phi_i: \quad & \mathbb{F}_q^n \to \mathbb{C} \text{ for } 1 \leq i \leq K \\ & \mathbf{v} \mapsto \sum_{\mathbf{c} \in \mathbb{F}_q^n} \varphi_i(\mathbf{c}) \eta^{T(\mathbf{c} \cdot \mathbf{v})}.\end{aligned} \tag{IV.5}$$

Let $\mathbf{v}_A, \mathbf{b}_A \in \mathbb{F}_q^{|A|}, \mathbf{v}_X \in \mathbb{F}_q^{|X|}$, and $\mathbf{b}_Z \in \mathbb{F}_q^{|Z|}$. For each d such that $1 \leq d \leq \min\{d_x, d_z\}$ and a partition of $\{1, 2, \ldots, n\}$ given in (IV.2), we show that

$$\begin{aligned} S = & \sum_{\substack{\mathbf{v}_Z \in \mathbb{F}_q^{|Z|}, \\ \mathbf{v}_B \in \mathbb{F}_q^{|B|}}} \overline{\Phi_i(\mathbf{v})} \Phi_j(\mathbf{v}_A + \mathbf{b}_A, \mathbf{v}_X, \mathbf{v}_Z + \mathbf{b}_Z, \mathbf{v}_B) \\ & = \begin{cases} 0 & \text{for } i \neq j \\ I'(\mathbf{v}_A, \mathbf{b}_A, \mathbf{b}_Z, \mathbf{v}_X) & \text{for } i = j, \end{cases} \end{aligned} \tag{IV.6}$$

where $I'(\mathbf{v}_A, \mathbf{b}_A, \mathbf{b}_Z, \mathbf{v}_X)$ is an element of $\mathbb{C}$ which is independent of i.

Let $\mathbf{t} = (\mathbf{v}_A + \mathbf{b}_A, \mathbf{v}_X, \mathbf{v}_Z + \mathbf{b}_Z, \mathbf{v}_B)$. Applying (IV.5) yields

$$S = \sum_{\substack{\mathbf{v}_Z \in \mathbb{F}_q^{|Z|}, \\ \mathbf{v}_B \in \mathbb{F}_q^{|B|}}} \sum_{\mathbf{c}, \mathbf{d} \in \mathbb{F}_q^n} \overline{\varphi_i(\mathbf{c})} \varphi_j(\mathbf{d}) \eta^{T((-\mathbf{c} \cdot \mathbf{v}) + (\mathbf{d} \cdot \mathbf{t}))}. \tag{IV.7}$$

By carefully rearranging the summations and grouping the terms, we get

$$S = \sum_{\mathbf{c}, \mathbf{d} \in \mathbb{F}_q^n} \overline{\varphi_i(\mathbf{c})} \varphi_j(\mathbf{d}) \cdot \kappa \cdot \lambda \tag{IV.8}$$

where

$$\begin{aligned}\kappa &= \eta^{T(\mathbf{v}_A \cdot (\mathbf{d}_A - \mathbf{c}_A) + \mathbf{v}_X \cdot (\mathbf{d}_X - \mathbf{c}_X) + \mathbf{d}_A \cdot \mathbf{b}_A + \mathbf{d}_Z \cdot \mathbf{b}_Z)} \\ \lambda &= \sum_{\substack{\mathbf{v}_Z \in \mathbb{F}_q^{|Z|}, \\ \mathbf{v}_B \in \mathbb{F}_q^{|B|}}} \eta^{T(\mathbf{v}_B \cdot (\mathbf{d}_B - \mathbf{c}_B) + \mathbf{v}_Z \cdot (\mathbf{d}_Z - \mathbf{c}_Z))}.\end{aligned}$$

By orthogonality of characters

$$\lambda = \begin{cases} q^{|Z| + |B|} & \text{if } \mathbf{d}_B = \mathbf{c}_B \text{ and } \mathbf{d}_Z = \mathbf{c}_Z \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$S = \sum_{\substack{\mathbf{c} \in \mathbb{F}_q^n \\ \mathbf{d}_A \in \mathbb{F}_q^{|A|}, \mathbf{d}_X \in \mathbb{F}_q^{|X|}}} q^{|Z| + |B|} \cdot \overline{\varphi_i(\mathbf{c})} \varphi_j(\mathbf{d}_A, \mathbf{d}_X, \mathbf{c}_Z, \mathbf{c}_B) \cdot \pi \tag{IV.9}$$

where

$$\pi = \eta^{T(\mathbf{v}_A \cdot (\mathbf{d}_A - \mathbf{c}_A) + \mathbf{v}_X \cdot (\mathbf{d}_X - \mathbf{c}_X) + \mathbf{d}_A \cdot \mathbf{b}_A + \mathbf{c}_Z \cdot \mathbf{b}_Z)}.$$

Now, we let $k = n - d_x + 1, \mathbf{a}_A = \mathbf{d}_A - \mathbf{c}_A$, and $\mathbf{a}_X = \mathbf{d}_X - \mathbf{c}_X$. Splitting up the summation once again yields

$$\begin{aligned} S = q^k & \sum_{\substack{\mathbf{c}_A, \mathbf{a}_A \in \mathbb{F}_q^{|A|} \\ \mathbf{c}_Z \in \mathbb{F}_q^{|Z|}, \mathbf{a}_X \in \mathbb{F}_q^{|X|}}} \eta^{T(\mathbf{v}_A \cdot \mathbf{a}_A + \mathbf{v}_X \cdot \mathbf{a}_X + \mathbf{b}_A \cdot (\mathbf{c}_A + \mathbf{a}_A) + \mathbf{c}_Z \cdot \mathbf{b}_Z)} \\ & \cdot \sum_{\substack{\mathbf{c}_X \in \mathbb{F}_q^{|X|}, \\ \mathbf{c}_B \in \mathbb{F}_q^{|B|}}} \overline{\varphi_i(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B)} \varphi_j(\mathbf{c}_A + \mathbf{a}_A, \mathbf{c}_X + \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B). \end{aligned} \tag{IV.10}$$

Invoking (IV.3) concludes the proof for the first part with I' given by

$$I' = q^k I \sum_{\substack{\mathbf{c}_A, \mathbf{a}_A \in \mathbb{F}_q^{|A|} \\ \mathbf{c}_Z \in \mathbb{F}_q^{|Z|}, \mathbf{a}_X \in \mathbb{F}_q^{|X|}}} \eta^{T(\mathbf{v}_A \cdot \mathbf{a}_A + \mathbf{v}_X \cdot \mathbf{a}_X + \mathbf{b}_A \cdot (\mathbf{c}_A + \mathbf{a}_A) + \mathbf{c}_Z \cdot \mathbf{b}_Z)}. \tag{IV.11}$$

For the second part, let us assume the existence of a pure $((n, K, a/b))_q$ asymmetric quantum code Q. Note that the Fourier transformations Φ_i for $1 \leq i \leq K$ are linearly independent. We use (IV.10) and (IV.4) to establish the equality

$$\begin{aligned} S = & \sum_{\substack{\mathbf{v}_Z \in \mathbb{F}_q^{|Z|}, \\ \mathbf{v}_B \in \mathbb{F}_q^{|B|}}} \overline{\Phi_i(\mathbf{v})} \Phi_j(\mathbf{v}_A + \mathbf{b}_A, \mathbf{v}_X, \mathbf{v}_Z + \mathbf{b}_Z, \mathbf{v}_B) \\ & = \begin{cases} 0 & \text{for } (\mathbf{b}_A, \mathbf{b}_Z) \neq (\mathbf{0}, \mathbf{0}) \\ q^n \frac{(\varphi_i, \varphi_j)}{q^{d_x - 1}} & \text{for } (\mathbf{b}_A, \mathbf{b}_Z) = (\mathbf{0}, \mathbf{0}). \end{cases} \end{aligned} \tag{IV.12}$$

Consider the term

$$M := \sum_{\substack{\mathbf{c}_X \in \mathbb{F}_q^{|X|}, \\ \mathbf{c}_B \in \mathbb{F}_q^{|B|}}} \overline{\varphi_i(\mathbf{c})} \varphi_j(\mathbf{c}_A + \mathbf{a}_A, \mathbf{c}_X + \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B)$$

in (IV.10). By the purity assumption, for $(\mathbf{a}_A, \mathbf{a}_X) \neq (\mathbf{0}, \mathbf{0})$, $M = 0$. For $(\mathbf{a}_A, \mathbf{a}_X) = (\mathbf{0}, \mathbf{0})$, $M = \frac{(\varphi_i, \varphi_j)}{q^{d_z - 1}}$. Hence

$$S = q^k \sum_{\substack{\mathbf{c}_A \in \mathbb{F}_q^{|A|}, \\ \mathbf{c}_Z \in \mathbb{F}_q^{|Z|}}} \eta^{T(\mathbf{b}_A \cdot \mathbf{c}_A + \mathbf{b}_Z \cdot \mathbf{c}_Z)} \cdot \frac{(\varphi_i, \varphi_j)}{q^{d_z - 1}}. \tag{IV.13}$$

By orthogonality of characters, if $(\mathbf{b}_A, \mathbf{b}_Z) \neq (\mathbf{0}, \mathbf{0})$, then

$$\sum_{\substack{\mathbf{c}_A \in \mathbb{F}_q^{|A|}, \\ \mathbf{c}_Z \in \mathbb{F}_q^{|Z|}}} \eta^{T(\mathbf{b}_A \cdot \mathbf{c}_A + \mathbf{b}_Z \cdot \mathbf{c}_Z)} = 0$$

making $S = 0$. If $(\mathbf{b}_A, \mathbf{b}_Z) = (\mathbf{0}, \mathbf{0})$, then

$$S = q^k \cdot q^{|A| + |Z|} \cdot \frac{(\varphi_i, \varphi_j)}{q^{d_z - 1}}.$$

This completes the proof of the second part. ■

With this result, without loss of generality, $d_z \geq d_x$ is henceforth assumed.

Remark 4.3: If we examine closely the proof of Theorem 4.1 above as presented in Theorem 3.1 of [35], only the additive property (instead of linearity) is used. We will show that the conclusion of the theorem with an adjusted value for K still follows if we use classical additive codes instead of linear codes.

Theorem 4.4: Let d_x and d_z be positive integers. Let C be a classical additive code in $\mathbb{F}_q^n$. Assume that $d^{\perp_{\mathrm{tr}}} = d(C^{\perp_{\mathrm{tr}}})$ is the minimum distance of the dual code $C^{\perp_{\mathrm{tr}}}$ of C under the trace Hermitian inner product. For a set $V := \{\mathbf{v}_i : 1 \leq i \leq K\}$ of K distinct vectors in $\mathbb{F}_q^n$, let $d_v := \min\{\mathrm{wt}_H(\mathbf{v}_i - \mathbf{v}_j + \mathbf{c}) : 1 \leq i \neq j \leq K, \mathbf{c} \in C\}$. If $d^{\perp_{\mathrm{tr}}} \geq d_z$ and $d_v \geq d_x$, then there exists an asymmetric quantum code Q with parameters $((n, K, d_z/d_x))_q$.

Proof: Define the following functions:

$$\begin{aligned} \varphi_i : \quad & \mathbb{F}_q^n \to \mathbb{C} \text{ for } 1 \leq i \leq K \\ & \mathbf{u} \mapsto \begin{cases} 1 & \text{if } \mathbf{u} \in \mathbf{v}_i + C \\ 0 & \text{if } \mathbf{u} \notin \mathbf{v}_i + C. \end{cases} \end{aligned} \tag{IV.14}$$

For each d such that $1 \leq d \leq \min\{d_x, d_z\}$ and a partition of $\{1, 2, \ldots, n\}$ given in (IV.2),

$$\overline{\varphi_i(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B)}\varphi_j(\mathbf{c}_A + \mathbf{a}_A, \mathbf{c}_X + \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B) \neq 0$$

if and only if

$$\begin{cases} (\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B) & \in \mathbf{v}_i + C \\ (\mathbf{c}_A + \mathbf{a}_A, \mathbf{c}_X + \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B) & \in \mathbf{v}_j + C \end{cases}$$

which, in turn, is equivalent to

$$\begin{cases} (\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B) & \in \mathbf{v}_i + C \\ (\mathbf{a}_A, \mathbf{a}_X, \mathbf{0}_Z, \mathbf{0}_B) & \in \mathbf{v}_j - \mathbf{v}_i + C. \end{cases} \tag{IV.15}$$

Note that since $\mathrm{wt}_H(\mathbf{a}_A, \mathbf{a}_X, \mathbf{0}_Z, \mathbf{0}_B) \leq |A| + |X| = d_x - 1$, we know that $(\mathbf{a}_A, \mathbf{a}_X, \mathbf{0}_Z, \mathbf{0}_B) \in \mathbf{v}_j - \mathbf{v}_i + C$ means $i = j$ by the definition of d_v above. Thus, if $i \neq j$

$$\sum_{\substack{\mathbf{c}_X \in \mathbb{F}_q^{|X|} \\ \mathbf{c}_B \in \mathbb{F}_q^{|B|}}} \overline{\varphi_i(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B)}\varphi_j(\mathbf{c}_A + \mathbf{a}_A, \mathbf{c}_X + \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B) = 0. \tag{IV.16}$$

Now, consider the case of $i = j$. By (IV.15), if $(\mathbf{a}_A, \mathbf{a}_X, \mathbf{0}_Z, \mathbf{0}_B) \notin C$, then it has no contribution to the sum we are interested in. If $(\mathbf{a}_A, \mathbf{a}_X, \mathbf{0}_Z, \mathbf{0}_B) \in C$, then

$$\sum_{\substack{\mathbf{c}_X \in \mathbb{F}_q^{|X|} \\ \mathbf{c}_B \in \mathbb{F}_q^{|B|}}} \overline{\varphi_i(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B)}\varphi_i(\mathbf{c}_A + \mathbf{a}_A, \mathbf{c}_X + \mathbf{a}_X, \mathbf{c}_Z, \mathbf{c}_B) = \sum_{\substack{\mathbf{c}_X \in \mathbb{F}_q^{|X|}, \mathbf{c}_B \in \mathbb{F}_q^{|B|} \\ (\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B) \in \mathbf{v}_i + C}} 1. \tag{IV.17}$$

Proposition 2.5 above tells us that, if C is any classical q-ary code of length n and size M such that the minimum distance $d^{\perp}$ of its dual is greater than or equal to d_z, then any coset of C is an orthogonal array of level q and of strength exactly $d_z - 1$. In other words, there are exactly $\frac{|C|}{q^{d_z-1}}$ vectors $(\mathbf{c}_A, \mathbf{c}_X, \mathbf{c}_Z, \mathbf{c}_B) \in \mathbf{v}_i + C$ for any fixed $(\mathbf{c}_A, \mathbf{c}_Z) \in \mathbb{F}_q^{d_z-1}$. Thus, for $i = j$, the sum on the right hand side of (IV.17) is $\frac{|C|}{q^{d_z-1}}$, which is independent of i. By Theorem 4.1 we have an asymmetric quantum code Q with parameters $((n, K, d_z/d_x))_q$. ∎

Theorem 4.5: Let $q = r^2$ be an even power of a prime p. For $i = 1, 2$, let C_i be a classical additive code with parameters $(n, K_i, d_i)_q$. If $C_1^{\perp_{\mathrm{tr}}} \subseteq C_2$, then there exists an asymmetric quantum code Q with parameters $((n, \frac{|C_2|}{|C_1^{\perp_{\mathrm{tr}}}|}, d_z/d_x))_q$ where $\{d_z, d_x\} = \{d_1, d_2\}$.

Proof: We take $C = C_1^{\perp_{\mathrm{tr}}}$ in Theorem 4.4 above. Since $C_1^{\perp_{\mathrm{tr}}} \subseteq C_2$, we have $C_2 = C_1^{\perp_{\mathrm{tr}}} \oplus C'$, where C' is an $\mathbb{F}_r$-submodule of C_2 and $\oplus$ is the direct sum so that $|C'| = \frac{|C_2|}{|C_1^{\perp_{\mathrm{tr}}}|}$. Let $C' = \{\mathbf{v}_1, \ldots, \mathbf{v}_K\}$, where $K = \frac{|C_2|}{|C_1^{\perp_{\mathrm{tr}}}|}$. Then

$$\begin{aligned} d^{\perp_{\mathrm{tr}}} &= d(C^{\perp_{\mathrm{tr}}}) = d(C_1) = d_1 \text{ and} \\ d_v &= \min\{\mathrm{wt}_H(\mathbf{v}_i - \mathbf{v}_j + \mathbf{c}) : 1 \leq i \neq j \leq K, \mathbf{c} \in C\} \\ &= \min\left\{\mathrm{wt}_H(\mathbf{v} + \mathbf{c}) : \mathbf{0} \neq \mathbf{v} \in C', \mathbf{c} \in C_1^{\perp_{\mathrm{tr}}}\right\} \geq d_2. \end{aligned}$$

∎

Theorem 4.5 can now be used to construct quantum codes. In this paper, all computations are done in MAGMA [5] version V2.16-5.

The construction method of Theorem 4.5 falls into what some have labelled the *CSS-type construction*. It is noted in [32, Lemma 3.3] that any CSS-type $\mathbb{F}_q$-linear $[[n, k, d_z/d_x]]_q$-code satisfies the quantum version of the Singleton bound

$$k \leq n - d_x - d_z + 2.$$

This bound is conjectured to hold for all asymmetric quantum codes. Some of our codes in later sections attain $k = n - d_x - d_z + 2$. They are printed in boldface throughout the tables and examples.

V. Additive Codes Over $\mathbb{F}_4$

Let $\mathbb{F}_4 := \{0, 1, \omega, \omega^2 = \overline{\omega}\}$. For $x \in \mathbb{F}_4$, $\overline{x} = x^2$, the conjugate of x. By definition, an additive code C of length n over $\mathbb{F}_4$ is a free $\mathbb{F}_2$-module. It has size 2^l for some $0 \leq l \leq 2n$. As an $\mathbb{F}_2$-module, C has a basis consisting of l basis vectors. A *generator matrix* of C is an $l \times n$ matrix with entries elements of $\mathbb{F}_4$ whose rows form a basis of C.

Additive codes over $\mathbb{F}_4$ equipped with the trace Hermitian inner product have been studied primarily in connection to designs (e.g., [26]) and to stabilizer quantum codes (e.g., [16] and [24, Sec. 9.10]). It is well known that if C is an additive $(n, 2^l)_4$-code, then $C^{\perp_{\mathrm{tr}}}$ is an additive $(n, 2^{2n-l})_4$-code.

To compute the weight enumerator of $C^{\perp_{\mathrm{tr}}}$ we use (II.5) with $q = 4$

$$W_{C^{\perp_{\mathrm{tr}}}}(X, Y) = \frac{1}{|C|} W_C(X + 3Y, X - Y). \tag{V.1}$$

Remark 5.1: If the code C is $\mathbb{F}_4$-linear with parameters $[n, k, d]_4$, then $C^{\perp_{\mathrm{H}}} = C^{\perp_{\mathrm{tr}}}$. This is because $C^{\perp_{\mathrm{H}}}$ is of size $4^{n-k} = 2^{2n-2k}$ which is also the size of $C^{\perp_{\mathrm{tr}}}$. Alternatively, one can invoke [11, Th. 3].

TABLE I
BEST-KNOWN ADDITIVE SELF-DUAL CODES OVER $\mathbb{F}_4$ FOR $n \leq 30$ AND THE RESULTING ASYMMETRIC QUANTUM CODES

n	d_I	**num**$_I$	**Ref.**	**Code** Q	d_{II}	**num**$_{II}$	**Ref.**	**Code** Q	n	d_I	**num**$_I$	**Ref.**	**Code** Q
2	1^o	1	[13]	$[[2,0,1/1]]_4$	2^e	1	[13]	$[[\mathbf{2},\mathbf{0},\mathbf{2}/\mathbf{2}]]_\mathbf{4}$	3	2^e	1	[13]	$[[3,0,2/2]]_4$
4	2^e	1	[13]	$[[4,0,2/2]]_4$	2^e	2	[13]	$[[4,0,2/2]]_4$	5	3^e	1	[13]	$[[5,0,3/3]]_4$
6	3^e	1	[13]	$[[6,0,3/3]]_4$	4^e	1	[13]	$[[\mathbf{6},\mathbf{0},\mathbf{4}/\mathbf{4}]]_\mathbf{4}$	7	3^o	4	[13]	$[[7,0,3/3]]_4$
8	4^e	2	[13]	$[[8,0,4/4]]_4$	4^e	3	[13]	$[[8,0,4/4]]_4$	9	4^e	8	[13]	$[[9,0,4/4]]_4$
10	4^e	101	[13]	$[[10,0,4/4]]_4$	4^e	19	[13]	$[[10,0,4/4]]_4$	11	5^e	1	[13]	$[[11,0,5/5]]_4$
12	5^e	63	[13]	$[[12,0,5/5]]_4$	6^e	1	[13]	$[[12,0,6/6]]_4$	13	5^o	85845	[33]	$[[13,0,5/5]]_4$
14	6^e	2	[33]	$[[14,0,6/6]]_4$	6^e	1020	[13]	$[[14,0,6/6]]_4$	15	6^e	≥ 2118	[33]	$[[15,0,6/6]]_4$
16	6^e	≥ 8369	[33]	$[[16,0,6/6]]_4$	6^e	≥ 112	[33]	$[[16,0,6/6]]_4$	17	7^e	≥ 2	[33]	$[[17,0,7/7]]_4$
18	7^e	≥ 2	[33]	$[[18,0,7/7]]_4$	8^e	≥ 1	[33]	$[[18,0,8/8]]_4$	19	7^b	≥ 17	[33]	$[[19,0,7/7]]_4$
20	8^e	≥ 3	[19]	$[[20,0,8/8]]_4$	8^e	≥ 5	[19]	$[[20,0,8/8]]_4$	21	8^e	≥ 2	[33]	$[[21,0,8/8]]_4$
22	8^e	≥ 1	[19]	$[[22,0,8/8]]_4$	8^e	≥ 67	[19]	$[[22,0,8/8]]_4$	23	8^b	≥ 2	[19]	$[[23,0,8/8]]_4$
24	8^b	≥ 5	[34]	$[[24,0,8/8]]_4$	8^b	≥ 51	[19]	$[[24,0,8/8]]_4$	25	8^b	≥ 30	[19]	$[[25,0,8/8]]_4$
26	8^b	≥ 49	[34]	$[[26,0,8/8]]_4$	8^b	≥ 161	[34]	$[[26,0,8/8]]_4$	27	8^b	≥ 15	[19]	$[[27,0,9/9]]_4$
28	10	?	[19]		10^e	≥ 1	[19]	$[[28,0,10/10]]_4$	29	11^e	≥ 1	[19]	$[[29,0,11/11]]_4$
30	11	?	[19]		12^e	≥ 1	[19]	$[[30,0,12/12]]_4$					

From here on, we assume the trace Hermitian inner product whenever additive $\mathbb{F}_4$ codes are discussed and the Hermitian inner product whenever $\mathbb{F}_4$-linear codes are used.

Two additive codes C_1 and C_2 over $\mathbb{F}_4$ are said to be *equivalent* if there is a map sending the codewords of one code onto the codewords of the other where the map consists of a permutation of coordinates, followed by a scaling of coordinates by elements of $\mathbb{F}_4$, followed by a conjugation of the entries of some of the coordinates.

VI. CONSTRUCTION FROM EXTREMAL OR OPTIMAL ADDITIVE SELF-DUAL CODES OVER $\mathbb{F}_4$

As a direct consequence of Theorem 4.5, we have the following result.

Theorem 6.1: If C is an additive self-dual code of parameters $(n, 2^n, d)_4$, then there exists an $[[n, 0, d_z/d_x]]_4$ asymmetric quantum code Q with $d_z = d_x = d(C^{\perp_{\rm tr}})$.

Additive self-dual codes over $\mathbb{F}_4$ exist for any length n since the identity matrix I_n clearly generates a self-dual $(n, 2^n, 1)_4$-code. Any linear self-dual $[n, n/2, d]_4$-code is also an additive self-dual $(n, 2^n, d)_4$-code.

Definition 6.2: A self-dual $(n, 2^n, d)_4$-code C is *Type II* if all of its codewords have even weight. If C has a codeword of odd weight, then C is *Type I*.

It is known (see [31, Sec. 4.2]) that Type II codes of length n exist only if n is even and that a Type I code is not $\mathbb{F}_4$-linear. There is a bound in [31, Th. 33] on the minimum weight of an additive self-dual code. If d_I and d_{II} are the minimum weights of Type I and Type II codes of length n, respectively, then

$$d_I \leq \begin{cases} 2\lfloor \frac{n}{6} \rfloor + 1, & \text{if } n \equiv 0 \pmod 6 \\ 2\lfloor \frac{n}{6} \rfloor + 3, & \text{if } n \equiv 0 \pmod 6 \\ 2\lfloor \frac{n}{6} \rfloor + 2, & \text{otherwise} \end{cases}$$
$$d_{II} \leq 2\lfloor \frac{n}{6} \rfloor + 2. \qquad \text{(VI.1)}$$

A code that meets the appropriate bound is called *extremal*. If a code is not extremal yet no code of the given type can exist with a larger minimum weight, then we call the code *optimal*.

The complete classification, up to equivalence, of additive self-dual codes over $\mathbb{F}_4$ up to $n = 12$ can be found in [13]. The classification of extremal codes of lengths $n = 13$ and $n = 14$ is presented in [33]. Many examples of good additive codes for larger values of n are presented in [19], [33], and [34].

Table I summarizes the results thus far and lists down the resulting asymmetric quantum codes for lengths up to $n = 30$. The subscripts $_I$ and $_{II}$ indicates the types of the codes. The superscripts ^{e},^{o} ,^{b} indicate the fact that the minimum distance d is extremal, optimal, and best-known (not necessarily extremal or optimal), respectively. The number of codes for each set of given parameters is listed in the column under the heading **num**.

Remark 6.3:

1) The unique additive $(12, 2^{12}, 6)_4$-code is also known as *dodecacode*. It is well known that the best Hermitian self-dual linear code is of parameters $[12, 6, 4]_4$.
2) In [34], four so-called additive circulant graph codes of parameters $(30, 2^{30}, 12)_4$ are constructed without classification. It is yet unknown if any of these four codes is inequivalent to the one listed in [19].

VII. CONSTRUCTION FROM SELF-ORTHOGONAL LINEAR CODES

It is well known (see [24, Th. 1.4.10]) that a linear code C having the parameters $[n, k, d]_4$ is Hermitian self-orthogonal if and only if the weights of its codewords are all even.

Theorem 7.1: If C is a Hermitian self-orthogonal code of parameters $[n, k, d]_4$, then there exists an asymmetric quantum code Q with parameters $[[n, n-2k, d_z/d_x]]_4$, where

$$d_x = d_z = d(C^{\perp_{\rm H}}). \qquad \text{(VII.1)}$$

Proof: Seen as an additive code, C is of parameters $(n, 2^{2k}, d)_4$ with $C^{\perp_{\rm tr}}$ being the code $C^{\perp_{\rm H}}$ seen as an $(n, 2^{2n-2k}, d^{\perp_{\rm tr}})$ additive code (see Remark 5.1). Applying Theorem 4.5 by taking $C_1 = C^{\perp_{\rm tr}} = C_2$ to satisfy $C_1^{\perp_{\rm tr}} \subseteq C_2$ completes the proof. ■

Example 7.2: Let n be an even positive integer. Consider the repetition code $[n, 1, n]_4$ with weight enumerator $1+3Y^n$. Since the weights are all even, this $\mathbb{F}_4$-linear code is Hermitian self-

TABLE II
ASYMMETRIC QECC FROM CLASSIFIED HERMITIAN SELF-ORTHOGONAL $\mathbb{F}_4$-LINEAR CODES IN [9]

No.	Code C	Code Q	num
1	$[6,3,4]_4$	$[[6,0,3/3]]_4$	1
2	$[7,3,4]_4$	$[[7,1,3/3]]_4$	1
3	$[8,3,4]_4$	$[[8,2,3/3]]_4$	1
4	$[8,4,4]_4$	$[[8,0,4/4]]_4$	1
5	$[9,3,6]_4$	$[[9,3,3/3]]_4$	1
6	$[9,4,4]_4$	$[[9,1,3/3]]_4$	2
7	$[10,3,6]_4$	$[[10,4,3/3]]_4$	1
8	$[10,4,4]_4$	$[[10,2,3/3]]_4$	3
9	$[10,5,4]_4$	$[[10,0,4/4]]_4$	2
10	$[11,3,6]_4$	$[[11,5,3/3]]_4$	1
11	$[11,4,4]_4$	$[[11,3,3/3]]_4$	3
12	$[11,4,6]_4$	$[[11,3,3/3]]_4$	1
13	$[11,5,4]_4$	$[[11,1,3/3]]_4$	6
14	$[12,3,8]_4$	$[[12,6,3/3]]_4$	1
15	$[12,4,6]_4$	$[[12,4,4/4]]_4$	1
16	$[12,5,6]_4$	$[[12,2,4/4]]_4$	1
17	$[12,6,4]_4$	$[[12,0,4/4]]_4$	5
18	$[13,3,8]_4$	$[[13,7,3/3]]_4$	1
19	$[13,4,8]_4$	$[[13,5,3/3]]_4$	5
20	$[13,5,6]_4$	$[[13,3,4/4]]_4$	1
21	$[13,6,6]_4$	$[[13,1,5/5]]_4$	1
22	$[14,3,10]_4$	$[[14,8,3/3]]_4$	1
23	$[14,4,8]_4$	$[[14,6,4/4]]_4$	1
24	$[14,5,8]_4$	$[[14,4,4/4]]_4$	4
25	$[14,6,6]_4$	$[[14,2,5/5]]_4$	1
26	$[14,7,6]_4$	$[[14,0,6/6]]_4$	1
27	$[15,3,10]_4$	$[[15,9,3/3]]_4$	1
28	$[15,4,8]_4$	$[[15,7,3/3]]_4$	189
29	$[15,5,8]_4$	$[[15,5,4/4]]_4$	26
30	$[15,6,8]_4$	$[[15,3,5/5]]_4$	3
31	$[16,3,12]_4$	$[[16,10,3/3]]_4$	1
32	$[16,4,10]_4$	$[[16,8,3/3]]_4$	38
33	$[16,5,8]_4$	$[[16,6,4/4]]_4$	519
34	$[16,6,8]_4$	$[[16,4,4/4]]_4$	697
35	$[17,3,12]_4$	$[[17,11,2/2]]_4$	4
36	$[17,4,12]_4$	$[[17,9,4/4]]_4$	1
37	$[17,5,10]_4$	$[[17,7,4/4]]_4$	27
38	$[18,3,12]_4$	$[[18,12,2/2]]_4$	45
39	$[18,4,12]_4$	$[[18,10,3/3]]_4$	11
40	$[18,6,10]_4$	$[[18,6,5/5]]_4$	2
41	$[19,3,12]_4$	$[[19,13,2/2]]_4$	185
42	$[19,4,12]_4$	$[[19,11,3/3]]_4$	2570
43	$[20,3,14]_4$	$[[20,14,2/2]]_4$	10
44	$[20,5,12]_4$	$[[20,10,3/3]]_4$	4
45	$[21,3,16]_4$	$[[21,15,3/3]]_4$	1
46	$[21,4,14]_4$	$[[21,13,3/3]]_4$	212
47	$[21,5,12]_4$	$[[21,11,3/3]]_4$	3
48	$[22,3,16]_4$	$[[22,16,3/3]]_4$	4
49	$[22,5,14]_4$	$[[22,12,4/4]]_4$	67
50	$[23,3,16]_4$	$[[23,17,2/2]]_4$	46
51	$[23,4,16]_4$	$[[23,15,3/3]]_4$	1
52	$[24,3,16]_4$	$[[24,18,2/2]]_4$	614
53	$[24,4,16]_4$	$[[24,16,3/3]]_4$	20456
54	$[25,3,18]_4$	$[[25,19,2/2]]_4$	6
55	$[25,4,16]_4$	$[[25,17,3/3]]_4$	19
56	$[26,3,18]_4$	$[[26,20,2/2]]_4$	185
57	$[26,4,18]_4$	$[[26,18,3/3]]_4$	14
58	$[27,3,20]_4$	$[[27,21,2/2]]_4$	2
59	$[28,3,20]_4$	$[[28,22,2/2]]_4$	46
60	$[28,4,20]_4$	$[[28,20,3/3]]_4$	1
61	$[29,3,20]_4$	$[[29,23,2/2]]_4$	850
62	$[29,4,20]_4$	$[[29,21,3/3]]_4$	11365
63	$[30,5,20]_4$	$[[30,20,3/3]]_4$	≥ 90
64	$[31,4,22]_4$	$[[31,23,3/3]]_4$	1

orthogonal. We then have a quantum code Q with parameters $[[\mathbf{n},\mathbf{n}-\mathbf{2},\mathbf{2}/\mathbf{2}]]_\mathbf{4}$.

Table II presents the resulting asymmetric quantum codes based on the classification of self-orthogonal $\mathbb{F}_4$-linear codes of length up to 29 and of dimensions 3 up to 6 as presented in [9]. Bouyukliev [6] shared with us the original data used in the said classification plus some additional results for lengths 30 and 31.

TABLE III
ASYMMETRIC QECC FROM HERMITIAN SELF-DUAL $\mathbb{F}_4$-LINEAR CODES BASED ON [18, TABLE7] FOR $32 \leq n \leq 80$

Code C	Code Q	Code C	Code Q
$[32,16,10]_4$	$[[32,0,10/10]]_4$	$[58,29,14]_4$	$[[58,0,14/14]]_4$
$[34,17,10]_4$	$[[34,0,10/10]]_4$	$[60,30,16]_4$	$[[60,0,16/16]]_4$
$[36,18,12]_4$	$[[36,0,12/12]]_4$	$[62,31,18]_4$	$[[62,0,18/18]]_4$
$[38,19,12]_4$	$[[38,0,12/12]]_4$	$[64,32,16]_4$	$[[64,0,16/16]]_4$
$[40,20,12]_4$	$[[40,0,12/12]]_4$	$[66,33,16]_4$	$[[66,0,16/16]]_4$
$[42,21,12]_4$	$[[42,0,12/12]]_4$	$[68,34,18]_4$	$[[68,0,18/18]]_4$
$[44,22,12]_4$	$[[44,0,12/12]]_4$	$[70,35,18]_4$	$[[70,0,18/18]]_4$
$[46,23,14]_4$	$[[46,0,14/14]]_4$	$[72,36,18]_4$	$[[72,0,18/18]]_4$
$[48,24,14]_4$	$[[48,0,14/14]]_4$	$[74,37,18]_4$	$[[74,0,18/18]]_4$
$[50,25,14]_4$	$[[50,0,14/14]]_4$	$[76,38,18]_4$	$[[76,0,18/18]]_4$
$[52,26,14]_4$	$[[52,0,14/14]]_4$	$[78,39,18]_4$	$[[78,0,18/18]]_4$
$[54,27,16]_4$	$[[54,0,16/16]]_4$	$[80,40,20]_4$	$[[80,0,20/20]]_4$
$[56,28,14]_4$	$[[56,0,14/14]]_4$		

Given fixed length n and dimension k, we only consider $[n,k,d]_4$-codes C with maximal possible value for the minimum distances of their duals. For example, among 12 self-orthogonal $[10,4,4]_4$-codes, there are 4 distinct codes with $d^{\perp_{\mathrm{H}}} = 3$ while the remaining 8 codes have $d^{\perp_{\mathrm{H}}} = 2$. We take only the first four codes.

The number of distinct codes that can be used for the construction of the asymmetric quantum codes for each set of given parameters is listed in the fourth column of the table.

Comparing some entries in Table II, say, numbers 5 and 6, we notice that the $[[9,3,3/3]]_4$-code has better parameters than the $[[9,1,3/3]]_4$-code does. Both codes are included in the table in the interest of preserving the information on precisely how many of such codes there are from the classification result.

In [18, Table 7], examples of $\mathbb{F}_4$-linear self-dual codes for even lengths $2 \leq n \leq 80$ are presented. Table III lists down the resulting asymmetric quantum codes for $32 \leq n \leq 80$.

For parameters other than those listed in Table II, we do not have complete classification just yet. The Q-extension program described in [7] can be used to extend the classification effort given sufficient resources. Some classifications based on the optimality of the minimum distances of the codes can be found in [8] and in [36], although when used in the construction of asymmetric quantum codes using our framework, they do not yield good $d_z = d_x$ relative to the length n.

Many other $\mathbb{F}_4$-linear self-orthogonal codes are known. Examples can be found in [11, Table II], [28], as well as from the list of best known linear codes (BKLC) over $\mathbb{F}_4$ as explained in [17].

Table IV presents more examples of asymmetric quantum codes constructed based on known self-orthogonal linear codes up to length $n = 40$. The list of codes in Table IV is by no means exhaustive. It may be possible to find asymmetric codes with better parameters.

For lengths larger than $n = 40$, [20] provides some known $\mathbb{F}_4$-linear codes of dimension 6 that belong to the class of *quasi-twisted codes*. Based on the weight distribution of these codes [20, Table 3], we know which ones of them are self-orthogonal.

TABLE IV
ASYMMETRIC QECC FROM HERMITIAN SELF-ORTHOGONAL $\mathbb{F}_4$-LINEAR CODES FOR $n \leq 40$

No.	Code C	Code Q	Ref.
1	$[5,2,4]_4$	$[[\mathbf{5},\mathbf{1},\mathbf{3}/\mathbf{3}]]_4$	[17, BKLC]
2	$[6,2,2]_4$	$[[6,2,2/2]]_4$	[6]
3	$[8,2,6]_4$	$[[8,4,2/2]]_4$	[17, BKLC]
4	$[10,2,8]_4$	$[[10,6,2/2]]_4$	[17, BKLC]
5	$[14,7,6]_4$	$[[14,0,6/6]]_4$	[18, Table 7]
6	$[15,2,12]_4$	$[[15,11,2/2]]_4$	[17, BKLC]
7	$[16,8,6]_4$	$[[16,0,6/6]]_4$	[18, Table 7]
8	$[20,2,16]_4$	$[[20,16,2/2]]_4$	[17, BKLC]
9	$[20,5,12]_4$	$[[20,10,4/4]]_4$	[11, Table II]
10	$[20,9,8]_4$	$[[20,2,6/6]]_4$	[28, p. 788]
11	$[20,10,8]_4$	$[[20,0,8/8]]_4$	[18, Table 7]
12	$[22,8,10]_4$	$[[22,6,5/5]]_4$	[27]
13	$[22,10,8]_4$	$[[22,2,6/6]]_4$	[27]
14	$[22,11,8]_4$	$[[22,0,8/8]]_4$	[18, Table 7]
15	$[23,8,10]_4$	$[[23,7,5/5]]_4$	[27]
16	$[23,8,12]_4$	$[[23,7,5/5]]_4$	[17, BKLC]
17	$[23,10,8]_4$	$[[23,3,6/6]]_4$	[27]
18	$[24,5,16]_4$	$[[24,14,3/3]]_4$	[17, BKLC]
19	$[24,8,10]_4$	$[[24,8,5/5]]_4$	[27]
20	$[24,9,12]_4$	$[[24,6,6/6]]_4$	[17, BKLC]
21	$[24,12,8]_4$	$[[24,0,8/8]]_4$	[18, Table 7]
22	$[25,2,20]_4$	$[[25,21,2/2]]_4$	[17, BKLC]
23	$[25,5,16]_4$	$[[25,15,4/4]]_4$	[11, Table II]
24	$[25,8,10]_4$	$[[25,9,5/5]]_4$	[27]
25	$[25,10,12]_4$	$[[25,5,7/7]]_4$	[27]
26	$[26,2,20]_4$	$[[26,22,2/2]]_4$	[17, BKLC]
27	$[26,6,16]_4$	$[[26,14,4/4]]_4$	[17, BKLC]
28	$[26,9,10]_4$	$[[26,8,5/5]]_4$	[27]
29	$[26,10,10]_4$	$[[26,6,6/6]]_4$	[27]
30	$[26,11,12]_4$	$[[26,4,8/8]]_4$	[17, BKLC]
31	$[27,6,16]_4$	$[[27,15,3/3]]_4$	[17, BKLC]
32	$[27,9,10]_4$	$[[27,9,5/5]]_4$	[27]
33	$[27,10,10]_4$	$[[27,7,6/6]]_4$	[27]
34	$[27,12,12]_4$	$[[27,3,9/9]]_4$	[17, BKLC]
35	$[28,7,16]_4$	$[[28,14,5/5]]_4$	[11, Table II]
36	$[28,8,12]_4$	$[[28,12,5/5]]_4$	[27]
37	$[28,10,10]_4$	$[[28,8,6/6]]_4$	[27]
38	$[28,13,12]_4$	$[[28,2,10/10]]_4$	[17, BKLC]
39	$[29,8,16]_4$	$[[29,13,5/5]]_4$	[17, BKLC]
40	$[29,11,10]_4$	$[[29,7,6/6]]_4$	[27]
41	$[29,14,12]_4$	$[[29,1,11/11]]_4$	[17, BKLC]
42	$[30,2,24]_4$	$[[30,26,2/2]]_4$	[17, BKLC]
43	$[30,5,20]_4$	$[[30,20,4/4]]_4$	[30, Table 13.2]
44	$[30,9,12]_4$	$[[30,12,5/5]]_4$	[27]
45	$[30,11,10]_4$	$[[30,8,6/6]]_4$	[27]
46	$[30,15,12]_4$	$[[30,0,12/12]]_4$	[18, Table 7]
47	$[31,9,16]_4$	$[[31,13,6/6]]_4$	[17, BKLC]
48	$[32,6,20]_4$	$[[32,20,4/4]]_4$	[20, Table 3]
49	$[32,9,14]_4$	$[[32,14,5/5]]_4$	[27]
50	$[32,11,12]_4$	$[[32,10,6/6]]_4$	[27]
51	$[33,7,20]_4$	$[[33,19,4/4]]_4$	[17, BKLC]
52	$[33,9,14]_4$	$[[33,15,5/5]]_4$	[27]
53	$[33,10,16]_4$	$[[33,13,6/6]]_4$	[17, BKLC]
54	$[33,12,14]_4$	$[[33,9,7/7]]_4$	[17, BKLC]
55	$[33,15,12]_4$	$[[33,3,9/9]]_4$	[17, BKLC]
56	$[34,9,18]_4$	$[[34,16,6/6]]_4$	[27]
57	$[34,13,14]_4$	$[[34,8,8/8]]_4$	[17, BKLC]
58	$[34,16,12]_4$	$[[34,2,10/10]]_4$	[17, BKLC]
59	$[35,5,24]_4$	$[[35,25,3/3]]_4$	[17, BKLC]
60	$[35,8,20]_4$	$[[35,19,5/5]]_4$	[17, BKLC]
61	$[35,11,14]_4$	$[[35,13,6/6]]_4$	[27]
62	$[35,17,12]_4$	$[[35,1,11/11]]_4$	[17, BKLC]
63	$[36,9,16]_4$	$[[36,18,4/4]]_4$	[27]
64	$[36,11,14]_4$	$[[36,14,6/6]]_4$	[27]
65	$[37,9,20]_4$	$[[37,19,5/5]]_4$	[17, BKLC]
66	$[37,18,12]_4$	$[[37,1,11/11]]_4$	[17, BKLC]
67	$[38,6,24]_4$	$[[38,26,3/3]]_4$	[17, BKLC]
68	$[38,11,18]_4$	$[[38,16,6/6]]_4$	[17, BKLC]
69	$[39,12,18]_4$	$[[39,15,7/7]]_4$	[17, BKLC]
70	$[39,4,28]_4$	$[[39,31,3/3]]_4$	[17, BKLC]
71	$[39,7,24]_4$	$[[39,25,4/4]]_4$	[17, BKLC]
72	$[40,5,28]_4$	$[[40,30,4/4]]_4$	[11, Table II]
73	$[40,15,16]_4$	$[[40,10,7/7]]_4$	[17, BKLC]

Applying Theorem 4.5 to them yields the 12 quantum codes listed in Table V.

TABLE V
ASYMMETRIC QECC FROM HERMITIAN SELF-ORTHOGONAL QUASI-TWISTED CODES FOUND IN [20]

Code C	Code Q	Code C	Code Q
$[48,6,32]_4$	$[[48,36,3/3]]_4$	$[144,6,104]_4$	$[[144,132,3/3]]_4$
$[78,6,56]_4$	$[[78,66,4/4]]_4$	$[150,6,108]_4$	$[[150,138,3/3]]_4$
$[102,6,72]_4$	$[[102,90,3/3]]_4$	$[160,6,116]_4$	$[[160,148,3/3]]_4$
$[112,6,80]_4$	$[[112,100,3/3]]_4$	$[182,6,132]_4$	$[[182,170,3/3]]_4$
$[120,6,86]_4$	$[[120,108,4/4]]_4$	$[192,6,138]_4$	$[[192,180,3/3]]_4$
$[132,6,94]_4$	$[[132,120,3/3]]_4$	$[200,6,144]_4$	$[[200,188,3/3]]_4$

Another family of codes that we can use is the *MacDonald codes*, commonly denoted by $C_{k,u}$ with $k > u > 0$. The MacDonald codes are linear codes with parameters $[(q^k - q^u)/(q-1), k, q^{k-1} - q^{u-1}]_q$. Some historical background and a construction of their generator matrices can be found in [4]. It is known that these codes are *two-weight codes*. That is, they have nonzero codewords of only two possible weights. In [10, Figures 1a and 2a], the MacDonald codes are labeled SU1. There are $q^k - q^{k-u}$ codewords of weight $q^{k-1} - q^{u-1}$ and $q^{k-u} - 1$ codewords of weight q^{k-1}.

The MacDonald codes satisfy the equality of the *Griesmer bound* which says that, for any $[n, k \geq 1, d]_q$-code,

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil. \qquad \text{(VII.2)}$$

Example 7.3: For $q = 4, k > u > 1$, the MacDonald codes are self-orthogonal since both $4^{k-1} - 4^{u-1}$ and 4^{k-1} are even. For a $[(4^k - 4^u)/3, k, 4^{k-1} - 4^{u-1}]_4$-code $C_{k,u}$, we know (see [4, Lemma 4]) that $d(C_{k,u}^{\perp_{\mathrm{H}}}) \geq 3$. Using $C_{k,u}$ and applying Theorem 7.1, we get an asymmetric quantum code Q with parameters

$$[[(4^k - 4^u)/3, ((4^k - 4^u)/3) - 2k, (\geq 3)/(\geq 3)]]_4.$$

For $k \leq 5$, we have the following more explicit examples. The weight enumerator is written in an abbreviated form. For instance, $(0,1),(12,60),(16,3)$ means that the corresponding code has 1 codeword of weight 0, 60 codewords of weight 12 and 3 codewords of weight 16.

1) For $k = 3, u = 2$, we have the $[16,3,12]_4$-code with weight enumerator $(0,1),(12,60),(16,3)$. The resulting asymmetric QECC is a $[[16,10,3/3]]_4$-code. This code is listed as number 31 in Table II.
2) For $k = 4, u = 3$, we have the $[64,4,48]_4$-code with weight enumerator $(0,1),(48,252),(64,3)$. The resulting asymmetric QECC is a $[[64,56,3/3]]_4$-code.
3) For $k = 4, u = 2$, we have the $[80,4,60]_4$-code with weight enumerator $(0,1),(60,240),(64,15)$. The resulting asymmetric QECC is a $[[80,72,3/3]]_4$-code.
4) For $k = 5, u = 4$, we have the $[256,5,192]_4$-code with weight enumerator $(0,1),(192,1020),(256,3)$. The resulting asymmetric QECC is a $[[256,246,3/3]]_4$-code.
5) For $k = 5, u = 3$, we have the $[320,5,240]_4$-code with weight enumerator $(0,1),(240,1008),(256,15)$. The resulting asymmetric QECC is a $[[320,310,3/3]]_4$-code.

TABLE VI
ASYMMETRIC QECC FROM NESTED CYCLIC CODES

No.	Codes C and D	Generator Polynomials of C and of D	Code Q
1	$[3,1,3]_4$	$(x+1)(x+\omega)$	$\mathbf{[[3,1,2/2]]_4}$
	$[3,2,2]_4$	$(x+1)$	
2	$[5,1,5]_4$	$(x^2+\omega x+1)(x^2+\omega^2 x+1)$	$\mathbf{[[5,2,3/2]]_4}$
	$[5,3,3]_4$	$(x^2+\omega^2 x+1)$	
3	$[7,1,7]_4$	$(x^3+x+1)(x^3+x^2+1)$	$[[7,3,3/2]]_4$
	$[7,4,3]_4$	(x^3+x+1)	
4	$[7,3,4]_4$	$(x^3+x+1)(x+1)$	$[[7,1,3/3]]_4$
	$[7,4,3]_4$	(x^3+x+1)	
5	$[9,1,9]_4$	$(x+\omega)(x+\omega^2)(x^3+\omega)(x^3+\omega^2)$	$[[9,1,6/2]]_4$
	$[9,2,6]_4$	$(x+\omega^2)(x^3+\omega)(x^3+\omega^2)$	
6	$[9,1,9]_4$	$(x+\omega)(x+\omega^2)(x^3+\omega)(x^3+\omega^2)$	$[[9,4,3/2]]_4$
	$[9,5,3]_4$	$(x+\omega)(x^3+\omega)$	
7	$[9,1,9]_4$	$(x+\omega)(x+\omega^2)(x^3+\omega)(x^3+\omega^2)$	$\mathbf{[[9,7,2/2]]_4}$
	$[9,8,2]_4$	$(x+\omega)$	
8	$[11,5,6]_4$	$(x+1)(x^5+\omega^2 x^4+x^3+x^2+\omega x+1)$	$[[11,1,5/5]]_4$
	$[11,6,5]_4$	$(x^5+\omega^2 x^4+x^3+x^2+\omega x+1)$	
9	$[11,1,11]_4$	$(x^{11}+1)/(x+1)$	$[[11,5,5/2]]_4$
	$[11,6,5]_4$	$(x^5+\omega x^4+x^3+x^2+\omega^2 x+1)$	
10	$[13,6,6]_4$	$(x+1)(x^6+\omega x^5+\omega^2 x^3+\omega x+1)$	$[[13,1,5/5]]_4$
	$[13,7,5]_4$	$(x^6+\omega x^5+\omega^2 x^3+\omega x+1)$	
11	$[13,1,13]_4$	$(x^{13}+1)/(x+1)$	$[[13,6,5/2]]_4$
	$[13,7,5]_4$	$(x^6+\omega x^5+\omega^2 x^3+\omega x+1)$	
12	$[15,3,11]_4$	$(x^{15}+1)/((x+1)(x^2+\omega^2 x+\omega^2))$	$[[15,1,9/3]]_4$
	$[15,4,9]_4$	$(x^{15}+1)/((x+1)(x+\omega)(x^2+\omega^2 x+\omega^2))$	
13	$[15,6,8]_4$	$(x^9+\omega x^8+x^7+x^5+\omega x^4+\omega^2 x^2+\omega^2 x+1)$	$[[15,1,7/5]]_4$
	$[15,7,7]_4$	$(x^8+\omega^2 x^7+\omega x^6+\omega x^5+\omega^2 x^4+x^3+x^2+\omega x+1)$	
14	$[15,7,7]_4$	$(x^8+\omega^2 x^7+\omega x^6+\omega x^5+\omega^2 x^4+x^3+x^2+\omega x+1)$	$[[15,1,6/6]]_4$
	$[15,8,6]_4$	$(x^7+x^6+\omega x^4+x^2+\omega^2 x+\omega^2)$	
15	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$[[15,2,11/2]]_4$
	$[15,3,11]_4$	$(x^{15}+1)/((x+1)(x^2+\omega^2 x+\omega^2))$	
16	$[15,3,11]_4$	$(x^{15}+1)/((x+1)(x^2+\omega^2 x+\omega^2))$	$[[15,2,8/3]]_4$
	$[15,5,8]_4$	$(x^{15}+1)/((x+1)(x^2+\omega^2 x+\omega^2)(x^2+\omega^2 x+1))$	
17	$[15,6,8]_4$	$(x^{15}+1)/((x^2+\omega x+\omega)(x^2+\omega^2 x+\omega^2)(x^2+\omega^2 x+1))$	$[[15,2,6/5]]_4$
	$[15,8,6]_4$	$(x^7+x^6+\omega x^4+x^2+\omega^2 x+\omega^2)$	
18	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$[[15,3,9/2]]_4$
	$[15,4,9]_4$	$(x^{15}+1)/((x+1)(x+\omega)(x^2+\omega^2 x+\omega^2))$	
19	$[15,8,6]_4$	$x^7+\omega x^6+\omega^2 x^4+\omega^2 x^2+\omega x+\omega^2$	$[[15,4,7/3]]_4$
	$[15,12,3]_4$	$x^3+x^2+\omega^2$	
20	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$[[15,4,8/2]]_4$
	$[15,5,8]_4$	$(x^{15}+1)/((x+1)(x^2+\omega^2 x+\omega^2)(x^2+\omega^2 x+1))$	
21	$[15,4,10]_4$	$(x^{15}+1)/(x^4+\omega^2 x^3+\omega x^2+\omega x+w)$	$[[15,5,5/4]]_4$
	$[15,9,5]_4$	$x^6+\omega^2 x^5+\omega^2 x^4+x^3+x^2+\omega x+1$	
22	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$[[15,6,7/2]]_4$
	$[15,7,7]_4$	$(x^8+\omega^2 x^7+\omega x^6+\omega x^5+\omega^2 x^4+x^3+x^2+\omega x+1)$	
23	$[15,3,11]_4$	$(x^{15}+1)/((x+1)(x^2+\omega^2 x+\omega^2))$	$[[15,6,5/3]]_4$
	$[15,9,5]_4$	$(x^6+\omega x^5+x^4+x^3+\omega^2 x^2+\omega^2 x+1)$	
24	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$[[15,7,6/2]]_4$
	$[15,8,6]_4$	$(x^7+x^6+\omega x^4+x^2+\omega^2 x+\omega^2)$	
25	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$[[15,8,5/2]]_4$
	$[15,9,5]_4$	$(x^6+\omega x^5+x^4+x^3+\omega^2 x^2+\omega^2 x+1)$	
26	$[15,3,11]_4$	$(x^{15}+1)/(x^3+\omega^2 x^2+\omega^2)$	$[[15,9,3/3]]_4$
	$[15,12,3]_4$	$x^3+x^2+\omega^2$	
27	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$[[15,11,3/2]]_4$
	$[15,12,3]_4$	$x^3+x^2+\omega^2$	
28	$[15,1,15]_4$	$(x^{15}+1)/(x+1)$	$\mathbf{[[15,13,2/2]]_4}$
	$[15,14,2]_4$	$(x+\omega)$	
29	$[17,12,4]_4$	$(x^5+\omega x^3+\omega x^2+1)$	$[[17,1,9/4]]_4$
	$[17,13,4]_4$	$(x^4+x^3+\omega^2 x^2+x+1)$	
30	$[17,8,8]_4$	$(x^9+\omega x^8+\omega^2 x^7+\omega^2 x^6+\omega^2 x^3+\omega^2 x^2+\omega x+1)$	$[[17,1,7/7]]_4$
	$[17,9,7]_4$	$(x^8+\omega^2 x^7+\omega^2 x^5+\omega^2 x^4+\omega^2 x^3+\omega^2 x+1)$	
31	$[17,1,17]_4$	$(x^{17}+1)/(x+1)$	$[[17,4,9/2]]_4$
	$[17,5,9]_4$	$(x^{17}+1)/(x^5+\omega^2 x^4+\omega^2 x^3+\omega^2 x^2+\omega^2 x+1)$	
32	$[17,4,12]_4$	$(x^{17}+1)/(x^4+x^3+\omega x^2+x+1)$	$[[17,4,8/4]]_4$
	$[17,8,8]_4$	$(x^9+\omega x^8+\omega^2 x^7+\omega^2 x^6+\omega^2 x^3+\omega^2 x^2+\omega x+1)$	
33	$[17,4,12]_4$	$(x^{17}+1)/(x^4+x^3+\omega x^2+x+1)$	$[[17,5,7/4]]_4$
	$[17,9,7]_4$	$(x^8+\omega^2 x^7+\omega^2 x^5+\omega^2 x^4+\omega^2 x^3+\omega^2 x+1)$	
34	$[17,1,17]_4$	$(x^{17}+1)/(x+1)$	$[[17,8,7/2]]_4$
	$[17,9,7]_4$	$(x^8+\omega x^7+\omega x^5+\omega x^4+\omega x^3+\omega x+1)$	
35	$[17,1,17]_4$	$(x^{17}+1)/(x+1)$	$[[17,12,4/2]]_4$
	$[17,13,4]_4$	$(x^4+x^3+\omega^2 x^2+x+1)$	

6) For $k=5, u=2$, we have the $[336,5,252]_4$-code with weight enumerator $(0,1),(252,960),(256,63)$. The resulting asymmetric QECC is a $[[336,326,3/3]]_4$-code.

TABLE VII
ASYMMETRIC QECC FROM NESTED CYCLIC CODES CONTINUED

No.	Codes C and D	Generator Polynomials of C and D	Code Q
36	$[19,9,8]_4$ $[19,10,7]_4$	$(x+1)(x^9+\omega^2x^8+\omega^2x^6+\omega^2x^5+\omega x^4+\omega x^3+\omega x+1)$ $(x^9+\omega^2x^8+\omega^2x^6+\omega^2x^5+\omega x^4+\omega x^3+\omega x+1)$	$[[19,1,7/7]]_4$
37	$[19,1,19]_4$ $[19,10,7]_4$	$(x^{19}+1)/(x+1)$ $(x^9+\omega x^8+\omega x^6+\omega x^5+\omega^2x^4+\omega^2x^3+\omega^2x+1)$	$[[19,9,7/2]]_4$
38	$[21,1,21]_4$ $[21,2,14]_4$	$(x^{21}+1)/(x+1)$ $(x^{21}+1)/(x^2+\omega^2x+\omega)$	$[[21,1,14/2]]_4$
39	$[21,4,12]_4$ $[21,7,11]_4$	$(x^{21}+1)/(x^4+\omega x^3+\omega^2x^2+x+1)$ $x^{14}+\omega x^{13}+\omega^2x^{12}+\omega^2x^{10}+x^8+\omega^2x^7+x^6+\omega^2x^4+\omega^2x^2+\omega x+1$	$[[21,3,11/3]]_4$
40	$[21,4,12]_4$ $[21,8,9]_4$	$(x^{21}+1)/(x^4+\omega^2x^3+\omega x^2+x+1)$ $(x^{21}+1)/(x^8+x^7+\omega x^6+\omega x^5+x^4+\omega^2x^3+x^2+x+\omega)$	$[[21,4,9/3]]_4$
41	$[21,1,21]_4$ $[21,4,12]_4$	$(x^{21}+1)/(x+1)$ $(x^{21}+1)/((x+1)(x^3+\omega^2x^2+1))$	$[[21,3,12/2]]_4$
42	$[21,7,11]_4$ $[21,10,8]_4$	$(x^{21}+1)/(x^7+\omega^2x^6+x^4+x^3+\omega^2x+1)$ $(x^{11}+\omega x^{10}+x^8+x^7+x^6+x^5+\omega x^4+\omega x^3+\omega^2x^2+\omega^2x+1)$	$[[21,3,8/5]]_4$
43	$[21,4,12]_4$ $[21,8,9]_4$	$(x^{21}+1)/(x^4+x^3+\omega x^2+\omega^2x+1)$ $(x^{21}+1)/(x^8+\omega x^7+\omega^2x^6+x^5+\omega^2x^4+\omega x^3+\omega x^2+\omega)$	$[[21,4,9/3]]_4$
44	$[21,7,11]_4$ $[21,11,6]_4$	$(x^{21}+1)/(x^7+\omega^2x^6+x^4+x^3+\omega^2x+1)$ $(x^{21}+1)/(x^{11}+x^8+\omega^2x^7+x^2+\omega)$	$[[21,4,6/5]]_4$
45	$[21,1,21]_4$ $[21,7,11]_4$	$(x^{21}+1)/(x+1)$ $(x^{21}+1)/(x^7+\omega^2x^6+x^4+x^3+\omega^2x+1)$	$[[21,6,11/2]]_4$
46	$[21,4,12]_4$ $[21,10,8]_4$	$(x^{21}+1)/(x^4+x^3+\omega x^2+\omega^2x+1)$ $(x^{11}+\omega x^{10}+x^8+x^7+x^6+x^5+\omega x^4+\omega x^3+\omega^2x^2+\omega^2x+1)$	$[[21,6,8/3]]_4$
47	$[21,7,11]_4$ $[21,14,5]_4$	$(x^{21}+1)/(x^7+\omega^2x^6+x^4+x^3+\omega^2x+1)$ $x^7+x^6+x^4+\omega x^3+\omega^2x+\omega$	$[[21,7,5/5]]_4$
48	$[21,1,21]_4$ $[21,8,9]_4$	$(x^{21}+1)/(x+1)$ $(x^{21}+1)/(x^8+\omega x^7+\omega^2x^6+x^5+\omega^2x^4+\omega x^3+\omega x^2+\omega)$	$[[21,7,9/2]]_4$
49	$[21,4,12]_4$ $[21,11,6]_4$	$(x^{21}+1)/(x^4+x^3+\omega x^2+\omega^2x+1)$ $(x^{10}+\omega x^9+x^8+\omega x^7+x^6+x^5+x^4+\omega^2x^2+\omega^2)$	$[[21,7,6/3]]_4$
50	$[21,1,21]_4$ $[21,10,8]_4$	$(x^{21}+1)/(x+1)$ $(x^{11}+\omega x^{10}+x^8+x^7+x^6+x^5+\omega x^4+\omega x^3+\omega^2x^2+\omega^2x+1)$	$[[21,9,8/2]]_4$
51	$[21,1,21]_4$ $[21,11,6]_4$	$(x^{21}+1)/(x+1)$ $(x^{10}+x^7+\omega^2x^6+x^4+\omega x^2+\omega^2)$	$[[21,10,6/2]]_4$
52	$[21,4,12]_4$ $[21,14,5]_4$	$(x^{21}+1)/(x^4+x^3+\omega x^2+\omega^2x+1)$ $(x^7+x^6+x^4+\omega x^3+\omega^2x+\omega)$	$[[21,10,5/3]]_4$
53	$[21,1,21]_4$ $[21,14,5]_4$	$(x^{21}+1)/(x+1)$ $(x^7+x^6+x^4+\omega x^3+\omega^2x+\omega)$	$[[21,13,5/2]]_4$
54	$[21,1,21]_4$ $[21,17,3]_4$	$(x^{21}+1)/(x+1)$ $(x+\omega)(x^3+\omega^2x^2+1)$	$[[21,16,3/2]]_4$
55	$[21,1,21]_4$ $[21,20,2]_4$	$(x^{21}+1)/(x+1)$ $(x+\omega)$	$\mathbf{[[21,19,2/2]]_4}$
56	$[23,11,8]_4$ $[23,12,7]_4$	$(x+1)(x^{11}+x^{10}+x^6+x^5+x^4+x^2+1)$ $(x^{11}+x^{10}+x^6+x^5+x^4+x^2+1)$	$[[23,1,7/7]]_4$
57	$[23,1,23]_4$ $[23,12,7]_4$	$(x^{23}+1)/(x+1)$ $(x^{11}+x^9+x^7+x^6+x^5+x+1)$	$[[23,11,7/2]]_4$
58	$[25,1,25]_4$ $[25,3,15]_4$	$(x^{25}+1)/(x+1)$ $(x^{25}+1)/(x^3+\omega x^2+\omega x+1)$	$[[25,2,15/2]]_4$
59	$[25,12,4]_4$ $[25,14,4]_4$	$(x^{13}+\omega^2x^{12}+\omega^2x^{11}+x^{10}+\omega x^8+x^7+x^6+\omega x^5+x^3+\omega^2x^2+\omega^2x+1)$ $(x^{11}+x^{10}+\omega x^6+\omega x^5+x+1)$	$[[25,2,4/4]]_4$
60	$[25,1,25]_4$ $[25,5,5]_4$	$(x^{25}+1)/(x+1)$ $(x^{25}+1)/(x^5+1)$	$[[25,4,5/2]]_4$
61	$[25,10,4]_4$ $[25,14,4]_4$	$(x^{15}+\omega^2x^{10}+\omega^2x^5+1)$ $(x^{11}+x^{10}+\omega x^6+\omega x^5+x+1)$	$[[25,4,4/3]]_4$
62	$[25,10,4]_4$ $[25,15,3]_4$	$(x^{15}+\omega^2x^{10}+\omega^2x^5+1)$ $(x^{10}+\omega x^5+1)$	$[[25,5,3/3]]_4$
63	$[25,1,25]_4$ $[25,13,4]_4$	$(x^{23}+1)/(x+1)$ $(x^{12}+\omega x^{11}+x^{10}+\omega^2x^7+x^6+\omega^2x^5+x^2+\omega x+1)$	$[[25,12,4/2]]_4$
64	$[25,1,25]_4$ $[25,15,3]_4$	$(x^{23}+1)/(x+1)$ $(x^{10}+\omega x^5+1)$	$[[25,14,3/2]]_4$
65	$[25,1,25]_4$ $[25,23,2]_4$	$(x^{23}+1)/(x+1)$ $(x^2+\omega x+1)$	$[[25,22,2/2]]_4$

VIII. CONSTRUCTION FROM NESTED LINEAR CYCLIC CODES

The asymmetric quantum codes that we have constructed so far have $d_z = d_x$. From this section onward, we construct asymmetric quantum codes with $d_z \geq d_x$. In most cases, $d_z > d_x$.

It is well established that, under the natural correspondence of vectors and polynomials, the study of cyclic codes in $\mathbb{F}_q^n$ is equivalent to the study of ideals in the residue class ring

$$\mathcal{R}_n = \mathbb{F}_q[x]/(x^n - 1).$$

The study of ideals in $\mathcal{R}_n$ depends on factoring $x^n - 1$. Basic results concerning and the properties of cyclic codes can be found in [24, Ch. 4] or [29, Ch. 7]. A cyclic code C is a subset of a cyclic code D of equal length over $\mathbb{F}_q$ if and only if the generator polynomial of D divides the generator polynomial of C. Both polynomials divide $x^n - 1$. Once the factorization of $x^n - 1$ into irreducible polynomials is known, the nestedness property becomes apparent.

We further require that n be relatively prime to 4 to exclude the so-called repeated-root cases since the resulting cyclic codes when n is not relatively prime to 4 have inferior parameters. See [12, p. 976] for comments and references regarding this matter.

Theorem 8.1: Let C and D be cyclic codes of parameters $[n, k_1, d_1]_4$ and $[n, k_2, d_2]_4$, respectively, with $C \subseteq D$, then there exists an asymmetric quantum code Q with parameters $[[n, k_2 - k_1, d_z/d_x]]_4$, where

$$\{d_z, d_x\} = \{d(C^{\perp_{\mathrm{H}}}), d_2\}. \tag{VIII.1}$$

Proof: Apply Theorem 4.5 by taking $C_1 = C^{\perp_{\mathrm{tr}}}$ and $C_2 = D$. Since C is an $[n, k_1, d_1]_4$ code, C is an additive code of parameters $(n, 2^{2k_1}, d_1)_4$. Similarly, D is an additive code of parameters $(n, 2^{2k_2}, d_2)_4$. The values for d_z and d_x can be verified by simple calculations. ∎

Example 8.2: Let C be the repetition $[n, 1, n]_4$-code generated by the polynomial $(x^n + 1)/(x + 1)$. If we take $C = D$ in Theorem 8.1, then we get a quantum code Q with parameters $[[\mathbf{n}, \mathbf{0}, \mathbf{n/2}]]_\mathbf{4}$.

Tables VI and VII list examples of asymmetric quantum codes constructed from nested cyclic codes up to $n = 25$. We exclude the case $C = D$ since the parameters of the resulting quantum code Q are $[[n, 0, d(C)/2]]_4$ which are never better than those of the code Q in Example 8.2. Among the resulting codes Q of equal length and dimension, we choose one with the largest d_z, d_x values. For codes Q with equal length and distances, we choose one with the largest dimension.

IX. Construction From Nested Linear BCH Codes

It is well known (see [12, Sec. 3]) that finding the minimum distance or even finding a good lower bound on the minimum distance of a cyclic code is not a trivial problem. One important family of cyclic codes is the family of BCH codes. Their importance lies on the fact that their designed distance provides a reasonably good lower bound on the minimum distance. For more on BCH codes, [24, Ch. 5] can be consulted.

The BCH Code constructor in MAGMA can be used to find nested codes to produce more asymmetric quantum codes. Table VIII lists down the BCH codes over $\mathbb{F}_4$ for $n = 27$ to $n = 51$ with n coprime to 4. For a fixed length n, the codes are nested, i.e., a code C with dimension k_1 is a subcode of a code D with dimension $k_2 > k_1$. The construction process can be done for larger values of n if so desired.

The range of the designed distances that can be supplied into MAGMA to come up with the code C and the actual minimum distance of C are denoted by $\delta(C)$ and $d(C)$, respectively. The minimum distance of $C^{\perp_{\mathrm{tr}}}$, which is needed in the computation of d_z, d_x, is denoted by $d(C^{\perp_{\mathrm{tr}}})$. To save space, the BCH $[n, 1, n]_4$ repetition code generated by the all one vector $\mathbf{1}$ is not listed down in the table although this code is used in the construction of many asymmetric quantum codes presented in Table IX.

TABLE VIII
BCH Codes Over $\mathbb{F}_4$ With $2 \leq k < n$ for $27 \leq n \leq 51$

No.	n	$\delta(C)$	$d(C)$	**Code** C	$d(C^{\perp_{\mathrm{tr}}})$
1	27	2	2	$[27, 18, 2]_4$	3
2		3	3	$[27, 9, 3]_4$	2
3		$4-6$	6	$[27, 6, 6]_4$	2
4		$7-9$	9	$[27, 3, 9]_4$	2
5		$10-18$	18	$[27, 2, 18]_4$	2
6	29	2	11	$[29, 15, 11]_4$	12
7	31	$2-3$	3	$[31, 26, 3]_4$	16
8		$4-5$	5	$[31, 21, 5]_4$	12
9		$6-7$	7	$[31, 16, 7]_4$	8
10		$8-11$	11	$[31, 11, 11]_4$	6
11		$12-15$	15	$[31, 6, 15]_4$	4
12	33	2	2	$[33, 28, 2]_4$	18
13		3	3	$[33, 23, 3]_4$	12
14		$4-5$	8	$[33, 18, 8]_4$	11
15		6	10	$[33, 13, 10]_4$	6
16		7	11	$[33, 8, 11]_4$	4
17		$8-11$	11	$[33, 3, 11]_4$	2
18		$12-22$	22	$[33, 2, 22]_4$	2
19	35	2	3	$[35, 29, 3]_4$	16
20		3	3	$[35, 23, 3]_4$	8
21		$4-5$	5	$[35, 17, 5]_4$	8
22		6	7	$[35, 14, 7]_4$	8
23		7	7	$[35, 8, 7]_4$	4
24		$8-14$	15	$[35, 6, 15]_4$	4
25		15	15	$[35, 4, 15]_4$	2
26	37	2	11	$[37, 19, 11]_4$	12
27	39	2	2	$[39, 33, 2]_4$	18
28		3	3	$[39, 27, 3]_4$	12
29		$4-6$	9	$[39, 21, 9]_4$	12
30		7	10	$[39, 15, 10]_4$	6
31		$8-13$	13	$[39, 9, 13]_4$	4
32		14	15	$[39, 8, 15]_4$	3
33		$15-26$	26	$[39, 2, 26]_4$	2
34	41	2	6	$[41, 31, 6]_4$	20
35		3	9	$[41, 21, 9]_4$	10
36		$4-6$	20	$[41, 11, 20]_4$	7
37	43	2	5	$[43, 36, 5]_4$	27
38		3	6	$[43, 29, 6]_4$	14
39		$4-6$	11	$[43, 22, 11]_4$	12
40		7	13	$[43, 15, 13]_4$	6
41		$8-9$	26	$[43, 8, 26]_4$	5
42	45	2	2	$[45, 39, 2]_4$	12
43		3	3	$[45, 33, 3]_4$	8
44		$4-5$	5	$[45, 31, 5]_4$	8
45		6	6	$[45, 28, 6]_4$	8
46		7	7	$[45, 26, 7]_4$	8
47		$8-9$	9	$[45, 20, 9]_4$	6
48		10	10	$[45, 18, 10]_4$	6
49		11	11	$[45, 15, 11]_4$	3
50		$12-15$	15	$[45, 9, 15]_4$	2
51		$16-18$	18	$[45, 8, 18]_4$	2
52		$19-21$	21	$[45, 6, 21]_4$	2
53		$22-30$	30	$[45, 4, 30]_4$	2
54		$31-33$	33	$[45, 3, 33]_4$	2
55	47	$2-5$	11	$[47, 24, 11]_4$	12
56	49	$2-3$	3	$[49, 28, 3]_4$	4
57		$4-7$	7	$[49, 7, 7]_4$	2
58		$8-21$	21	$[49, 4, 21]_4$	2
59	51	2	2	$[51, 47, 2]_4$	36
60		3	3	$[51, 43, 3]_4$	24
61		$4-5$	5	$[51, 39, 5]_4$	24
62		6	9	$[51, 35, 9]_4$	22
63		7	9	$[51, 31, 9]_4$	14
64		$8-9$	9	$[51, 27, 9]_4$	10
65		$10-11$	14	$[51, 23, 14]_4$	10
66		$12-17$	17	$[51, 19, 17]_4$	8
67		18	18	$[51, 18, 18]_4$	8
68		19	19	$[51, 14, 19]_4$	6
69		$20-22$	27	$[51, 10, 27]_4$	6
70		$23-34$	34	$[51, 6, 34]_4$	4
71		35	35	$[51, 5, 35]_4$	3

Table IX presents the resulting asymmetric quantum codes from nested BCH Codes based on Theorem 4.5. The inner codes

TABLE IX
ASYMMETRIC QECC FROM BCH CODES

No.	n	Code $C_1^{\perp_{\mathrm{tr}}}$	Code C_2	Code Q	No.	n	Code $C_1^{\perp_{\mathrm{tr}}}$	Code C_2	Code Q
1	27	$[27,1,27]_4$	$[27,2,18]_4$	$[[27,1,18/2]]_4$	76	43	$[43,8,26]_4$	$[43,29,6]_4$	$[[43,21,6/5]]_4$
2		$[27,1,27]_4$	$[27,3,9]_4$	$[[27,2,9/2]]_4$	77		$[43,1,43]_4$	$[43,29,6]_4$	$[[43,28,6/2]]_4$
3		$[27,1,27]_4$	$[27,6,6]_4$	$[[27,5,6/2]]_4$	78		$[43,8,26]_4$	$[43,36,5]_4$	$[[43,28,5/5]]_4$
4		$[27,1,27]_4$	$[27,9,3]_4$	$[[27,8,3/2]]_4$	79		$[43,1,43]_4$	$[43,36,5]_4$	$[[43,35,5/2]]_4$
5		$[27,1,27]_4$	$[27,18,2]_4$	$[[27,17,2/2]]_4$	80	45	$[45,1,45]_4$	$[45,3,33]_4$	$[[45,2,33/2]]_4$
6	29	$[29,1,29]_4$	$[29,15,11]_4$	$[[29,14,11/2]]_4$	81		$[45,18,10]_4$	$[45,20,9]_4$	$[[45,2,9/6]]_4$
7	31	$[31,1,31]_4$	$[31,6,15]_4$	$[[31,5,15/2]]_4$	82		$[45,1,45]_4$	$[45,4,30]_4$	$[[45,3,30/2]]_4$
8		$[31,21,5]_4$	$[31,26,3]_4$	$[[31,5,12/3]]_4$	83		$[45,15,11]_4$	$[45,18,10]_4$	$[[45,3,10/3]]_4$
9		$[31,6,15]_4$	$[31,11,11]_4$	$[[31,5,11/4]]_4$	84		$[45,1,45]_4$	$[45,6,21]_4$	$[[45,5,21/2]]_4$
10		$[31,16,7]_4$	$[31,21,5]_4$	$[[31,5,8/5]]_4$	85		$[45,15,11]_4$	$[45,20,9]_4$	$[[45,5,9/3]]_4$
11		$[31,11,11]_4$	$[31,16,7]_4$	$[[31,5,7/6]]_4$	86		$[45,26,7]_4$	$[45,31,5]_4$	$[[45,5,8/5]]_4$
12		$[31,1,31]_4$	$[31,11,11]_4$	$[[31,10,11/2]]_4$	87		$[45,1,45]_4$	$[45,8,18]_4$	$[[45,7,18/2]]_4$
13		$[31,16,7]_4$	$[31,26,3]_4$	$[[31,10,8/3]]_4$	88		$[45,26,7]_4$	$[45,33,3]_4$	$[[45,7,8/3]]_4$
14		$[31,6,15]_4$	$[31,16,7]_4$	$[[31,10,7/4]]_4$	89		$[45,1,45]_4$	$[45,9,15]_4$	$[[45,8,15/2]]_4$
15		$[31,11,11]_4$	$[31,21,5]_4$	$[[31,10,6/5]]_4$	90		$[45,18,10]_4$	$[45,26,7]_4$	$[[45,8,7/6]]_4$
16		$[31,1,31]_4$	$[31,16,7]_4$	$[[31,15,7/2]]_4$	91		$[45,18,10]_4$	$[45,28,6]_4$	$[[45,10,6/6]]_4$
17		$[31,11,11]_4$	$[31,26,3]_4$	$[[31,15,6/3]]_4$	92		$[45,15,11]_4$	$[45,26,7]_4$	$[[45,11,7/3]]_4$
18		$[31,6,15]_4$	$[31,21,5]_4$	$[[31,15,5/4]]_4$	93		$[45,18,10]_4$	$[45,31,5]_4$	$[[45,13,6/5]]_4$
19		$[31,1,31]_4$	$[31,21,5]_4$	$[[31,20,5/2]]_4$	94		$[45,1,45]_4$	$[45,15,11]_4$	$[[45,14,11/2]]_4$
20		$[31,6,15]_4$	$[31,26,3]_4$	$[[31,20,4/3]]_4$	95		$[45,18,10]_4$	$[45,33,3]_4$	$[[45,15,6/3]]_4$
21		$[31,1,31]_4$	$[31,26,3]_4$	$[[31,25,3/2]]_4$	96		$[45,15,11]_4$	$[45,31,5]_4$	$[[45,16,5/3]]_4$
22	33	$[33,1,33]_4$	$[33,2,22]_4$	$[[33,1,22/2]]_4$	97		$[45,1,45]_4$	$[45,18,10]_4$	$[[45,17,10/2]]_4$
23		$[33,23,3]_4$	$[33,28,2]_4$	$[[33,5,12/2]]_4$	98		$[45,15,11]_4$	$[45,33,3]_4$	$[[45,18,3/3]]_4$
24		$[33,18,8]_4$	$[33,23,3]_4$	$[[33,5,11/3]]_4$	99		$[45,1,45]_4$	$[45,20,9]_4$	$[[45,19,9/2]]_4$
25		$[33,8,11]_4$	$[33,13,10]_4$	$[[33,5,10/4]]_4$	100		$[45,1,45]_4$	$[45,26,7]_4$	$[[45,25,7/2]]_4$
26		$[33,13,10]_4$	$[33,18,8]_4$	$[[33,5,8/6]]_4$	101		$[45,1,45]_4$	$[45,28,6]_4$	$[[45,27,6/2]]_4$
27		$[33,18,8]_4$	$[33,28,2]_4$	$[[33,10,11/2]]_4$	102		$[45,1,45]_4$	$[45,31,5]_4$	$[[45,30,5/2]]_4$
28		$[33,8,11]_4$	$[33,18,8]_4$	$[[33,10,8/4]]_4$	103		$[45,1,45]_4$	$[45,33,3]_4$	$[[45,32,3/2]]_4$
29		$[33,1,33]_4$	$[33,13,10]_4$	$[[33,12,10/2]]_4$	104		$[45,1,45]_4$	$[45,39,2]_4$	$[[45,38,2/2]]_4$
30		$[33,8,11]_4$	$[33,23,3]_4$	$[[33,15,4/3]]_4$	105	47	$[47,1,47]_4$	$[47,24,11]_4$	$[[47,23,11/2]]_4$
31		$[33,1,33]_4$	$[33,18,8]_4$	$[[33,17,8/2]]_4$	106	49	$[49,1,49]_4$	$[49,4,21]_4$	$[[49,3,21/2]]_4$
32		$[33,8,11]_4$	$[33,28,2]_4$	$[[33,20,4/2]]_4$	107		$[49,1,49]_4$	$[49,7,7]_4$	$[[49,6,7/2]]_4$
33		$[33,1,33]_4$	$[33,23,3]_4$	$[[33,22,3/2]]_4$	108		$[49,1,49]_4$	$[49,28,3]_4$	$[[49,27,3/2]]_4$
34		$[33,1,33]_4$	$[33,28,2]_4$	$[[33,27,2/2]]_4$	109	51	$[51,5,35]_4$	$[51,6,34]_4$	$[[51,1,34/3]]_4$
35	35	$[35,14,7]_4$	$[35,17,5]_4$	$[[35,3,8/5]]_4$	110		$[51,18,18]_4$	$[51,19,17]_4$	$[[51,1,17/8]]_4$
36		$[35,1,35]_4$	$[35,6,15]_4$	$[[35,5,15/2]]_4$	111		$[51,1,51]_4$	$[51,5,35]_4$	$[[51,4,35/2]]_4$
37		$[35,6,15]_4$	$[35,14,7]_4$	$[[35,8,7/4]]_4$	112		$[51,6,34]_4$	$[51,10,27]_4$	$[[51,4,27/4]]_4$
38		$[35,6,15]_4$	$[35,17,5]_4$	$[[35,11,5/4]]_4$	113		$[51,10,27]_4$	$[51,14,19]_4$	$[[51,4,19/6]]_4$
39		$[35,14,7]_4$	$[35,29,3]_4$	$[[35,15,8/3]]_4$	114		$[51,1,51]_4$	$[51,6,34]_4$	$[[51,5,34/2]]_4$
40		$[35,1,35]_4$	$[35,17,5]_4$	$[[35,16,5/2]]_4$	115		$[51,5,35]_4$	$[51,10,27]_4$	$[[51,5,27/3]]_4$
41		$[35,6,15]_4$	$[35,29,3]_4$	$[[35,23,4/3]]_4$	116		$[51,18,18]_4$	$[51,23,14]_4$	$[[51,5,14/8]]_4$
42		$[35,1,35]_4$	$[35,29,3]_4$	$[[35,28,3/2]]_4$	117		$[51,39,5]_4$	$[51,47,2]_4$	$[[51,8,24/2]]_4$
43	37	$[37,1,37]_4$	$[37,19,2]_4$	$[[37,18,11/2]]_4$	118		$[51,6,34]_4$	$[51,14,19]_4$	$[[51,8,19/4]]_4$
44	39	$[39,1,39]_4$	$[39,2,26]_4$	$[[39,1,26/2]]_4$	119		$[51,10,27]_4$	$[51,18,18]_4$	$[[51,8,18/6]]_4$
45		$[39,1,39]_4$	$[39,2,26]_4$	$[[39,1,26/2]]_4$	120		$[51,1,51]_4$	$[51,10,27]_4$	$[[51,9,27/2]]_4$
46		$[39,8,15]_4$	$[39,9,13]_4$	$[[39,1,13/3]]_4$	121		$[51,5,35]_4$	$[51,14,19]_4$	$[[51,9,19/3]]_4$
47		$[39,21,9]_4$	$[39,27,3]_4$	$[[39,6,12/3]]_4$	122		$[51,10,27]_4$	$[51,19,17]_4$	$[[51,9,17/6]]_4$
48		$[39,9,13]_4$	$[39,15,10]_4$	$[[39,6,10/4]]_4$	123		$[51,6,34]_4$	$[51,18,18]_4$	$[[51,12,18/4]]_4$
49		$[39,15,10]_4$	$[39,21,9]_4$	$[[39,6,9/6]]_4$	124		$[51,1,51]_4$	$[51,14,19]_4$	$[[51,13,19/2]]_4$
50		$[39,1,39]_4$	$[39,8,15]_4$	$[[39,7,15/2]]_4$	125		$[51,5,35]_4$	$[51,18,18]_4$	$[[51,13,18/3]]_4$
51		$[39,8,15]_4$	$[39,15,10]_4$	$[[39,7,10/3]]_4$	126		$[51,6,34]_4$	$[51,19,17]_4$	$[[51,13,17/4]]_4$
52		$[39,1,39]_4$	$[39,9,13]_4$	$[[39,8,13/2]]_4$	127		$[51,10,27]_4$	$[51,23,14]_4$	$[[51,13,14/6]]_4$
53		$[39,21,9]_4$	$[39,33,2]_4$	$[[39,12,12/2]]_4$	128		$[51,5,35]_4$	$[51,19,17]_4$	$[[51,14,17/3]]_4$
54		$[39,9,13]_4$	$[39,21,9]_4$	$[[39,12,9/4]]_4$	129		$[51,1,51]_4$	$[51,18,18]_4$	$[[51,17,18/2]]_4$
55		$[39,8,15]_4$	$[39,21,9]_4$	$[[39,13,9/3]]_4$	130		$[51,6,34]_4$	$[51,23,14]_4$	$[[51,17,14/4]]_4$
56		$[39,1,39]_4$	$[39,15,10]_4$	$[[39,14,10/2]]_4$	131		$[51,18,18]_4$	$[51,35,9]_4$	$[[51,17,9/8]]_4$
57		$[39,9,13]_4$	$[39,27,3]_4$	$[[39,18,4/3]]_4$	132		$[51,1,51]_4$	$[51,19,17]_4$	$[[51,18,17/2]]_4$
58		$[39,8,15]_4$	$[39,27,3]_4$	$[[39,19,3/3]]_4$	133		$[51,5,35]_4$	$[51,23,14]_4$	$[[51,18,14/3]]_4$
59		$[39,1,39]_4$	$[39,21,9]_4$	$[[39,20,9/2]]_4$	134		$[51,1,51]_4$	$[51,23,14]_4$	$[[51,22,14/2]]_4$
60		$[39,9,13]_4$	$[39,33,2]_4$	$[[39,24,4/2]]_4$	135		$[51,10,27]_4$	$[51,35,9]_4$	$[[51,25,9/6]]_4$
61		$[39,1,39]_4$	$[39,27,3]_4$	$[[39,26,3/2]]_4$	136		$[51,6,34]_4$	$[51,35,9]_4$	$[[51,29,9/4]]_4$
62		$[39,1,39]_4$	$[39,33,2]_4$	$[[39,32,2/2]]_4$	137		$[51,10,27]_4$	$[51,39,5]_4$	$[[51,29,6/5]]_4$
63	41	$[41,1,41]_4$	$[41,11,20]_4$	$[[41,10,20/2]]_4$	138		$[51,5,35]_4$	$[51,35,9]_4$	$[[51,30,9/3]]_4$
64		$[41,21,9]_4$	$[41,31,6]_4$	$[[41,10,10/6]]_4$	139		$[51,10,27]_4$	$[51,43,3]_4$	$[[51,33,6/3]]_4$
65		$[41,11,20]_4$	$[41,21,9]_4$	$[[41,10,9/7]]_4$	140		$[51,6,34]_4$	$[51,39,5]_4$	$[[51,33,5/4]]_4$
66		$[41,1,41]_4$	$[41,21,9]_4$	$[[41,20,9/2]]_4$	141		$[51,1,51]_4$	$[51,35,9]_4$	$[[51,34,9/2]]_4$
67		$[41,11,20]_4$	$[41,31,6]_4$	$[[41,20,7/6]]_4$	142		$[51,5,35]_4$	$[51,39,5]_4$	$[[51,34,5/3]]_4$
68		$[41,1,41]_4$	$[41,31,6]_4$	$[[41,30,6/2]]_4$	143		$[51,10,27]_4$	$[51,47,2]_4$	$[[51,37,6/2]]_4$
69	43	$[43,1,43]_4$	$[43,8,26]_4$	$[[43,7,26/2]]_4$	144		$[51,6,34]_4$	$[51,43,3]_4$	$[[51,37,4/3]]_4$
70		$[43,29,6]_4$	$[43,36,5]_4$	$[[43,7,14/5]]_4$	145		$[51,1,51]_4$	$[51,39,5]_4$	$[[51,38,5/2]]_4$
71		$[43,22,11]_4$	$[43,29,6]_4$	$[[43,7,12/6]]_4$	146		$[51,5,35]_4$	$[51,43,3]_4$	$[[51,38,3/3]]_4$
72	43	$[43,1,43]_4$	$[43,15,13]_4$	$[[43,14,13/2]]_4$	147		$[51,6,34]_4$	$[51,47,2]_4$	$[[51,41,4/2]]_4$
73		$[43,22,11]_4$	$[43,36,5]_4$	$[[43,14,12/5]]_4$	148		$[51,5,35]_4$	$[51,47,2]_4$	$[[51,42,3/2]]_4$
74		$[43,15,13]_4$	$[43,29,6]_4$	$[[43,14,6/6]]_4$	149		$[51,1,51]_4$	$[51,47,2]_4$	$[[51,46,2/2]]_4$
75		$[43,1,43]_4$	$[43,22,11]_4$	$[[43,21,11/2]]_4$					

are listed in the column denoted by Code $C_1^{\perp_{\mathrm{tr}}}$ while the corresponding larger codes are put in the column denoted by Code C_2. The values for d_z, d_x are derived from the last column of Table VIII while keeping Proposition 4.2 in mind.

X. Asymmetric Quantum Codes From Nested Additive Codes Over $\mathbb{F}_4$

To show the gain that we can get from Theorem 4.5 over the construction which is based solely on $\mathbb{F}_4$-linear codes, we exhibit asymmetric quantum codes which are derived from nested additive codes.

An example of asymmetric quantum code with $k > 0$ can be derived from a self-orthogonal additive cyclic code listed as Entry 3 in [11, Table I]. The code is of parameters $(21, 2^{16}, 9)_4$ yielding a $[[21, 5, 6/6]]_4$ quantum code Q by Theorem 4.5. In a similar manner, a $[[23, 12, 4/4]]_4$ quantum code can be derived from Entry 5 of the same table.

Another very interesting example is the $(12, 2^{12}, 6)_4$ dodecacode C mentioned in Remark 6.3. Its generator matrix G is given in (X.1).

Let G_D, G_E be matrices formed, respectively, by deleting the last 4 and 8 rows of G. Construct two additive codes $D, E \subset C$ with generator matrices G_D and G_E, respectively. Applying Theorem 4.5 with $C_1 = D^{\perp_{\mathrm{tr}}}$ and $C_2 = C$ yields an asymmetric quantum code Q with parameters $[[12, 2, 6/3]]_4$. Performing the same process to $E \subset C$ results in a $[[12, 4, 6/2]]_4$-code

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & \omega & \omega & \omega & \omega & \omega \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega & \omega & \omega & \omega & \omega & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \omega & \overline{\omega} & 0 & 0 & 0 & 1 & \omega & \overline{\omega} \\ 0 & 0 & 0 & \omega & \overline{\omega} & 1 & 0 & 0 & 0 & \omega & \overline{\omega} & 1 \\ 1 & \overline{\omega} & \omega & 0 & 0 & 0 & 1 & \overline{\omega} & \omega & 0 & 0 & 0 \\ \omega & 1 & \overline{\omega} & 0 & 0 & 0 & \omega & 1 & \overline{\omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \overline{\omega} & \omega & \omega & \overline{\omega} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 1 & \overline{\omega} & 1 & \omega & \overline{\omega} & 0 & 0 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\omega} & \omega & 1 \\ \overline{\omega} & 1 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \overline{\omega} & \omega \end{pmatrix}. \tag{X.1}$$

The next three subsections present more systematic approaches to finding good asymmetric quantum codes based on nested additive codes.

A. Construction From Circulant Codes

As is the case with linear codes, an additive code C is said to be cyclic if, given a codeword $\mathbf{v} \in C$, the cyclic shift of $\mathbf{v}$ is also in C. It is known (see [11, Th. 14]) that any additive cyclic $(n, 2^k)_4$-code C has at most two generators. A more detailed study of additive cyclic codes over $\mathbb{F}_4$ is given in [23].

Instead of using additive cyclic codes, a subfamily which is called additive circulant $(n, 2^n)_4$-code in [19] is used for ease of computation. An additive circulant code C has as a generator matrix G the complete cyclic shifts of just one codeword $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. We call G the cyclic development of $\mathbf{v}$. More explicitly, G is given by

$$G = \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_{n-1} & v_n \\ v_n & v_1 & v_2 & \cdots & v_{n-2} & v_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_2 & v_3 & v_4 & \cdots & v_n & v_1 \end{pmatrix}. \tag{X.2}$$

TABLE X
Asymmetric Quantum Codes From Additive Circulant Codes for $n \leq 30$

No.	n	**Generator v**	**del**	**Code** Q
1	8	$(\overline{\omega}, 1, \omega, 0, 1)$	1	$[[8, 0.5, 4/2]]_4$
2	10	$(\overline{\omega}, 1, \omega, \omega, 0, 0, 1)$	2	$[[10, 1, 4/3]]_4$
3			3	$[[10, 1.5, 4/2]]_4$
4	12	$(1, 0, \omega, 1, \omega, 0, 1)$	2	$[[12, 1, 5/3]]_4$
5	14	$(\omega, 1, 1, \overline{\omega}, 0, \omega, 0, 1)$	1	$[[14, 0.5, 6/3]]_4$
6			5	$[[14, 2.5, 6/2]]_4$
7	16	$(\omega, 1, \overline{\omega}, \omega, 0, 0, 0, \omega, 1)$	2	$[[16, 1, 6/4]]_4$
8	16	$(\omega, 1, 1, 0, 0, \overline{\omega}, \omega, 0, 1)$	6	$[[16, 3, 6/3]]_4$
9	16	$(\omega, 1, \overline{\omega}, \omega, 0, 0, 0, \omega, 1)$	7	$[[16, 3.5, 6/2]]_4$
10	19	$(1, 0, \omega, \overline{\omega}, 1, \overline{\omega}, \omega, 0, 1)$	4	$[[19, 2, 7/4]]_4$
11	20	$(\overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, \omega, 1, 0, 0, 0, 1, 1)$	2	$[[20, 1, 8/5]]_4$
12			3	$[[20, 1.5, 8/4]]_4$
13			7	$[[20, 3.5, 8/3]]_4$
14	22	$(\omega, \omega, \omega, \omega, 1, 1, \overline{\omega}, \omega, 0, \omega, \omega, 0, \omega, \omega, 1, 1)$	4	$[[22, 2, 8/5]]_4$
15			6	$[[22, 3, 8/4]]_4$
16			10	$[[22, 5, 8/3]]_4$
17			11	$[[22, 5.5, 8/2]]_4$
18	23	$(1, \omega, \omega, 1, \overline{\omega}, 1, \omega, \omega, 1)$	2	$[[23, 1, 8/4]]_4$
19			6	$[[23, 3, 8/3]]_4$
20	25	$(1, 1, \omega, 0, 1, \overline{\omega}, 1, 0, \omega, 1, 1)$	3	$[[25, 1.5, 8/5]]_4$
21			6	$[[25, 3, 8/4]]_4$
22	26	$(1, 0, \omega, \omega, \omega, \overline{\omega}, \omega, \omega, \omega, 0, 1)$	5	$[[26, 2.5, 8/4]]_4$
23			6	$[[26, 3, 8/3]]_4$
24	27	$(1, 0, \omega, 1, \omega, \overline{\omega}, \omega, 1, \omega, 0, 1)$	1	$[[27, 0.5, 8/5]]_4$
25	27	$(1, 0, \omega, \omega, 1, \overline{\omega}, 1, \omega, \omega, 0, 1)$	5	$[[27, 2.5, 8/4]]_4$
26			6	$[[27, 3, 8/3]]_4$
27	28	$(\overline{\omega}, \omega, \overline{\omega}, 1, \overline{\omega}, 1, \omega, \omega, \overline{\omega}, \overline{\omega}, \omega, \omega, 0, 1, 1)$	1	$[[28, 0.5, 10/7]]_4$
28			2	$[[28, 1, 10/5]]_4$
29			9	$[[28, 4.5, 10/4]]_4$
30			11	$[[28, 5.5, 10/3]]_4$
31	29	$(1, \omega, 0, \omega, \overline{\omega}, 1, \overline{\omega}, \omega, \overline{\omega}, 1, \overline{\omega}, \omega, 0, \omega, 1)$	1	$[[29, 0.5, 11/7]]_4$
32			3	$[[29, 1.5, 11/6]]_4$
33			8	$[[29, 4, 11/4]]_4$
34			12	$[[29, 6, 11/3]]_4$
35	30	$(\overline{\omega}, 0, \overline{\omega}, \omega, 1, \omega, 0, \overline{\omega}, \omega, 0, 1, \omega, 1, 1, 0, 1)$	5	$[[30, 2.5, 12/6]]_4$
36			6	$[[30, 3, 12/5]]_4$
37			10	$[[30, 5, 12/3]]_4$
38			11	$[[30, 5.5, 12/2]]_4$

To generate a subcode of a circulant extremal self-dual code C we delete the rows of its generator matrix G starting from the last row, the first row being the generating codeword $\mathbf{v}$. We record the best possible combinations of the size of the resulting code Q and $\{d_z, d_x\}$. To save space, only new codes or codes with better parameters than those previously constructed are presented. Table X summarizes the finding for $n \leq 30$. Zeros on the right of each generating codeword are omitted. The number of last rows to be deleted to obtain the desired subcode is given in the column denoted by **del**.

B. Construction From 4-Circulant and Bordered 4-Circulant Codes

Following [19], a 4-circulant additive $(n, 2^n)_4$-code of even length n has the following generator matrix:

$$G = \begin{pmatrix} I_{\frac{n}{2}} & A_{\frac{n}{2}} \\ B_{\frac{n}{2}} & I_{\frac{n}{2}} \end{pmatrix} \tag{X.3}$$

where $I_{\frac{n}{2}}$ is an identity matrix of size $n/2$ and $A_{\frac{n}{2}}, B_{\frac{n}{2}}$ are circulant matrices of the form given in (X.2).

Starting from a generator matrix G_C of an additive 4-circulant code C, a matrix G_D is constructed by deleting the last r rows of G_C to derive an additive subcode D of C. For $n \leq 30$ we found three asymmetric quantum codes which are either new or better than the ones previously constructed. Table XI presents the findings. Under the column denoted by A, B we list down the generating codewords for the matrices A and B, in that order.

TABLE XI
ASYMMETRIC QUANTUM CODES FROM ADDITIVE 4-CIRCULANT CODES FOR $n \leq 30$

n	A, B	**del**	**Code** Q
14	$(1, \omega, \omega, \omega, 1, 0, 0)$, $(1, 0, 0, 1, \omega, \omega, \omega)$	2	$[[14, 1, 6/3]]_4$
20	$(\overline{\omega}, \omega, \omega, \omega, \omega, \omega, \overline{\omega}, 0, \omega, 0)$, $(\omega, 0, \overline{\omega}, 0, \omega, \omega, \overline{\omega}, \omega, \overline{\omega}, \omega)$	8	$[[20, 4, 8/2]]_4$

TABLE XII
ASYMMETRIC QUANTUM CODES FROM ADDITIVE BORDERED 4-CIRCULANT CODES FOR $n \leq 30$

n	e	A, B	**del**	**Code** Q
23	ω	$(\overline{\omega}, 1, \overline{\omega}, \omega, \omega, 1, 1, 0, 0, 0, 0)$, $(0, \overline{\omega}, \omega, \omega, \omega, \omega, \overline{\omega}, 1, 0, 1, \omega)$	1	$[[23, 0.5, 8/5]]_4$
23	ω	$(\overline{\omega}, 1, \overline{\omega}, \omega, \omega, 1, 1, 0, 0, 0, 0)$, $(0, \overline{\omega}, \omega, \omega, \omega, \omega, \overline{\omega}, 1, 0, 1, \omega)$	4	$[[23, 2, 8/4]]_4$
23	ω	$(1, 1, \overline{\omega}, 1, \overline{\omega}, 1, 1, 0, 0, 0, 0)$, $(\overline{\omega}, \overline{\omega}, \omega, \omega, \overline{\omega}, \overline{\omega}, \omega, 0, \omega, 0, \omega)$	9	$[[23, 4.5, 8/2]]_4$
25	ω	$(\omega, 1, 1, \omega, \overline{\omega}, 1, 0, 1, 0, 0, 0, 0)$, $(1, \overline{\omega}, \overline{\omega}, \omega, \omega, \overline{\omega}, \omega, \overline{\omega}, 1, 0, \overline{\omega}, \omega)$	4	$[[25, 2, 8/5]]_4$
25	ω	$(\omega, 1, \overline{\omega}, 1, 1, \omega, 0, 1, 0, 0, 0, 0)$, $(1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, \omega, 1, \omega, \overline{\omega}, 0, \omega)$	10	$[[25, 5, 8/2]]_4$

Let $\mathbf{d} = (\omega, \ldots, \omega)$ and $\mathbf{c}$ be the transpose of $\mathbf{d}$. A bordered 4-circulant additive $(n, 2^n)_4$-code of odd length n has the following generator matrix:

$$G = \begin{pmatrix} e & \mathbf{1} & \mathbf{d} \\ \mathbf{1} & I_{\frac{n-1}{2}} & A_{\frac{n-1}{2}} \\ \mathbf{c} & B_{\frac{n-1}{2}} & I_{\frac{n-1}{2}} \end{pmatrix} \quad \text{(X.4)}$$

where e is one of $0, 1, \omega$, or $\overline{\omega}$, and $A_{\frac{n-1}{2}}, B_{\frac{n-1}{2}}$ are circulant matrices.

We perform the procedure of constructing a subcode D of C by deleting the rows of G_C, starting from the last row. For $n \leq 30$, the five asymmetric quantum codes, either new or of better parameters, found can be seen in Table XII. As before, under the column denoted by A, B we list down the generating codewords for the matrices A and B, in that order.

Remark 10.1: A similar procedure has been done to the generator matrices of s-extremal additive codes found in [3] and [34] as well as to the formally self-dual additive codes of [21]. So far we have found no new or better asymmetric codes from these sources.

Deleting the rows of G_C in a more careful way than just doing so consecutively starting from the last row may yield new or better asymmetric quantum codes. The process, however, is more time consuming.

Consider the following instructive example taken from bordered 4-circulant codes. Let

$$P := \{1, 2, 4, 5, 8, 10, 12, 13, 14, 15, 16\}.$$

Let C be a bordered 4-circulant code of length $n = 23$ with generator matrix G_C in the form given in (X.4) with $e = \omega$ and with the circulant matrices A, B generated by, respectively,

$$(\overline{\omega}, 1, \overline{\omega}, \omega, \omega, 1, 1, 0, 0, 0, 0), \text{ and}$$
$$(0, \overline{\omega}, \omega, \omega, \omega, \omega, \overline{\omega}, 1, 0, 1, \omega).$$

Use the rows of G_C indexed by the set P as the rows of G_D, the generator matrix of a subcode D of C. Using $D \subset C$, a $[[23, 6, 8/2]]_4$ asymmetric quantum code Q can be constructed.

If we use the same code C but G_D is now G_C with rows 3,6,7,9, and 11 deleted, then, in a similar manner, we get a $[[23, 2.5, 8/4]]_4$ code Q.

C. Construction From Two Proper Subcodes

In the previous two subsections, the larger code C is an additive self-dual code while the subcode D of C is constructed by deleting rows of G_C. New or better asymmetric quantum codes can be constructed from two nested proper subcodes of an additive self-dual code. The following two examples illustrate this fact.

Example 10.2: Let C be a self-dual Type II additive code of length 22 with generating vector

$$\mathbf{v} = (\omega, \omega, \omega, \omega, 1, 1, \overline{\omega}, \omega, 0, \omega, \omega, 0, \omega, \omega, 1, 1, 0, \ldots, 0).$$

Let G_C be the generator matrix of C from the cyclic development of $\mathbf{v}$. Derive the generator matrices G_D of D and G_E of E by deleting, respectively, the last 10 and 11 rows of G_C. Applying Theorem 4.5 on $E \subset D$ yields an asymmetric $[[22, 0.5, 10/2]]_4$-code Q.

Example 10.3: Let C be a self-dual Type I additive code of length 25 labeled $C_{25,4}$ in [19] with generating vector

$$\mathbf{v} = (1, 1, \omega, 0, 1, \overline{\omega}, 1, 0, \omega, 1, 1, 0, 0, \ldots, 0).$$

Let G_C be the generator matrix of C from the cyclic development of $\mathbf{v}$. Derive the generator matrices G_D of D and G_E of E by deleting, respectively, the last 5 and 6 rows of G_C. An asymmetric $[[25, 0.5, 9/4]]_4$-code Q is hence constructed.

XI. CONCLUSIONS AND OPEN PROBLEMS

In this paper, we establish a new method of deriving asymmetric quantum codes from additive, not necessarily linear, codes over the field $\mathbb{F}_q$ with q an even power of a prime p.

Many asymmetric quantum codes over $\mathbb{F}_4$ are constructed. These codes are different from those listed in prior works (see [1, Ch. 17] and [32]) on asymmetric quantum codes.

There are several open directions to pursue. On $\mathbb{F}_4$-additive codes, exploring the notion of nestedness in tandem with the dual distance of the inner code is a natural continuation if we are to construct better asymmetric quantum codes. An immediate project is to understand such relation in the class of cyclic (not merely circulant) codes studied in [23].

Extension to codes over $\mathbb{F}_9$ or $\mathbb{F}_{16}$ is another option worth considering. More generally, establishing propagation rules may help us find better bounds on the parameters of asymmetric quantum codes.

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